Basic Topology and Games

Determinacy

Jacques Duparc

UNIL | Université de Lausanne

the secret word is.....

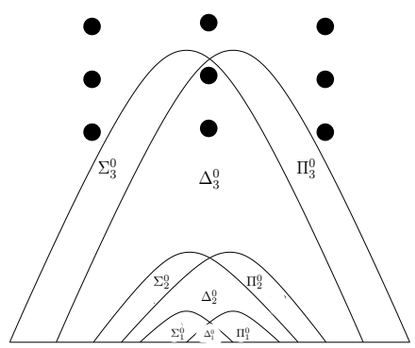
UNIL | Université de Lausanne

UNIL | Université de Lausanne

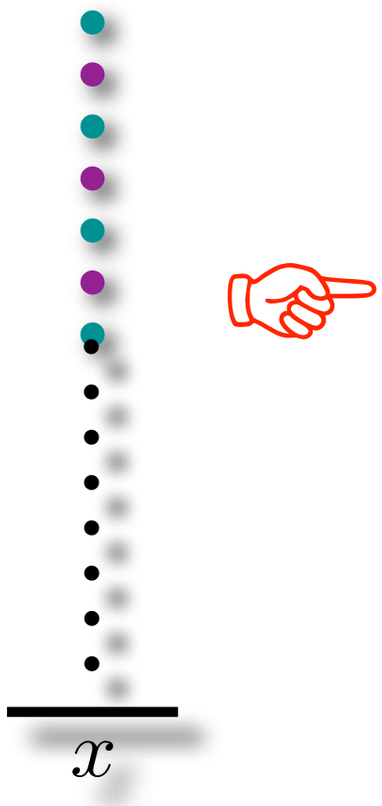


An example: evaluation game for 1st order logic

non determined game

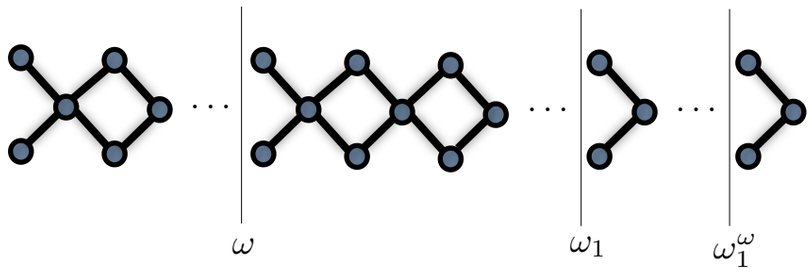


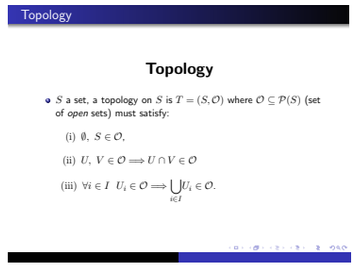
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

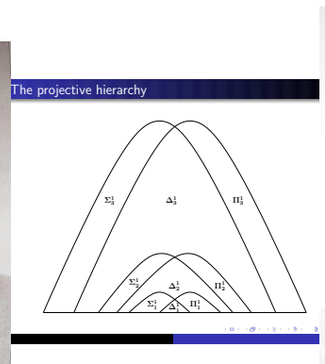
Games that construct mappings



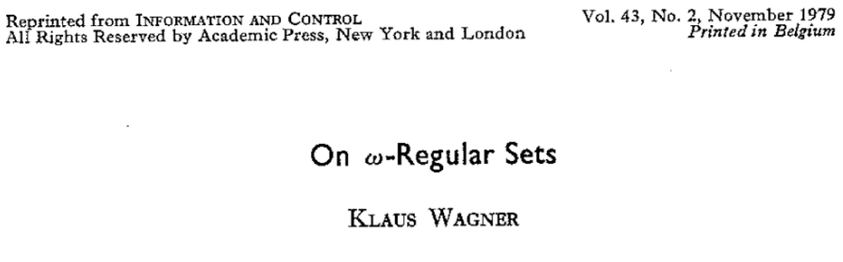










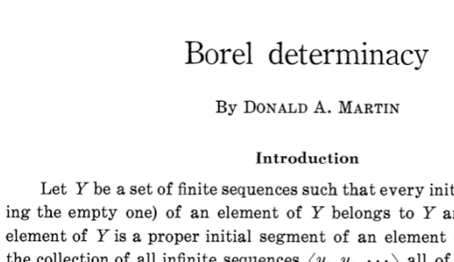


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between

UNIL | Université de Lausanne



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



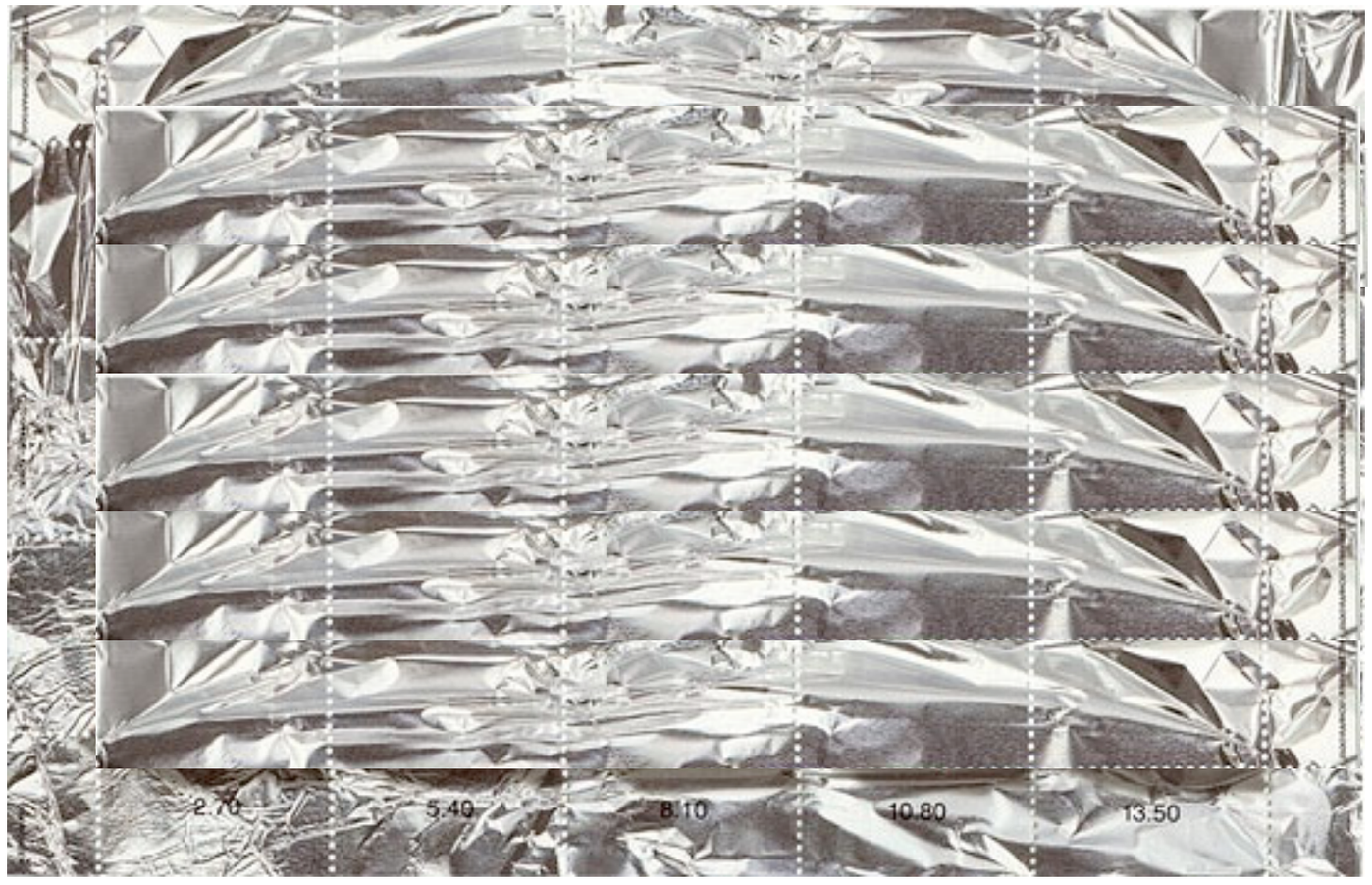
@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between



@ Imagine Alice and Bob alternately munch chocolate

@ when they take a piece off, they also take off all others located to the East and North of it and the ones in between

UNIL | Université de Lausanne



@ the one who takes the last piece of chocolate loses

UNIL | Université de Lausanne



@ Imagine the bar is a square and Alice starts



@ Alice has a winning strategy

UNIL | Université de Lausanne

Alice



@ Alice has a winning strategy:

- first take (1,1), then .

- when Bob picks (0,k), choose (k,0)
- when Bob picks (k,0), choose (0,k)

UNIL | Université de Lausanne

Bob



@ Alice has a winning strategy:

- first take (1,1), then .

- when Bob picks (0,k), choose (k,0)
- when Bob picks (k,0), choose (0,k)

UNIL | Université de Lausanne

Alice



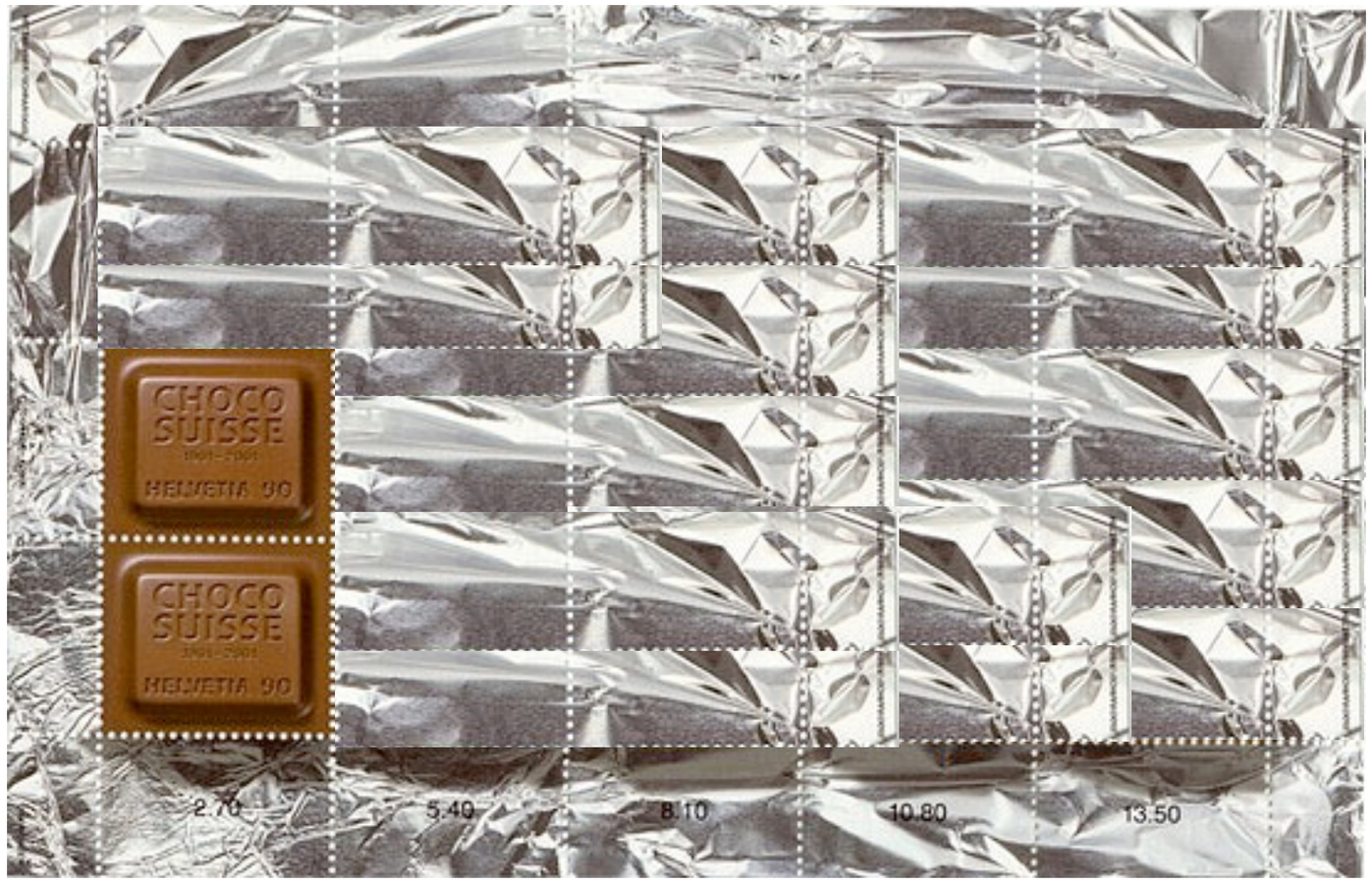
@ Alice has a winning strategy:

- first take (1,1), then .

- when Bob picks (0,k), choose (k,0)
- when Bob picks (k,0), choose (0,k)

UNIL | Université de Lausanne

Bob



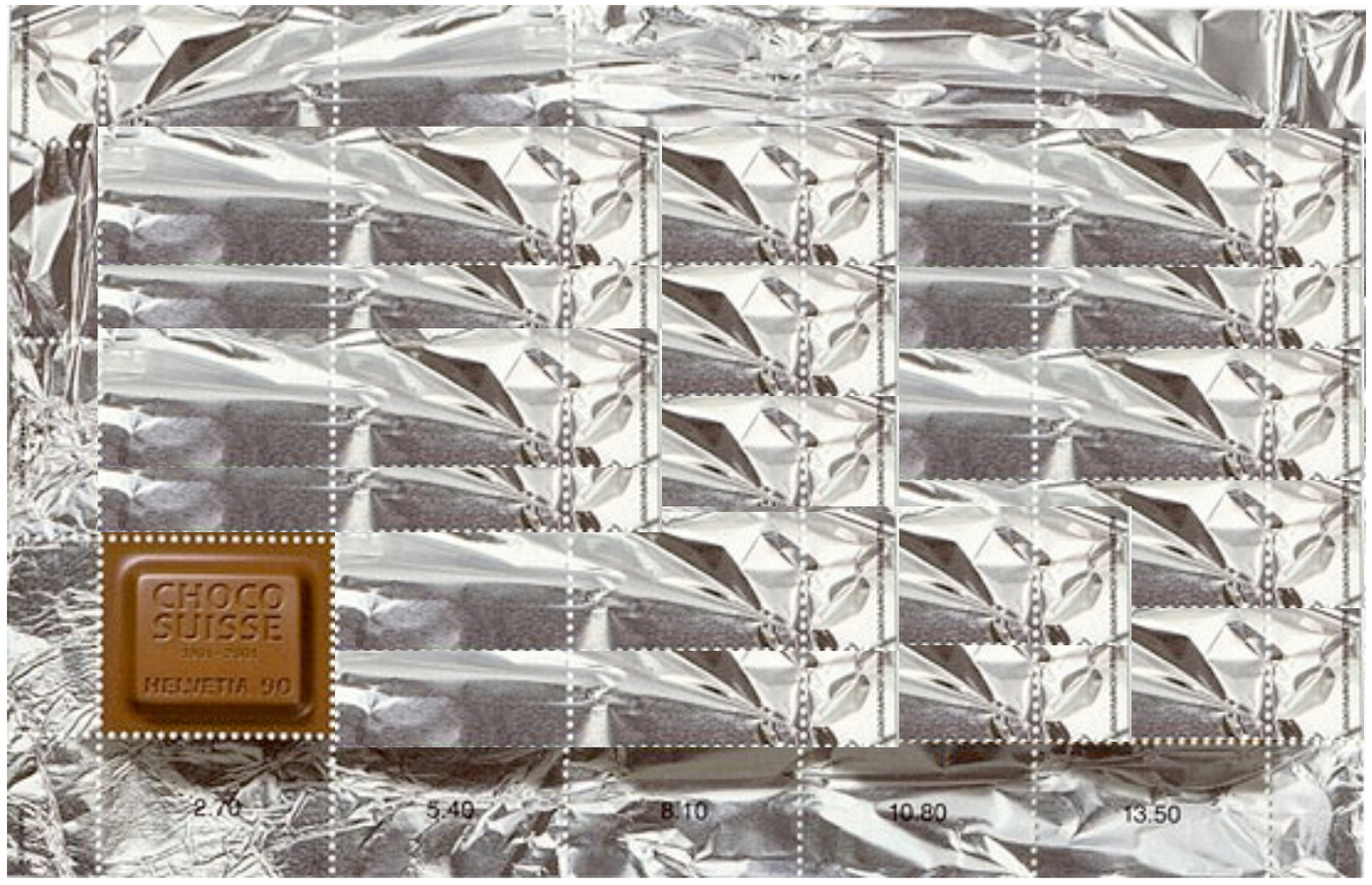
@ Alice has a winning strategy:

- first take (1,1), then .

- when Bob picks (0,k), choose (k,0)
- when Bob picks (k,0), choose (0,k)

UNIL | Université de Lausanne

Alice

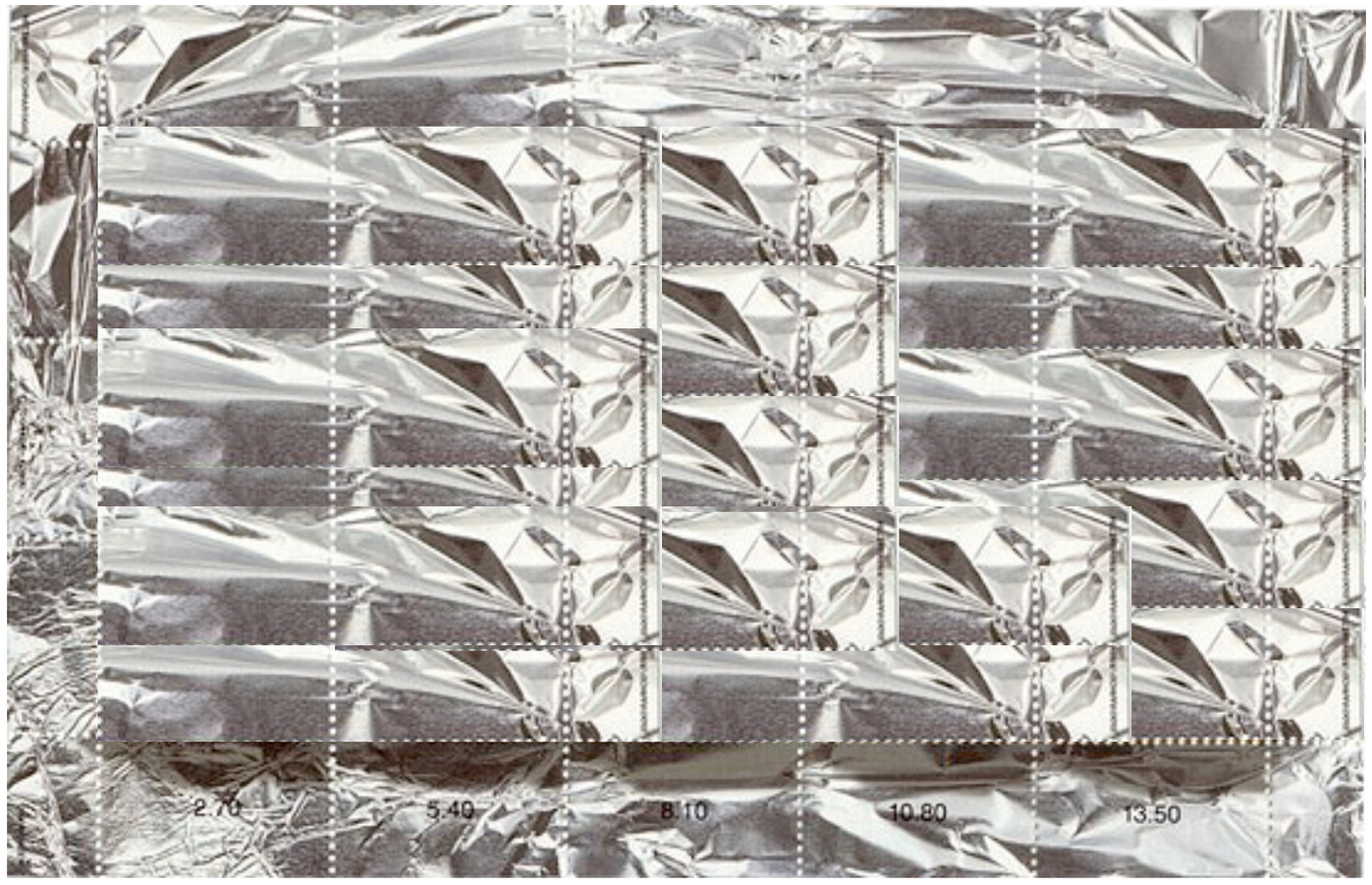


@ Alice has a winning strategy:

- first take (1,1), then .

- when Bob picks (0,k), choose (k,0)
- when Bob picks (k,0), choose (0,k)

Bob



@ Bob loses !!!!

UNIL | Université de Lausanne

- Same argument works for any (n,n) chocolate bar.
- Hence Alice always has a winning strategy.
- (Same argument works even for any (∞,∞) chocolate bar).
- What about rectangular chocolate bars?



@ Does Alice or Bob have a winning strategy ?

@ Does it depend on the size (n,k) ?

UNIL | Université de Lausanne



@ Does Alice or Bob have a winning strategy ?

Yes indeed,
for this is a finite 2-player game with perfect information

Yes indeed,

for this is a finite 2-player game with perfect information

- finite duration, the game stops after finitely many moves
- 2 as opposed to 3,4,... where coalition comes in the way
- at any time the whole configuration of the game is accessible to both players. As opposed to incomplete information
- win or lose

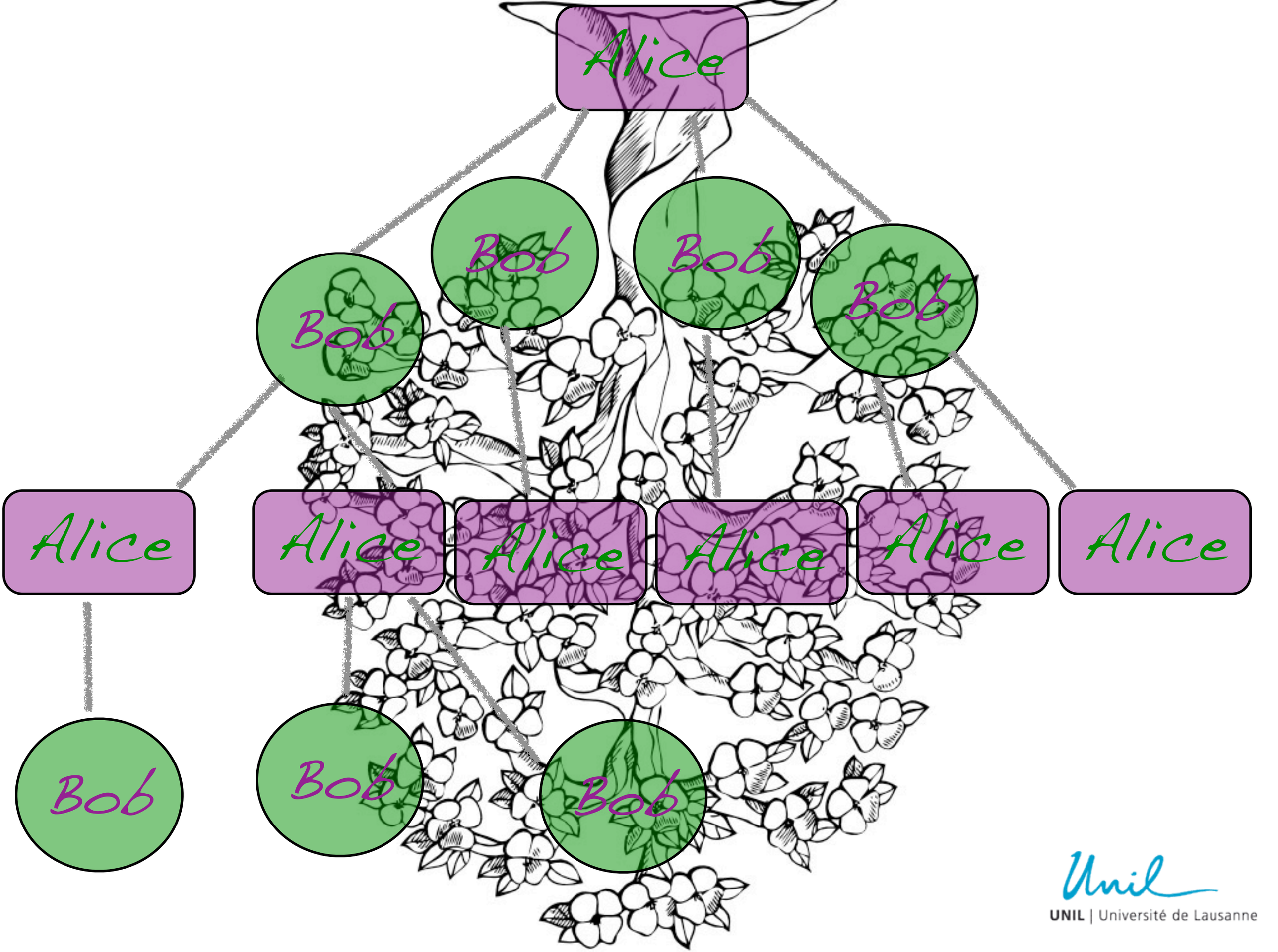
- such games are trees
- plays are branches
- strategies are sub-trees



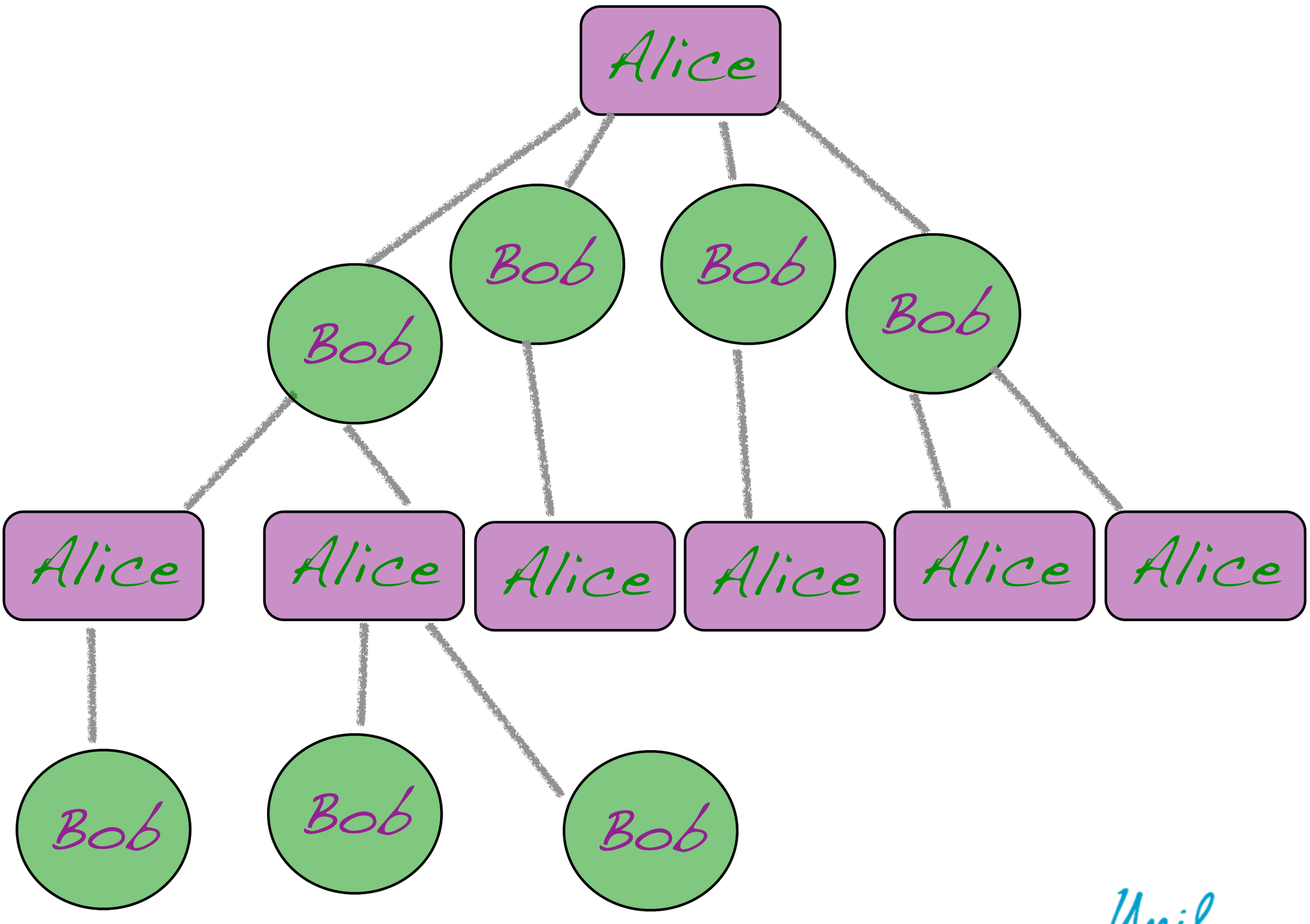
UNIL | Université de Lausanne



UNIL | Université de Lausanne



UNIL | Université de Lausanne



UNIL | Université de Lausanne

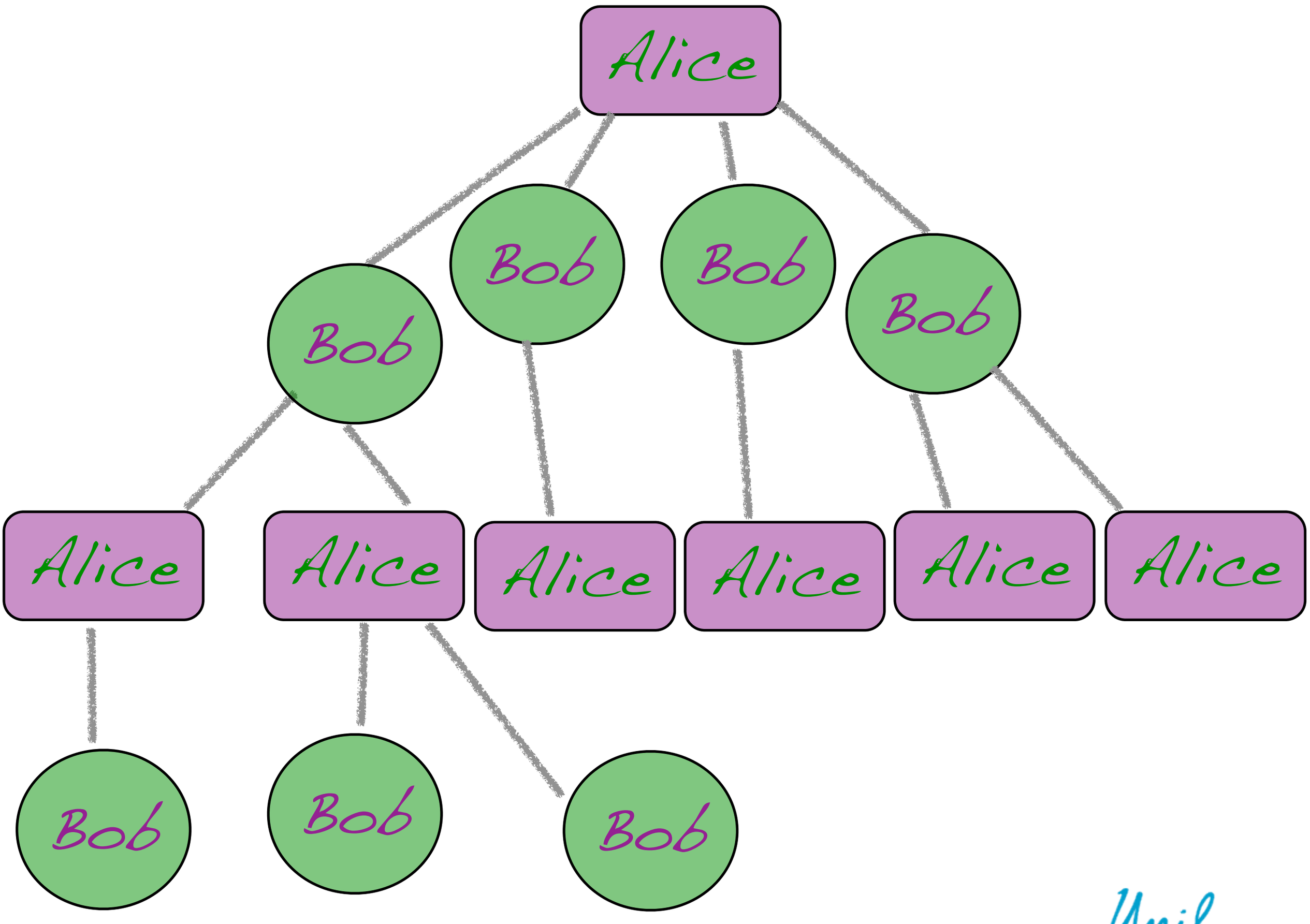
@ Does Alice or Bob have a winning strategy ?

Yes indeed,
for this is a finite 2-player game with perfect information

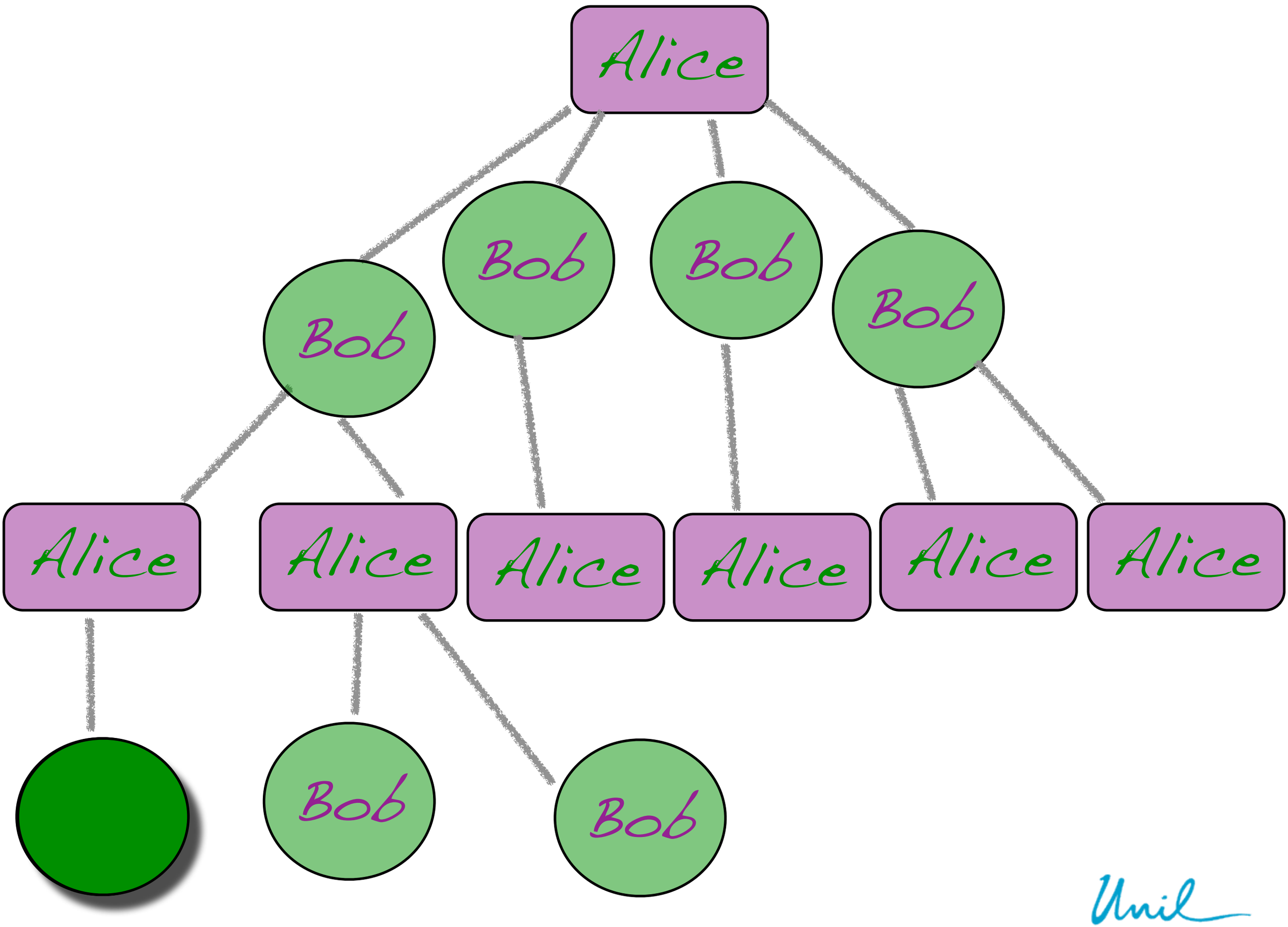
@ the game tree is well-founded.
- No infinite branch !.....
- A leaf sooner or later....

@ Solved by backward induction !

UNIL | Université de Lausanne



UNIL | Université de Lausanne



UNIL | Université de Lausanne

Alice

Bob Bob Bob Bob

Alice Alice Alice Alice Alice Alice

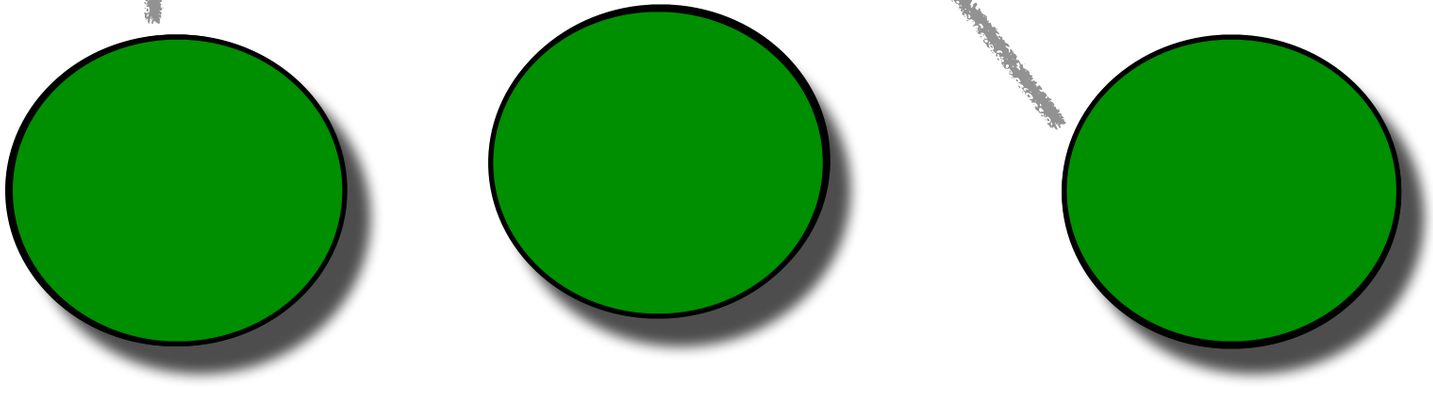
Bob

UNIL | Université de Lausanne

Alice

Bob Bob Bob Bob

Alice Alice Alice Alice Alice Alice



UNIL | Université de Lausanne

Alice

Bob

Bob

Bob

Bob

Alice

Alice

Alice

Alice

Alice

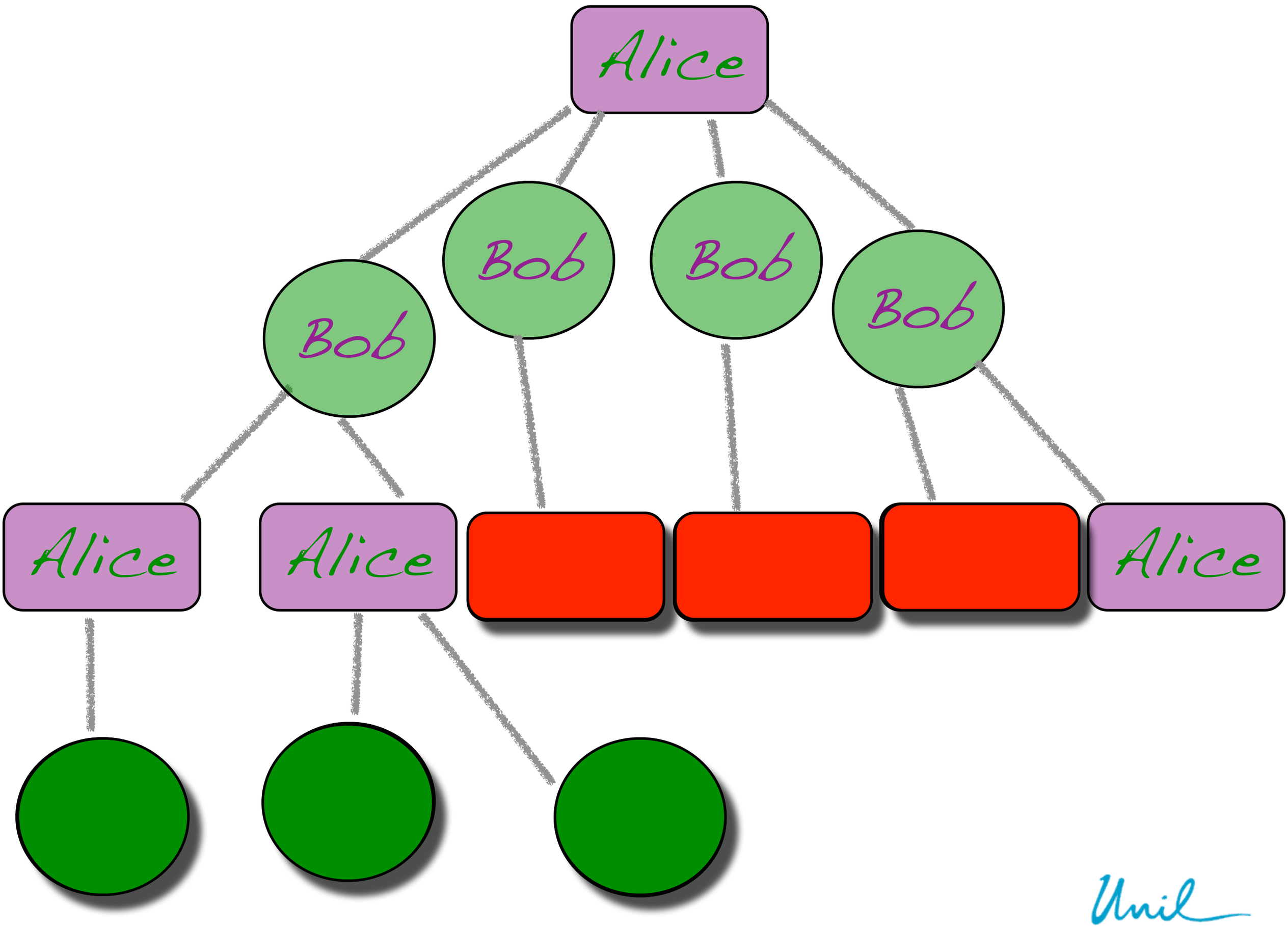
UNIL | Université de Lausanne

Alice

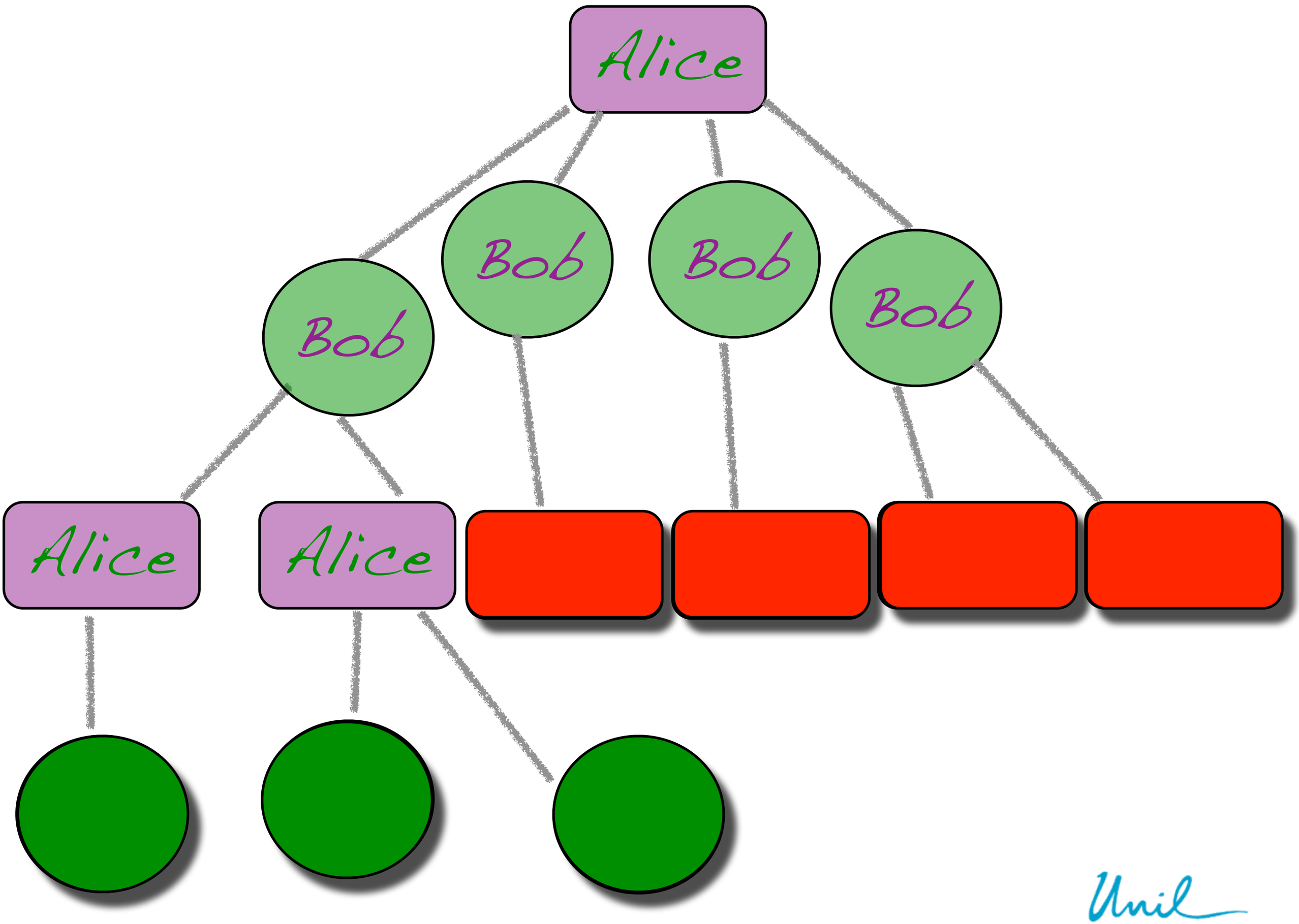
Bob Bob Bob Bob

Alice Alice Alice Alice

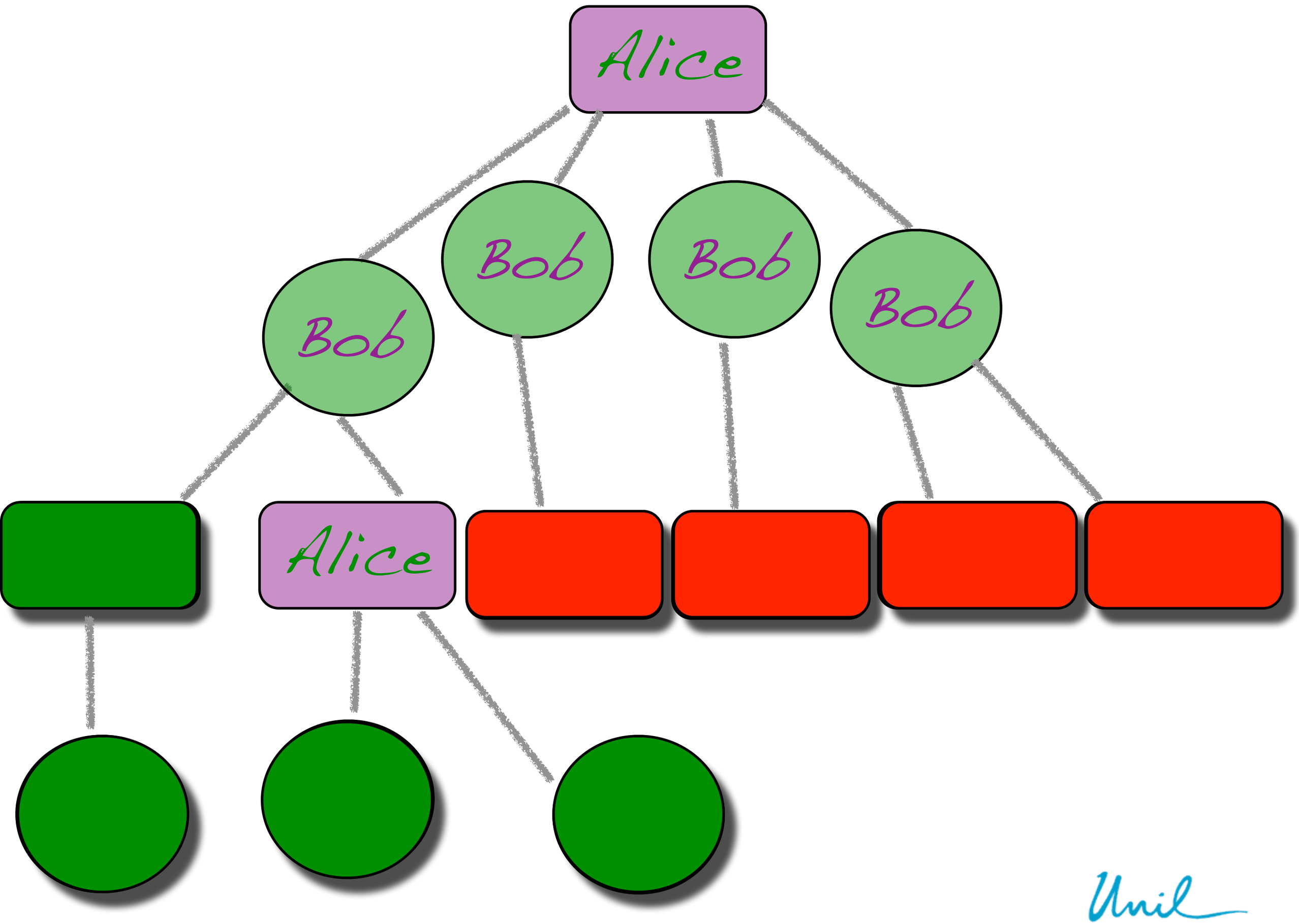
UNIL | Université de Lausanne



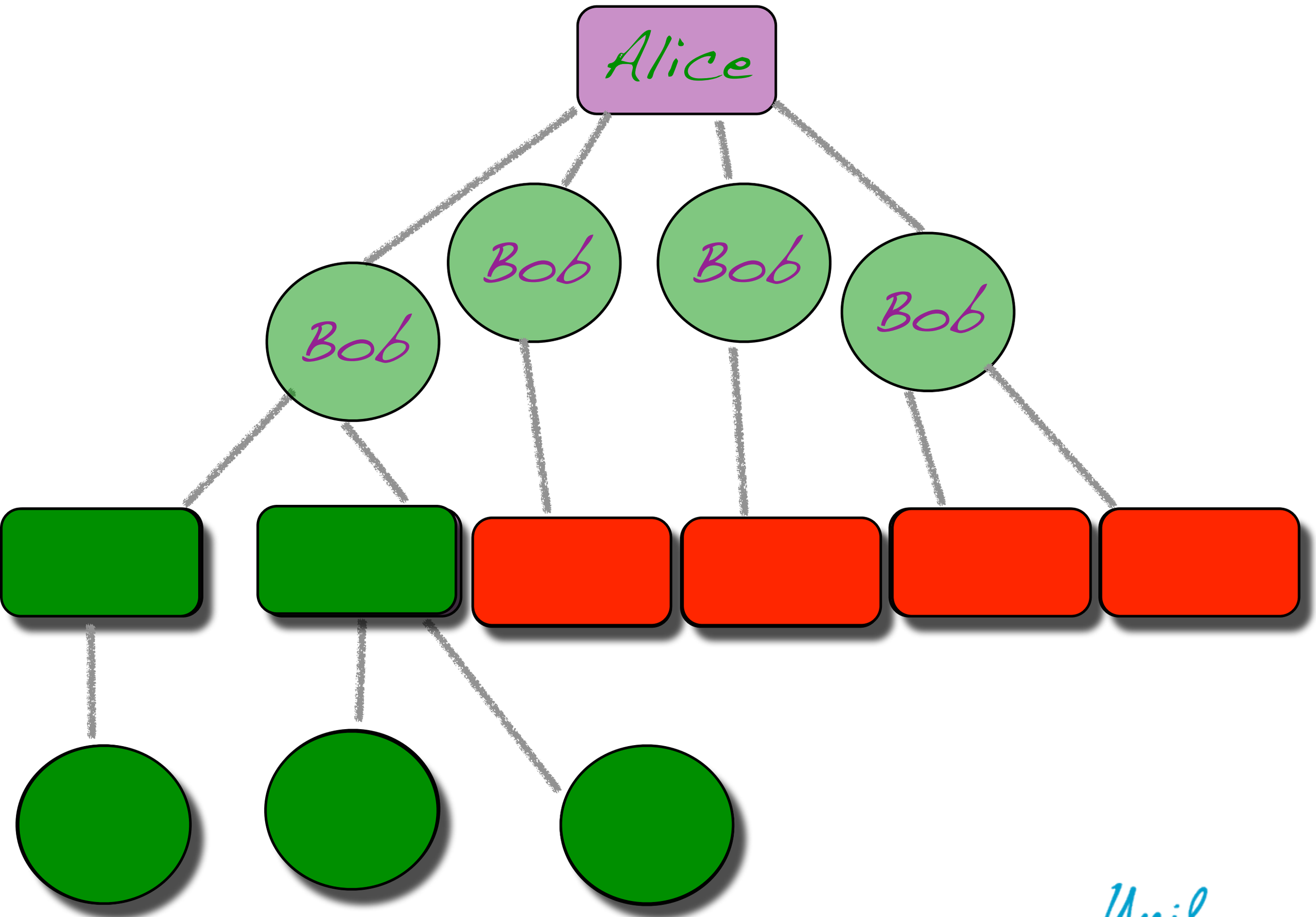
UNIL | Université de Lausanne



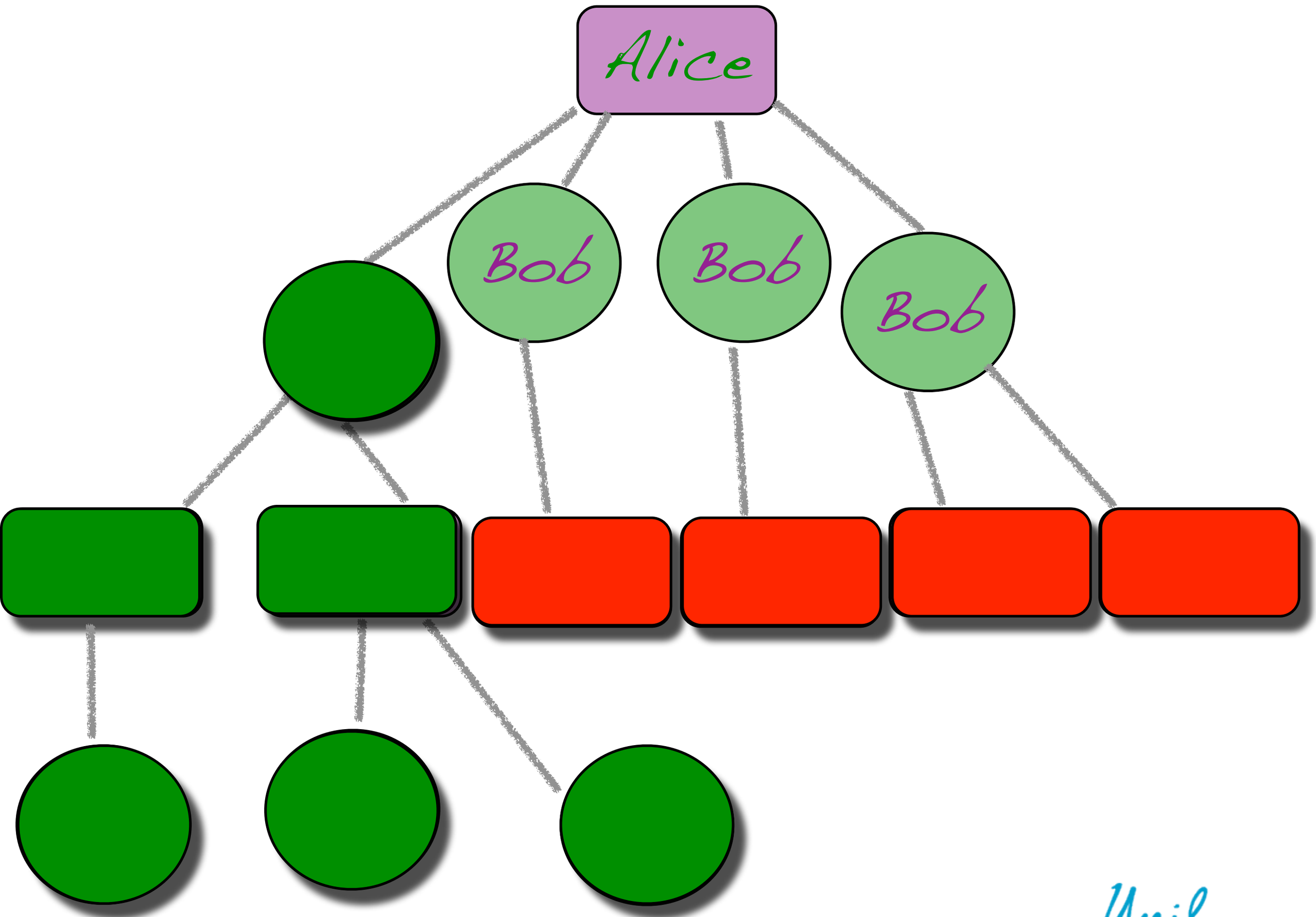
UNIL | Université de Lausanne



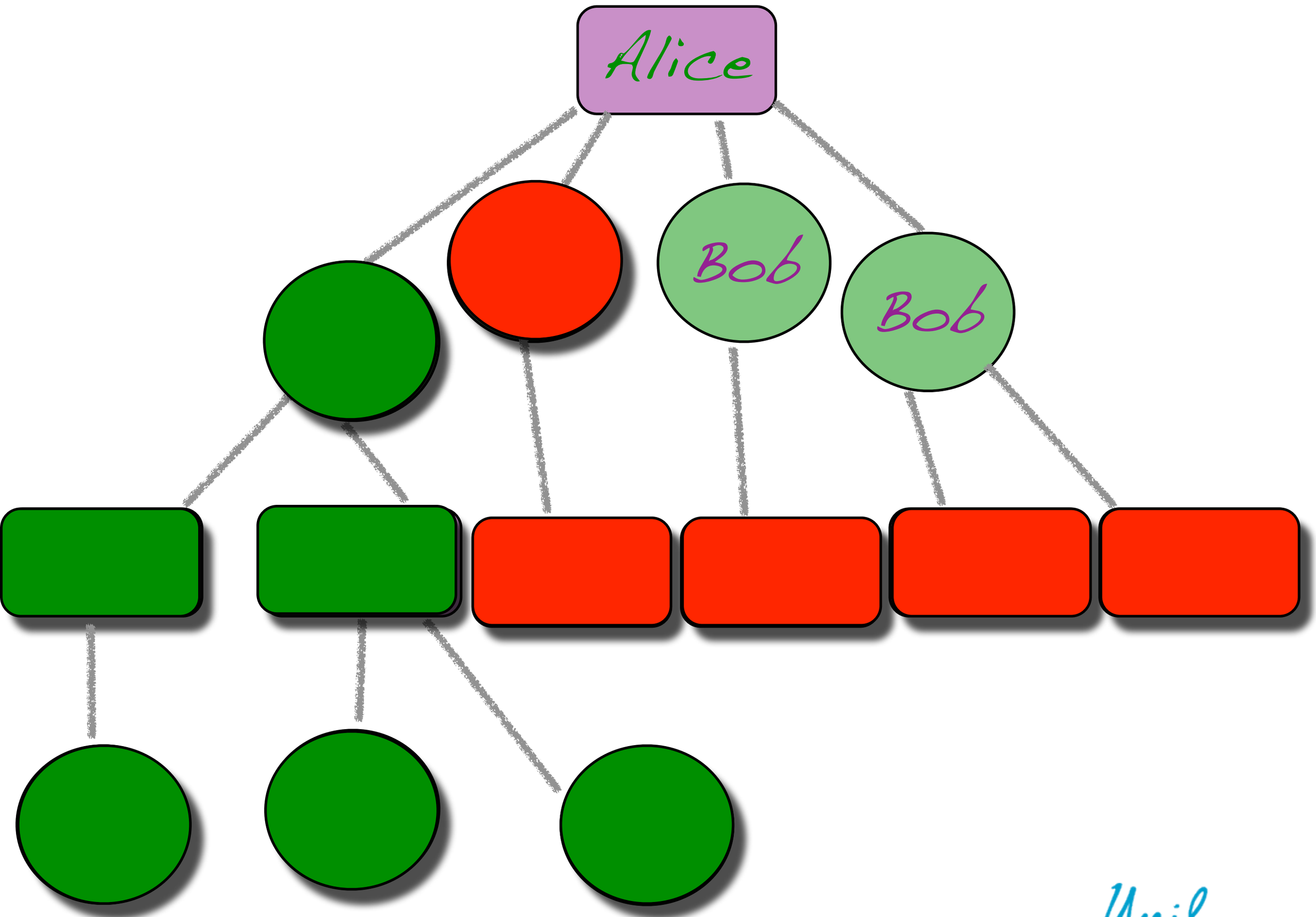
UNIL | Université de Lausanne



UNIL | Université de Lausanne



UNIL | Université de Lausanne

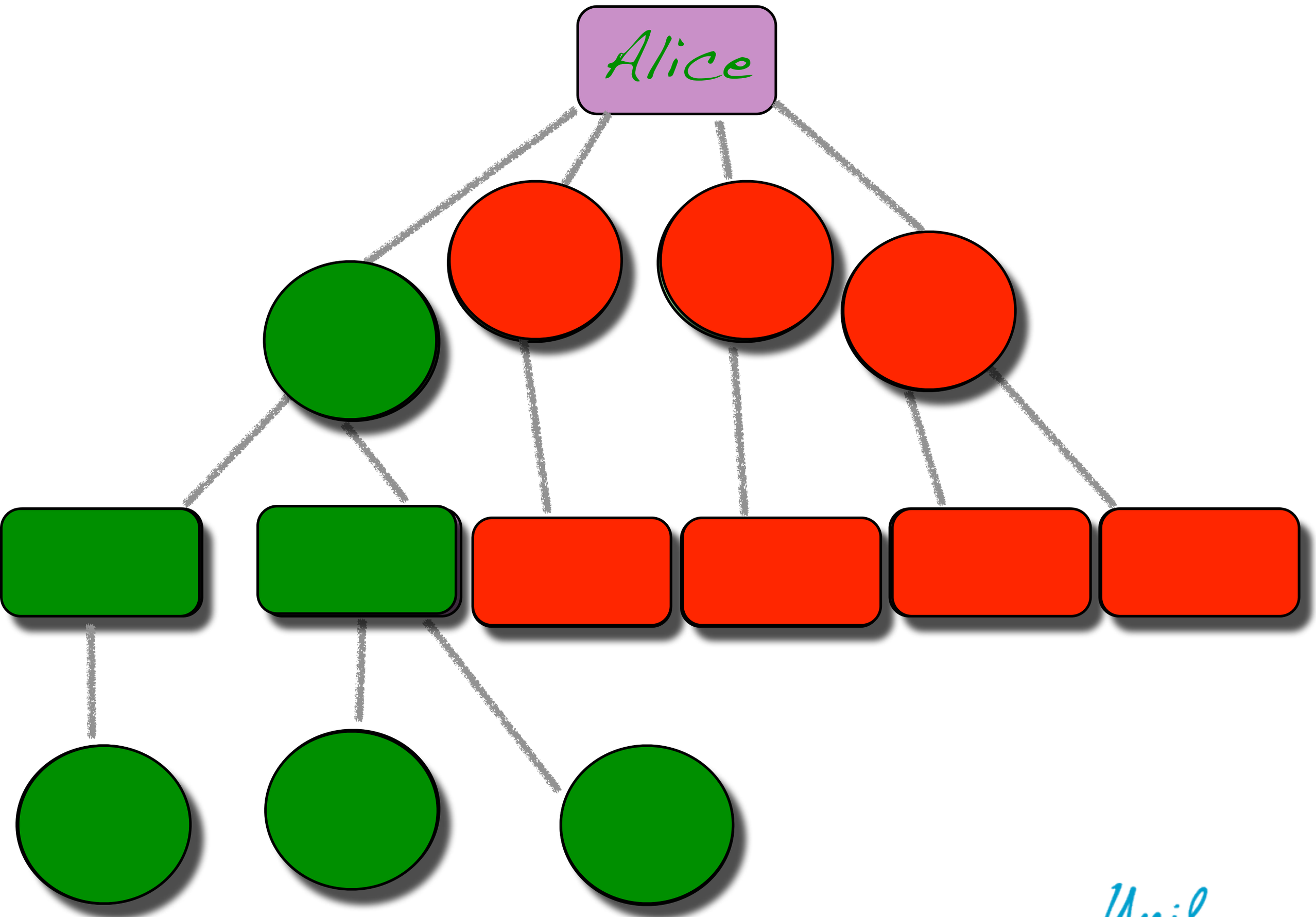


UNIL | Université de Lausanne

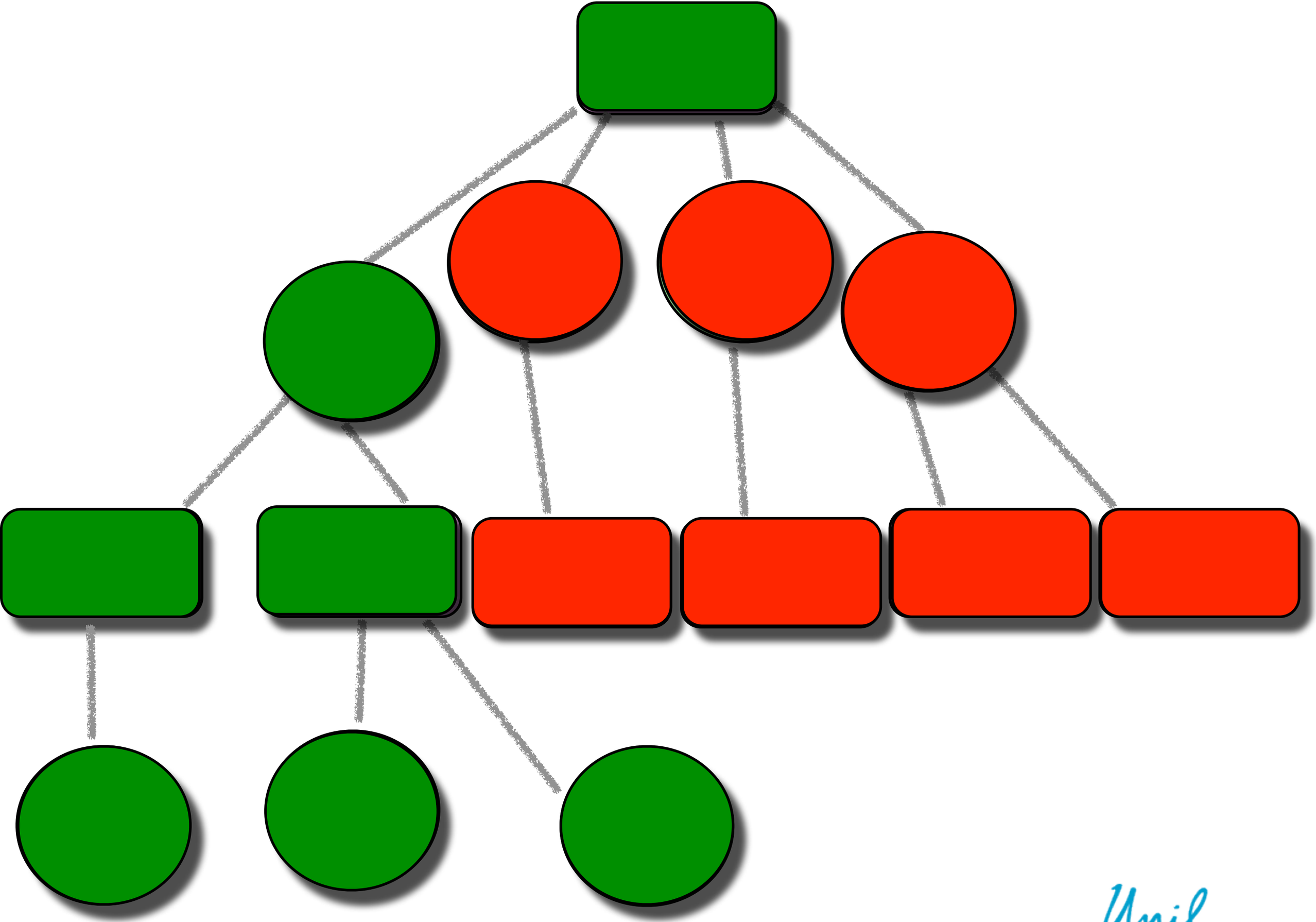
Alice

Bob

UNIL | Université de Lausanne



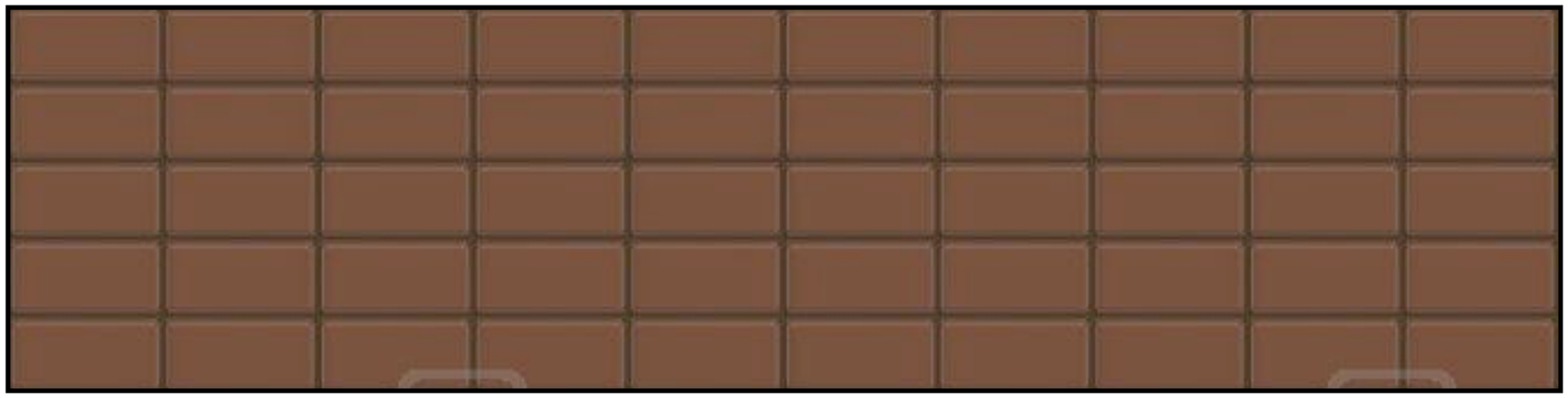
UNIL | Université de Lausanne



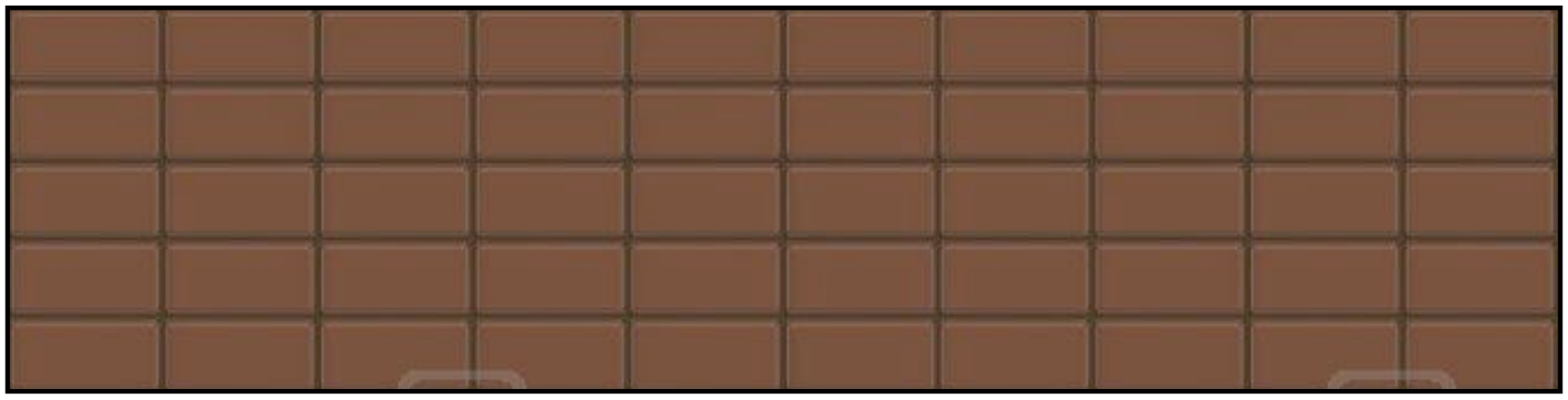


@ Does Alice or Bob have a winning strategy ?

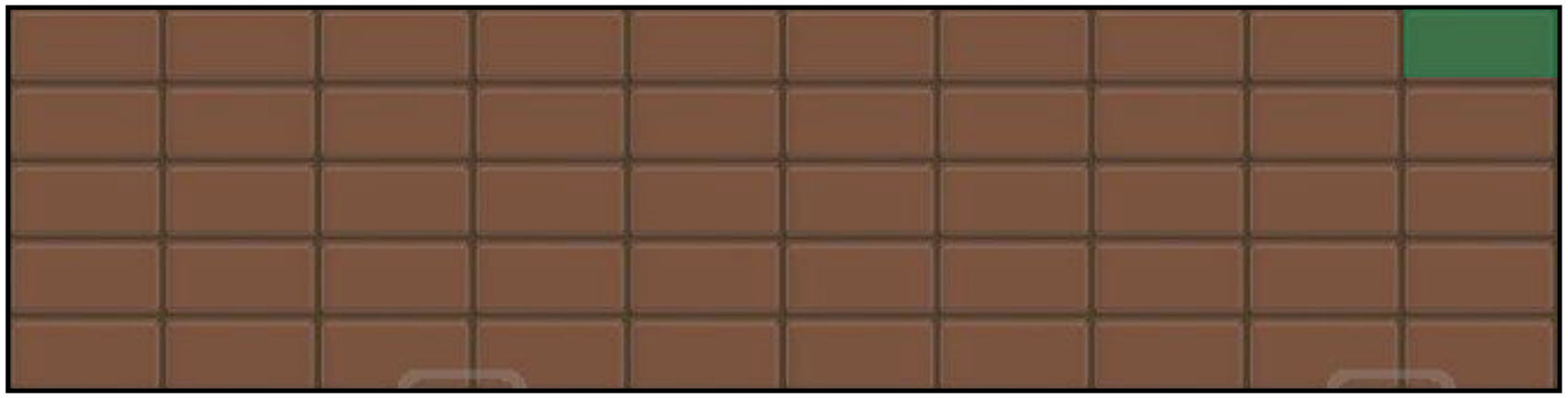
Yes indeed,
for this is a finite 2-player game with perfect information



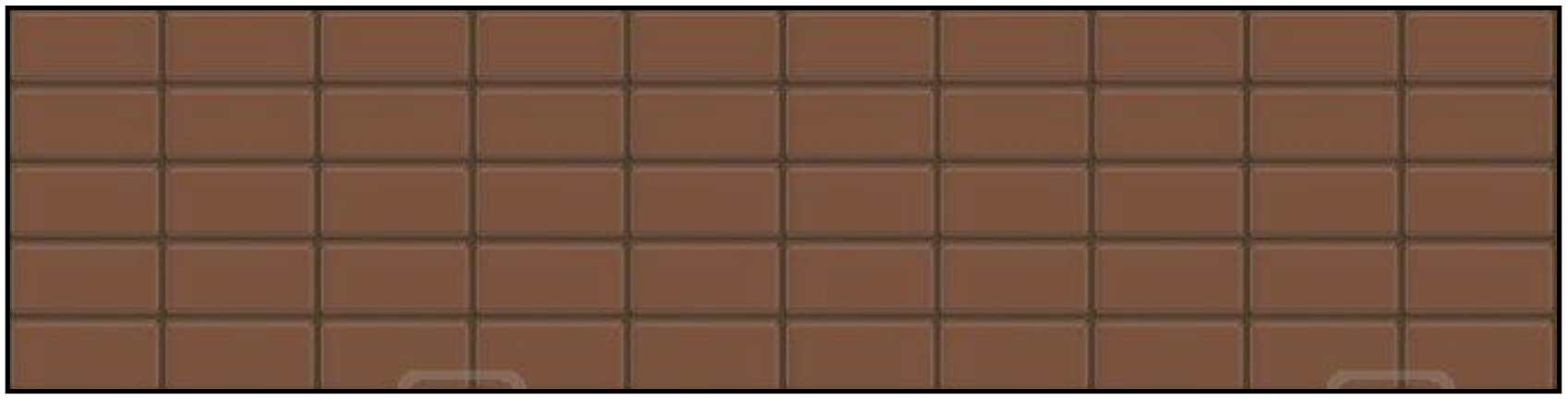
@ Alice plays against Bob



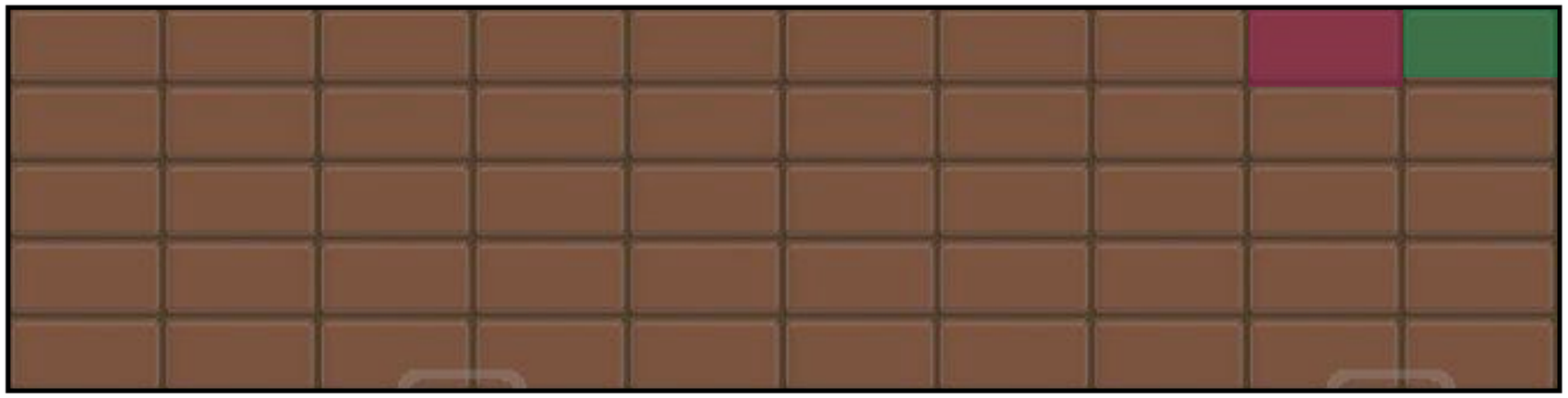
@ Bob applies his winning strategy



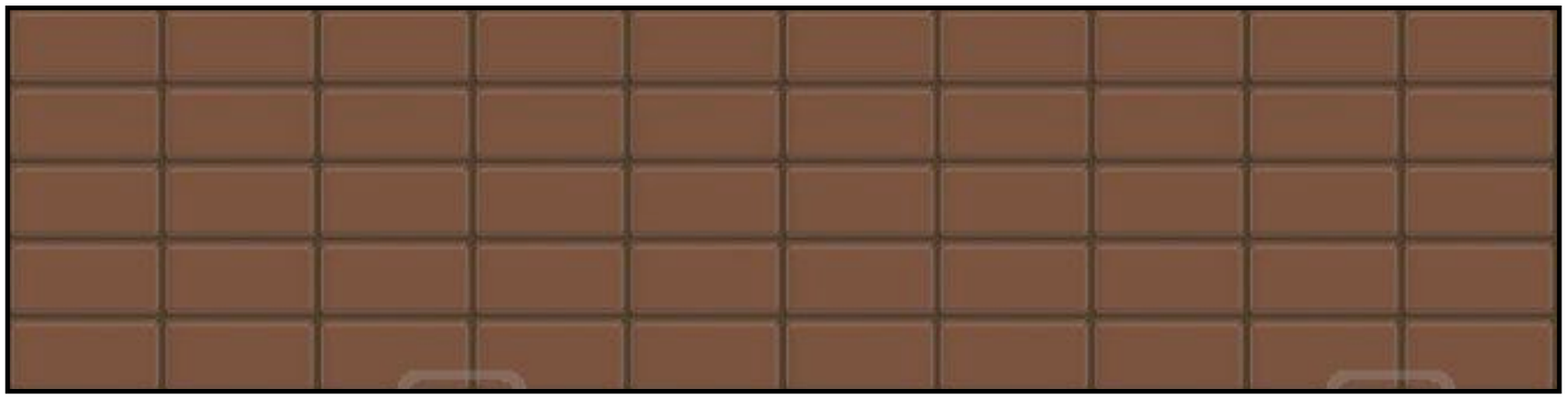
@ Alice plays against Bob



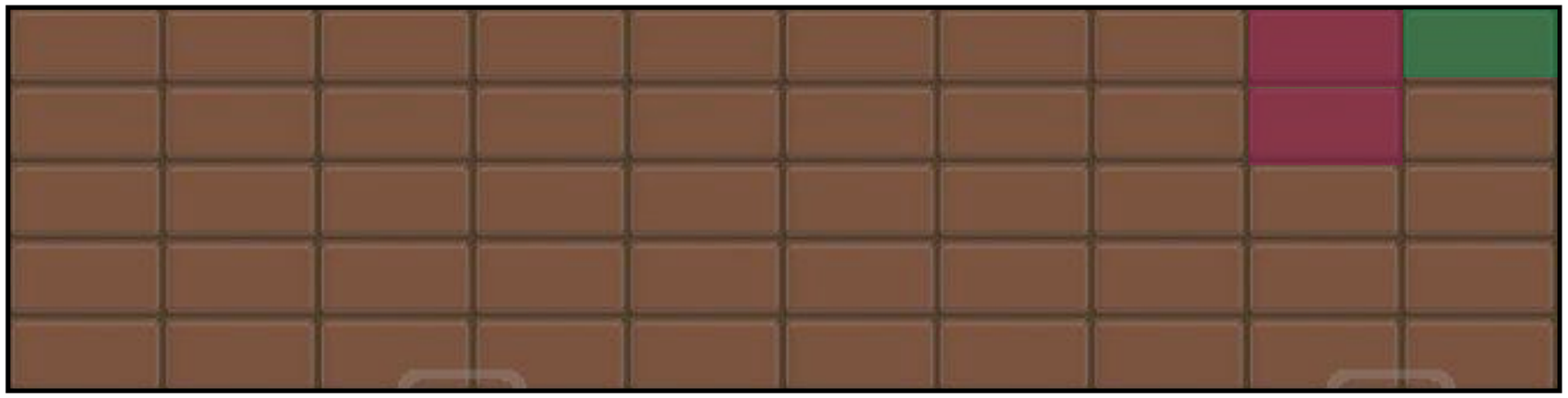
@ Bob applies his winning strategy



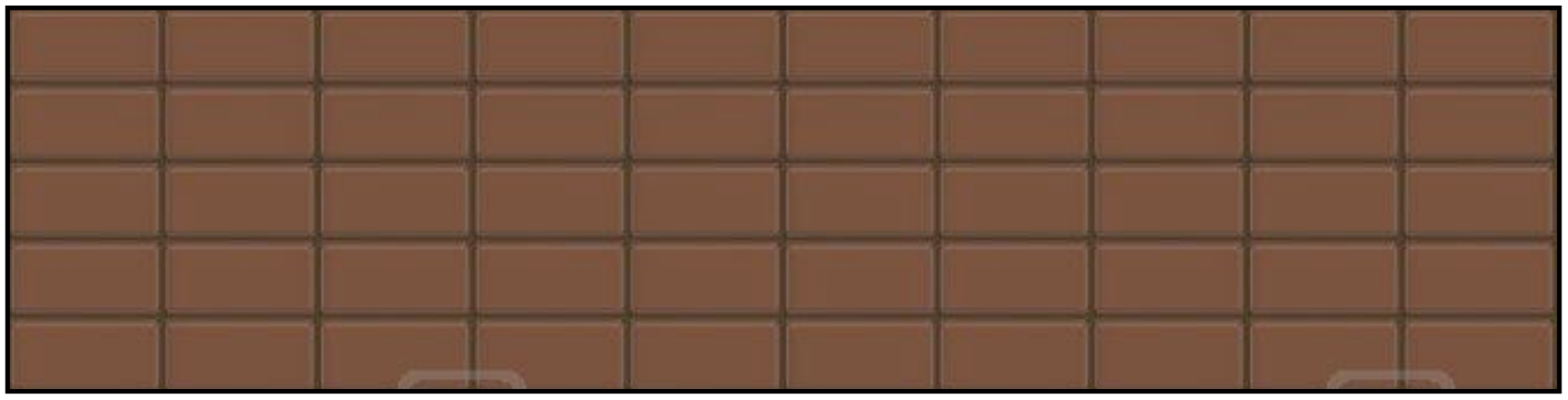
@ Alice plays against Bob



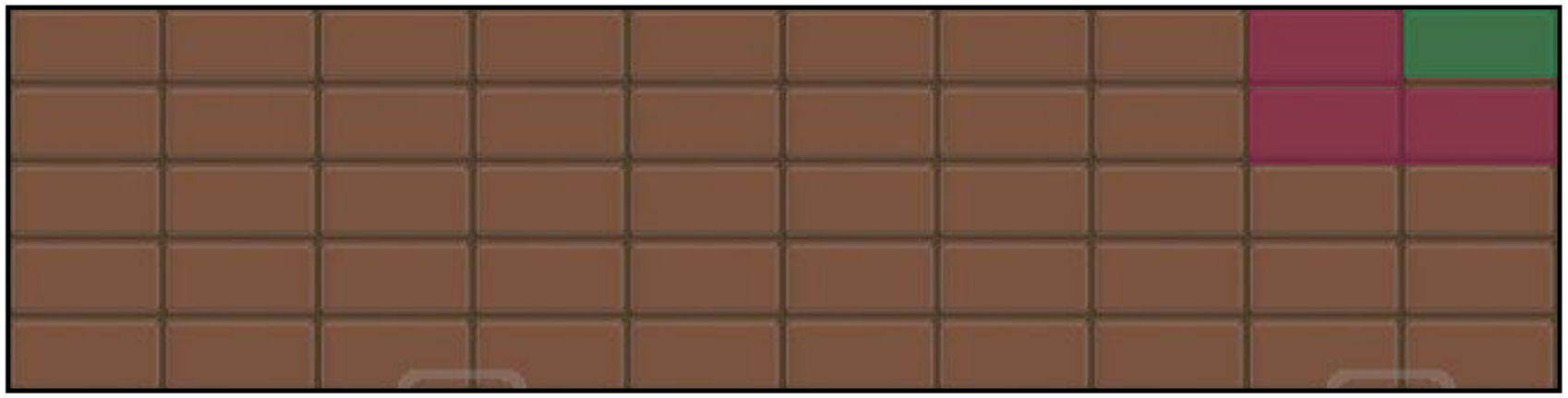
@ Bob applies his winning strategy



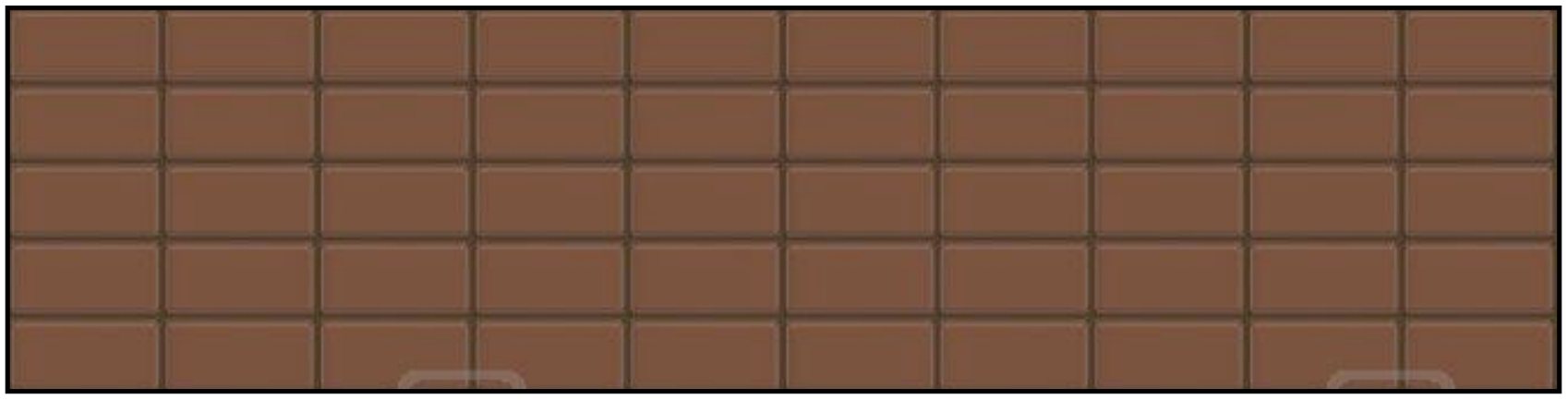
@ Alice plays against Bob



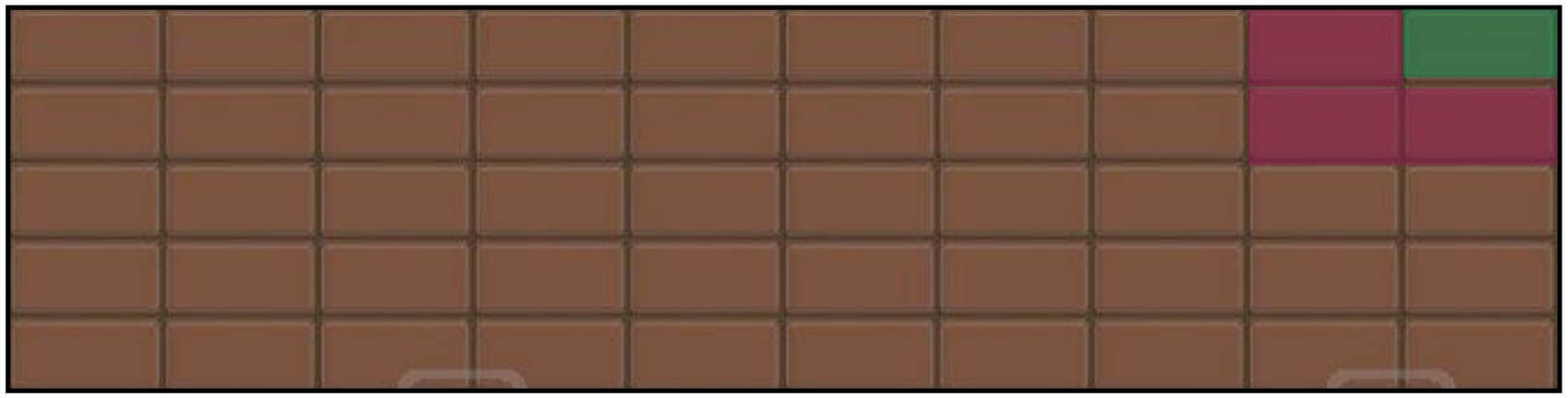
@ Bob applies his winning strategy



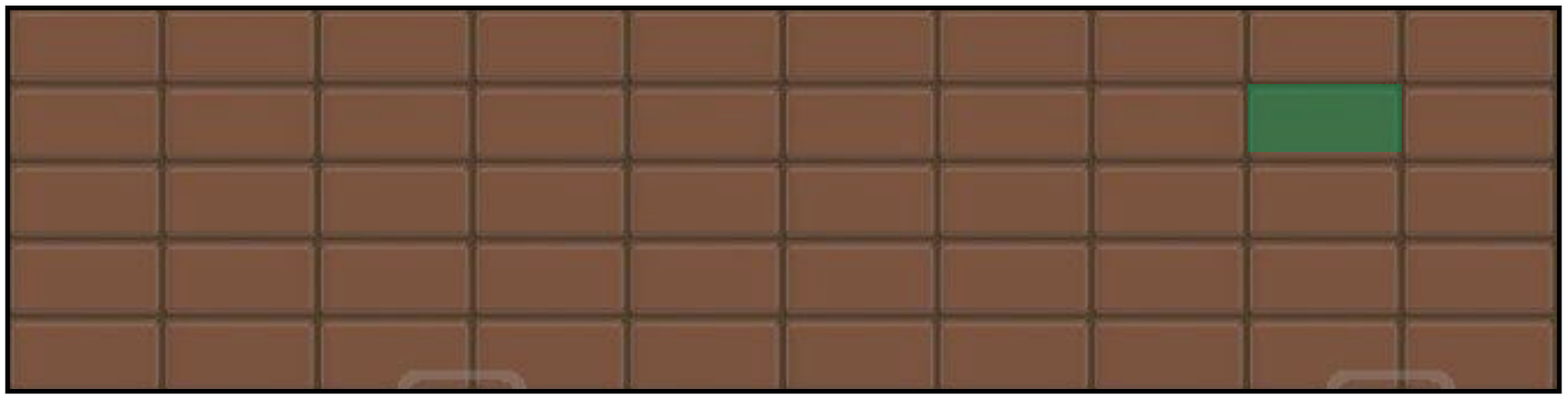
@ Alice plays against Bob



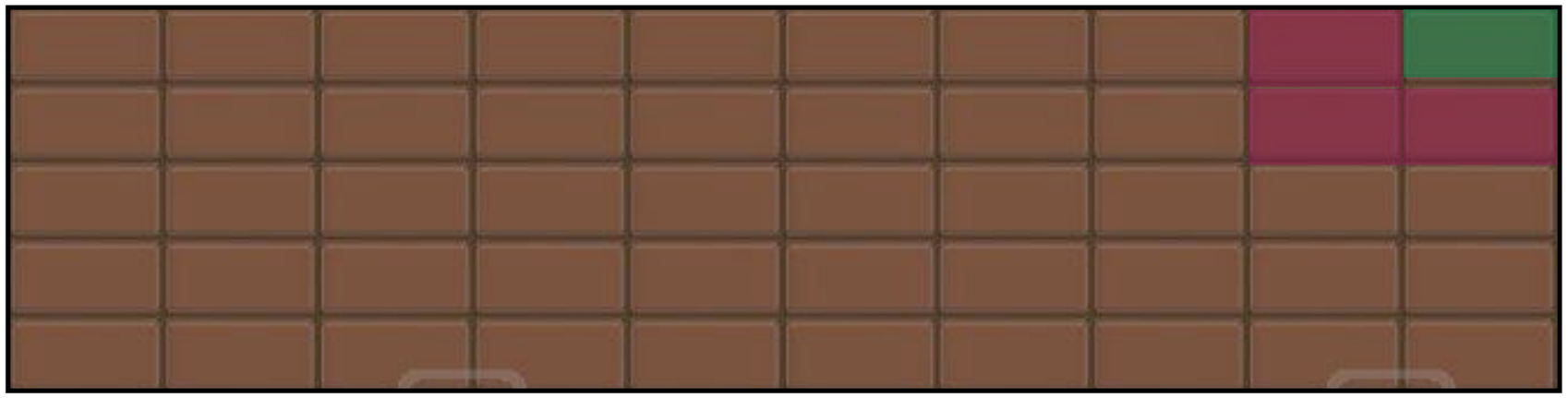
@ Bob applies his winning strategy



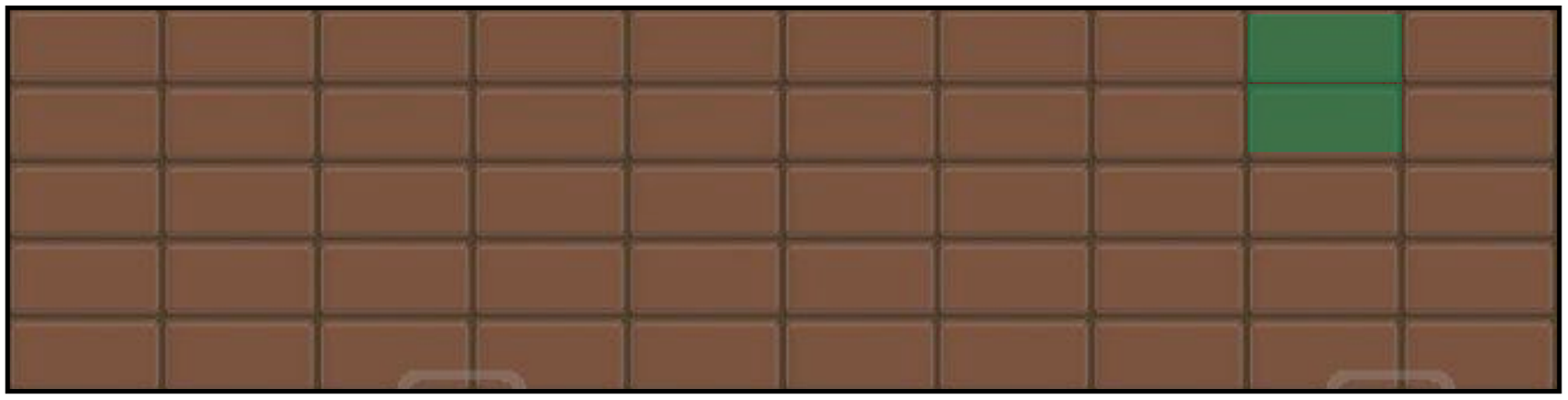
@ Alice plays against Bob



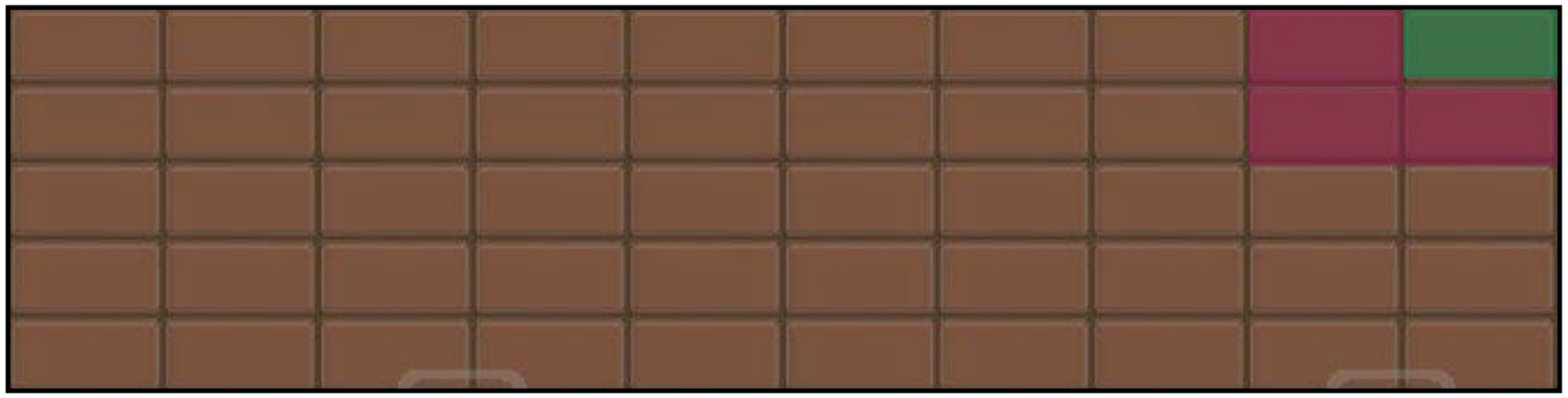
@ Bob applies his winning strategy



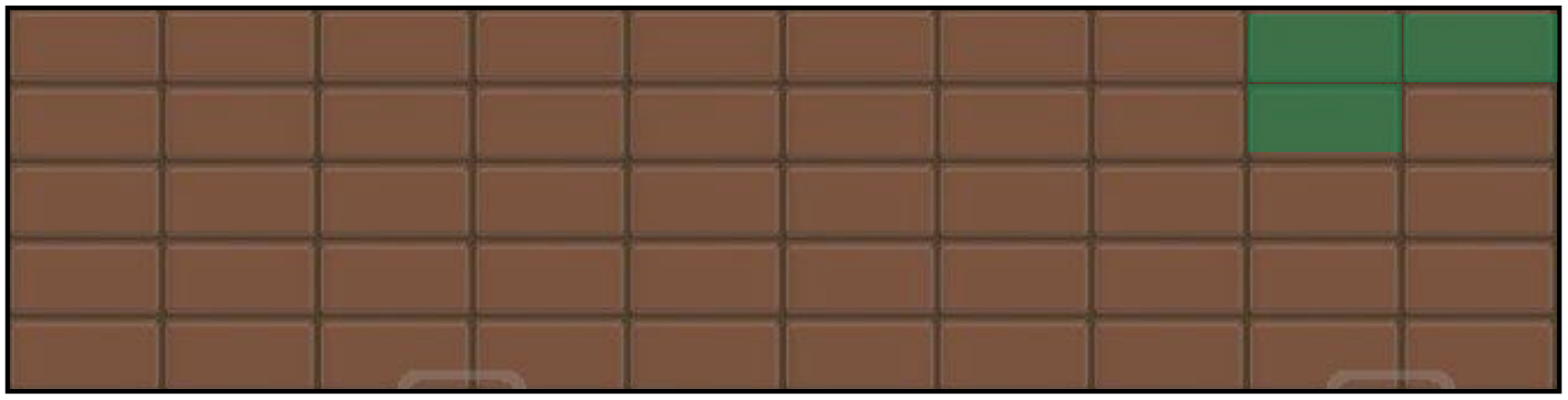
@ Alice plays against Bob



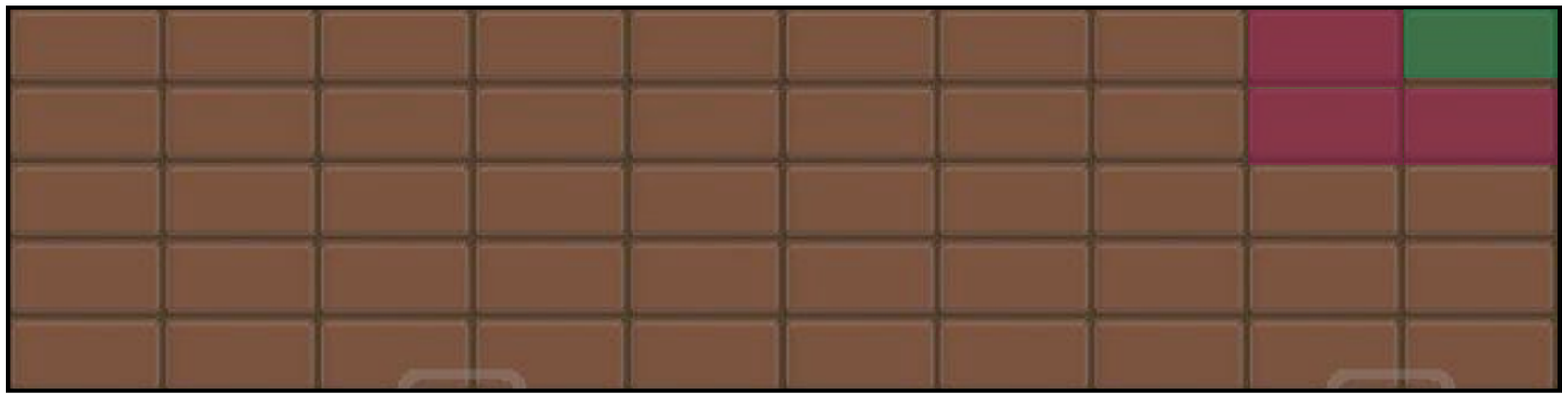
@ Bob applies his winning strategy



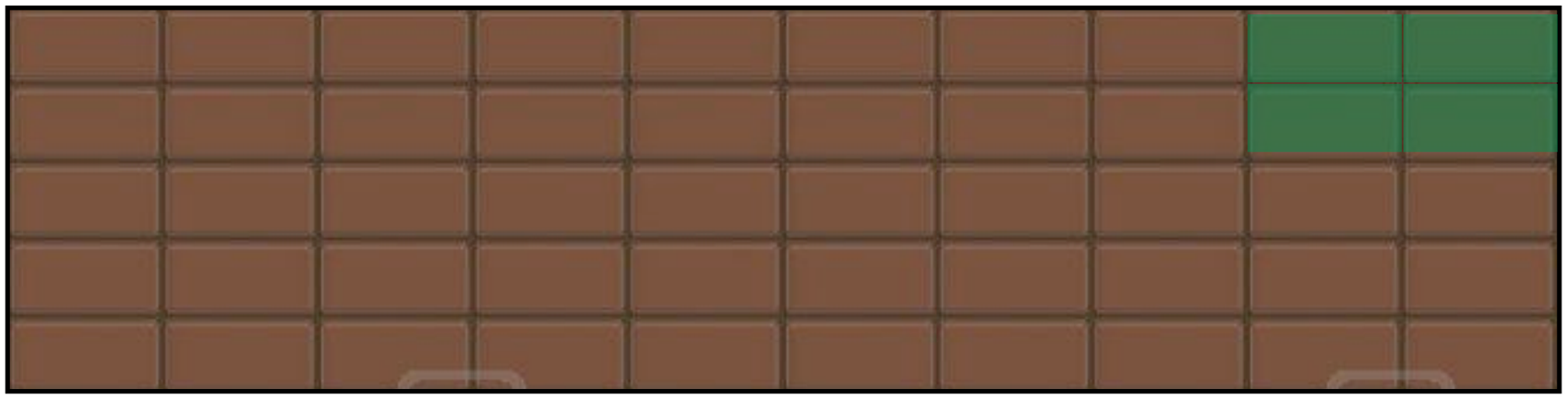
@ Alice plays against Bob



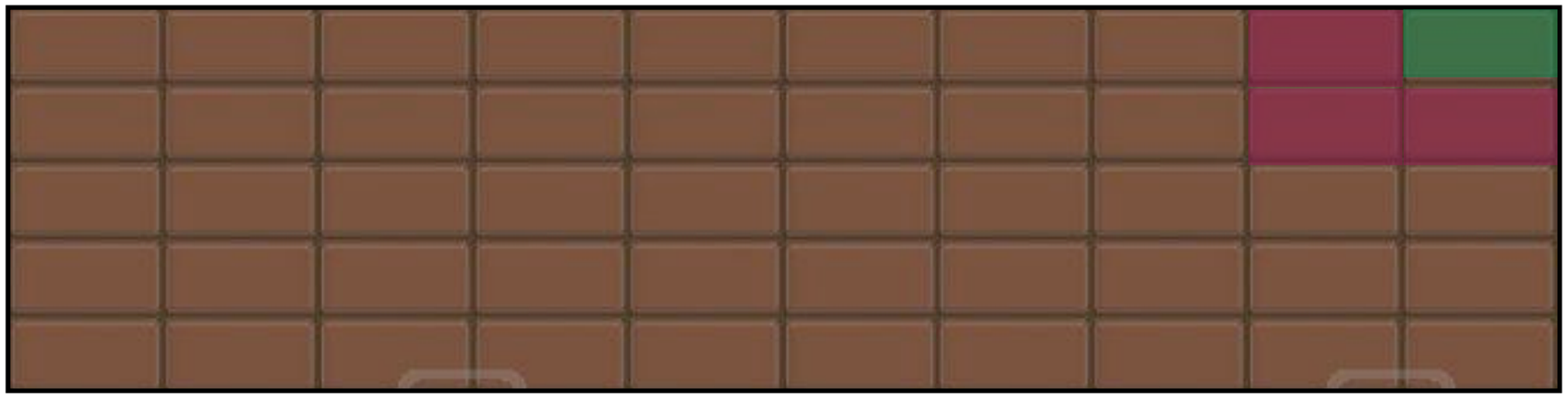
@ Bob applies his winning strategy



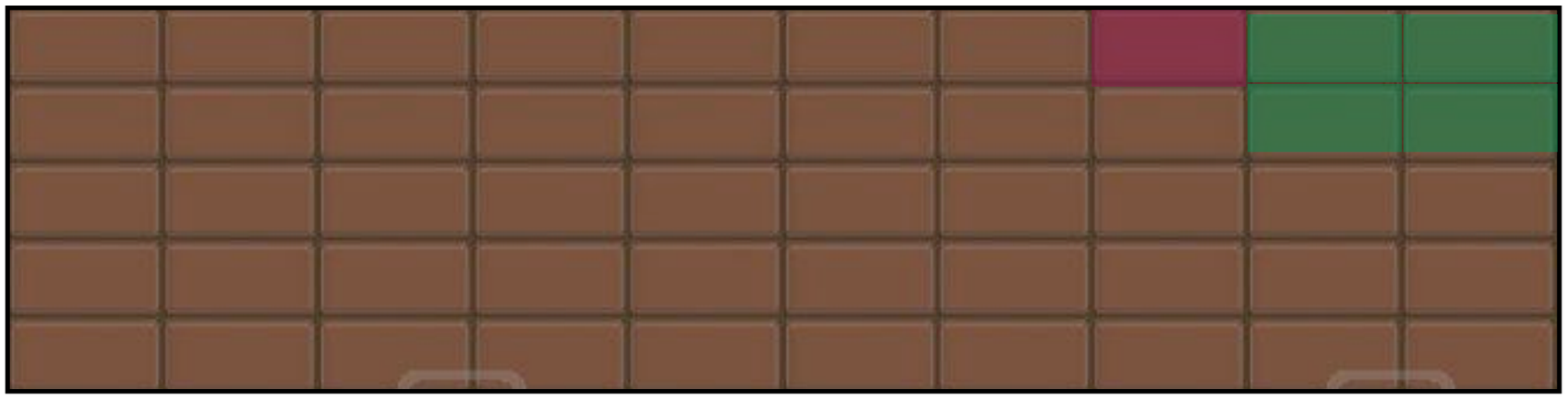
@ Alice plays against Bob



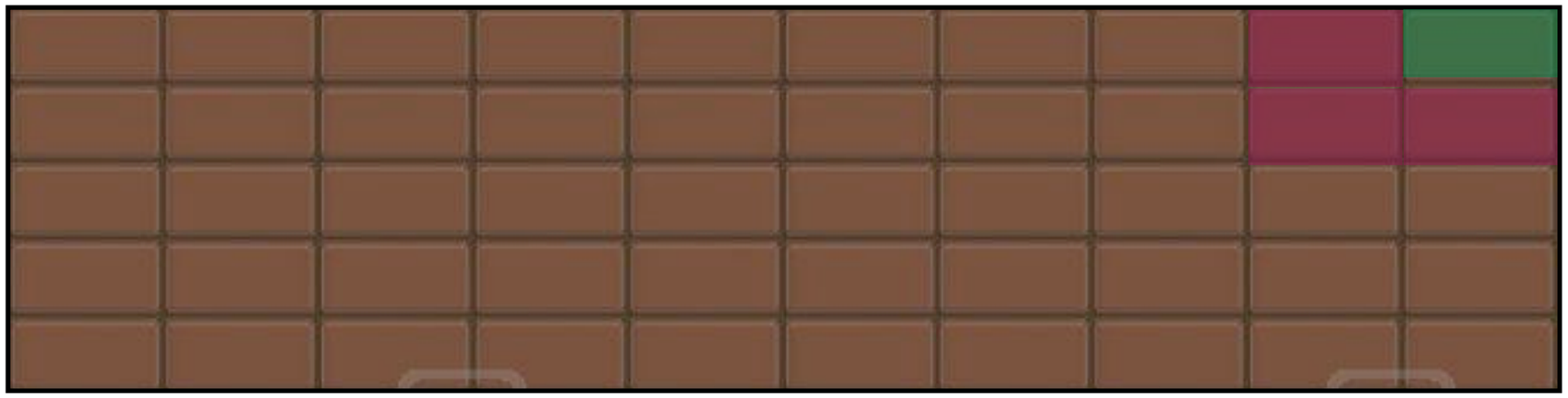
@ Bob applies his winning strategy



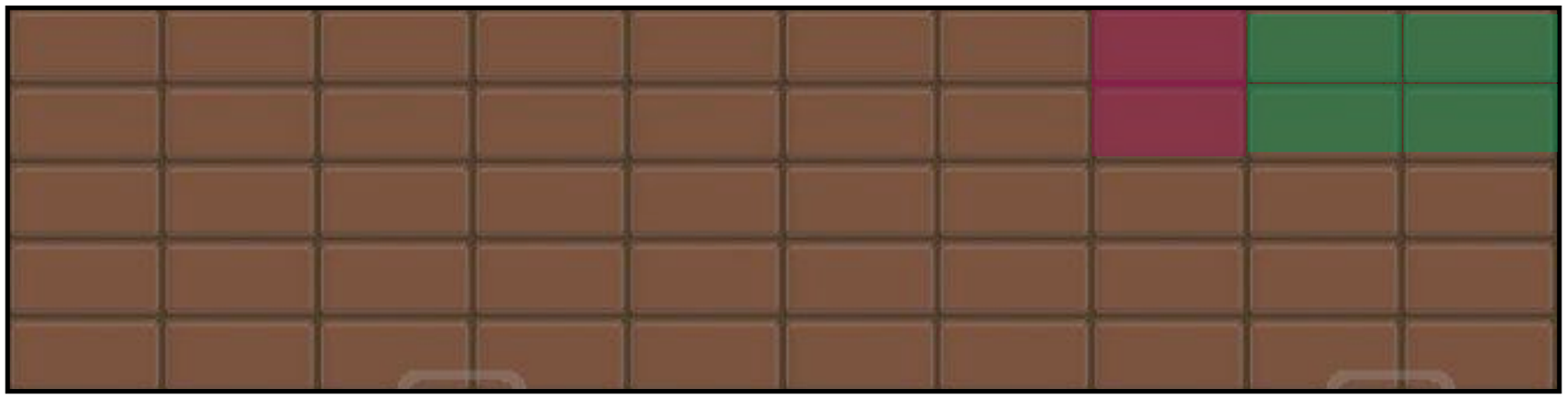
@ Alice plays against Bob



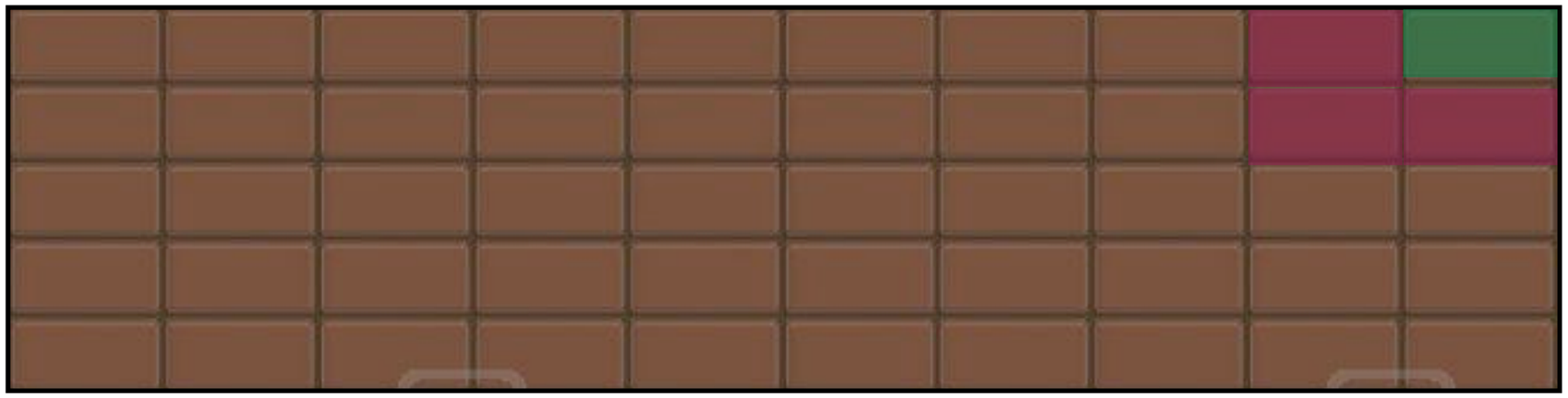
@ Bob applies his winning strategy



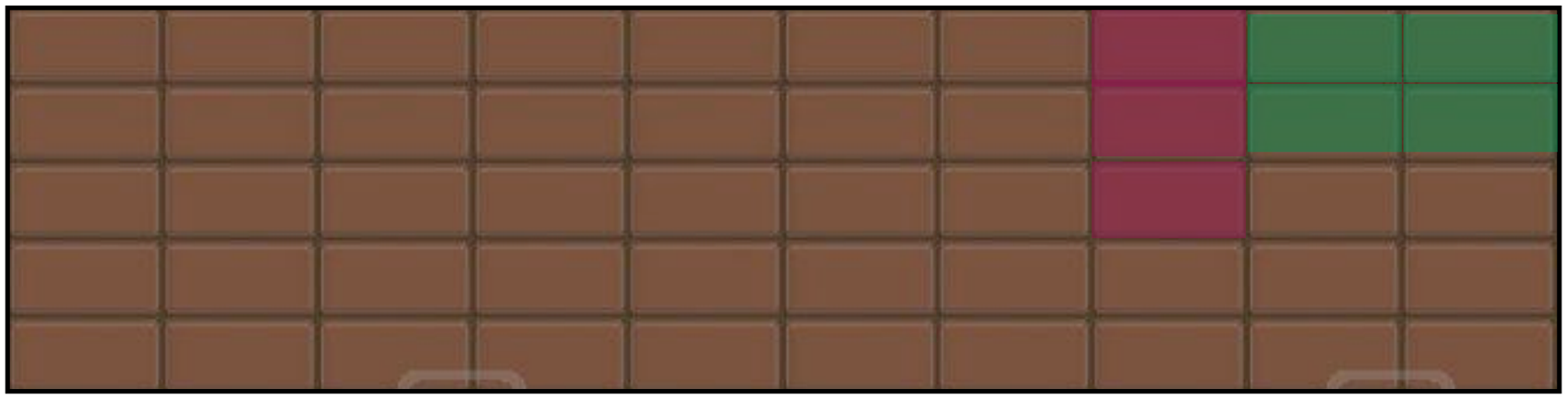
@ Alice plays against Bob



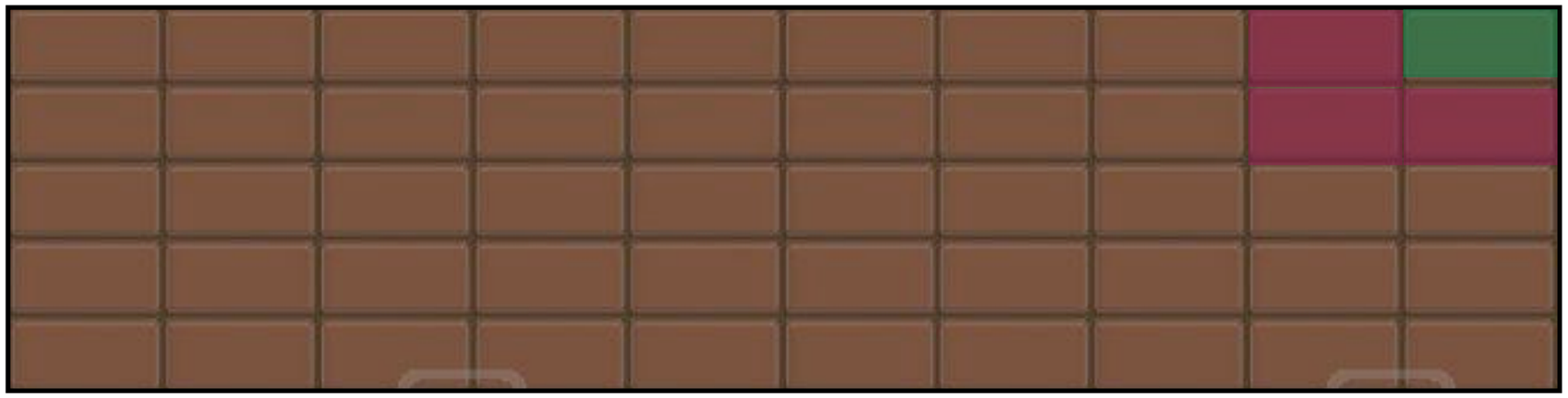
@ Bob applies his winning strategy



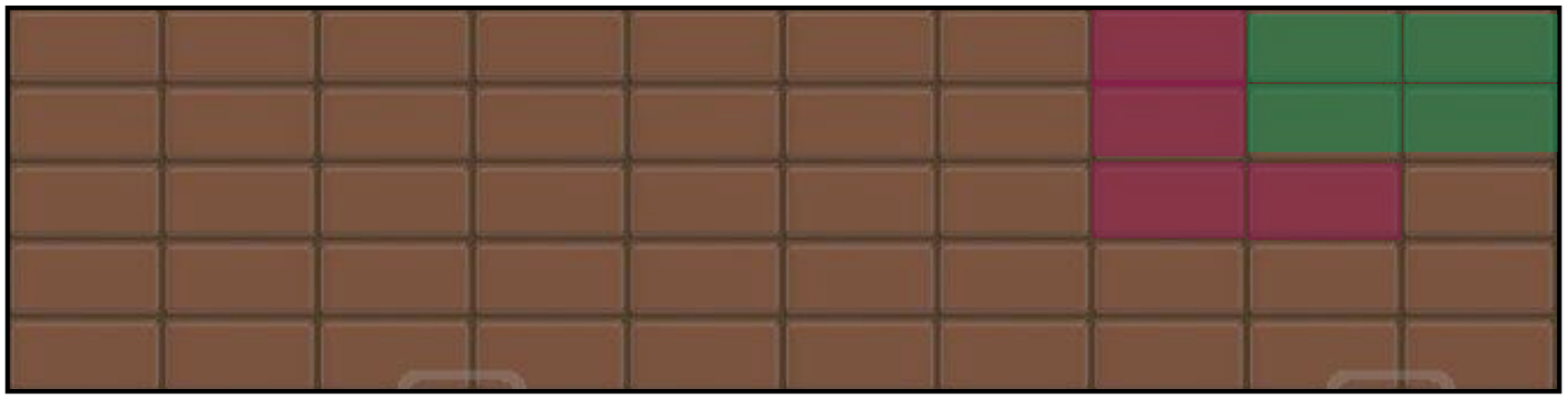
@ Alice plays against Bob



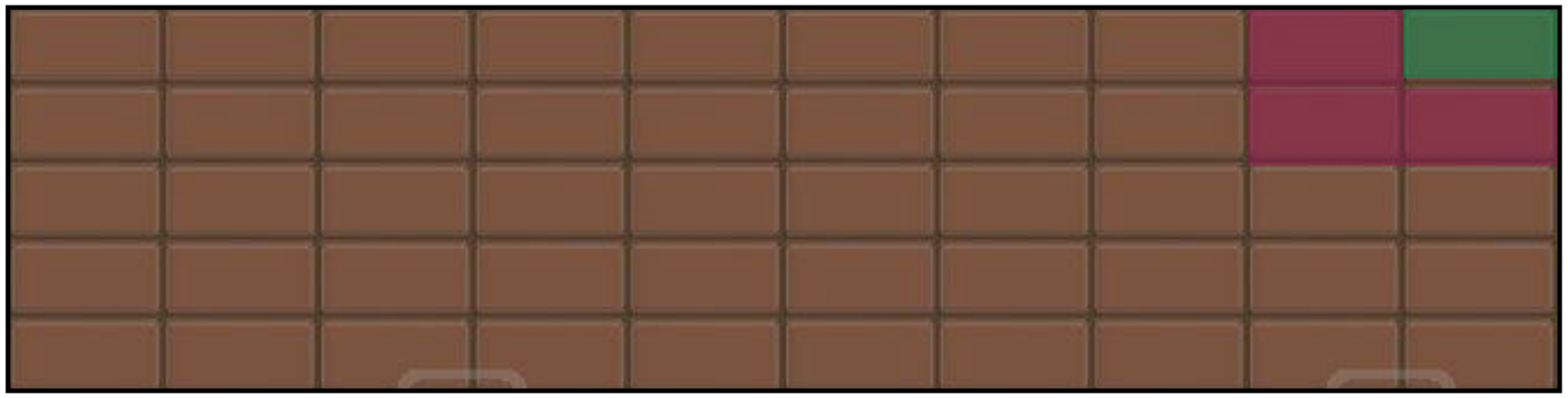
@ Bob applies his winning strategy



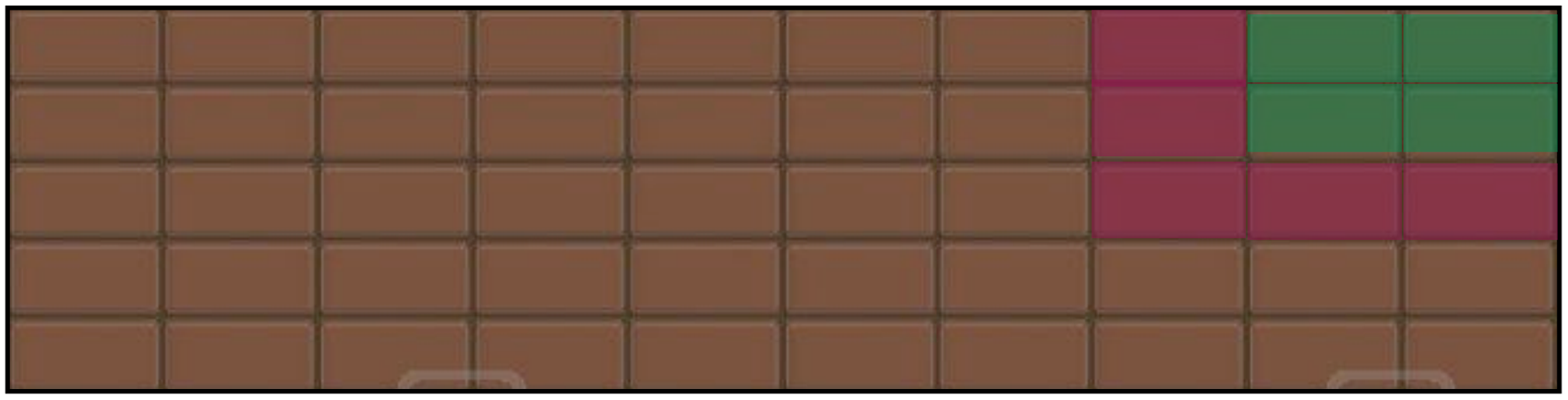
@ Alice plays against Bob



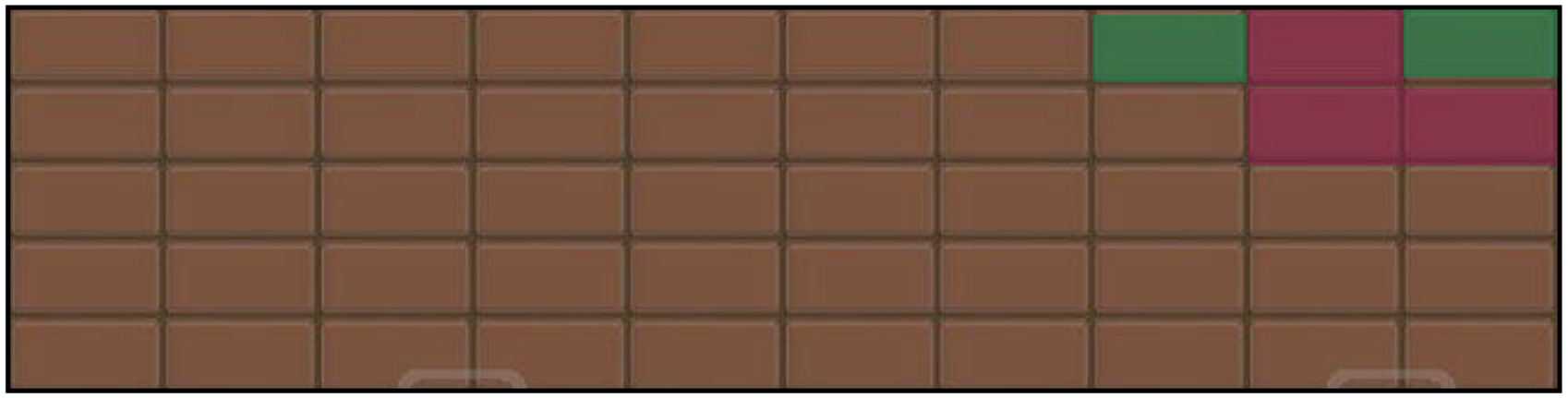
@ Bob applies his winning strategy



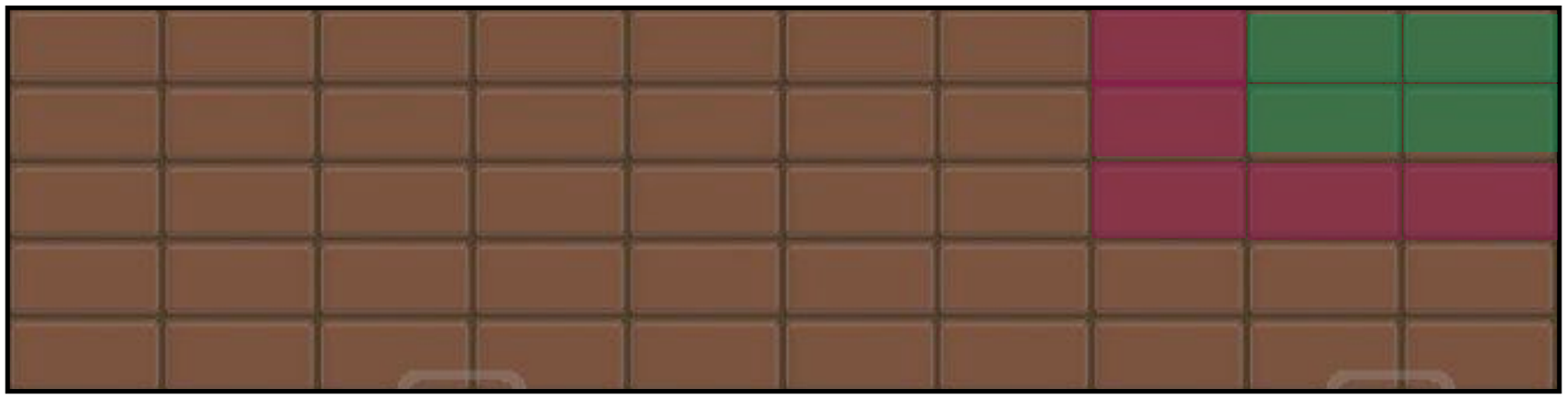
@ Alice plays against Bob



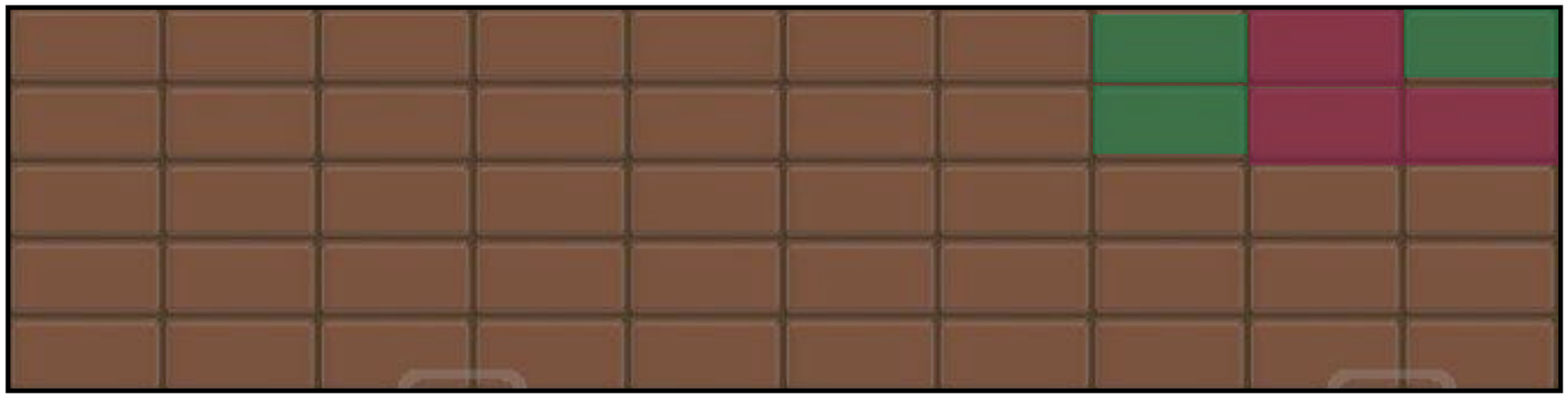
@ Bob applies his winning strategy



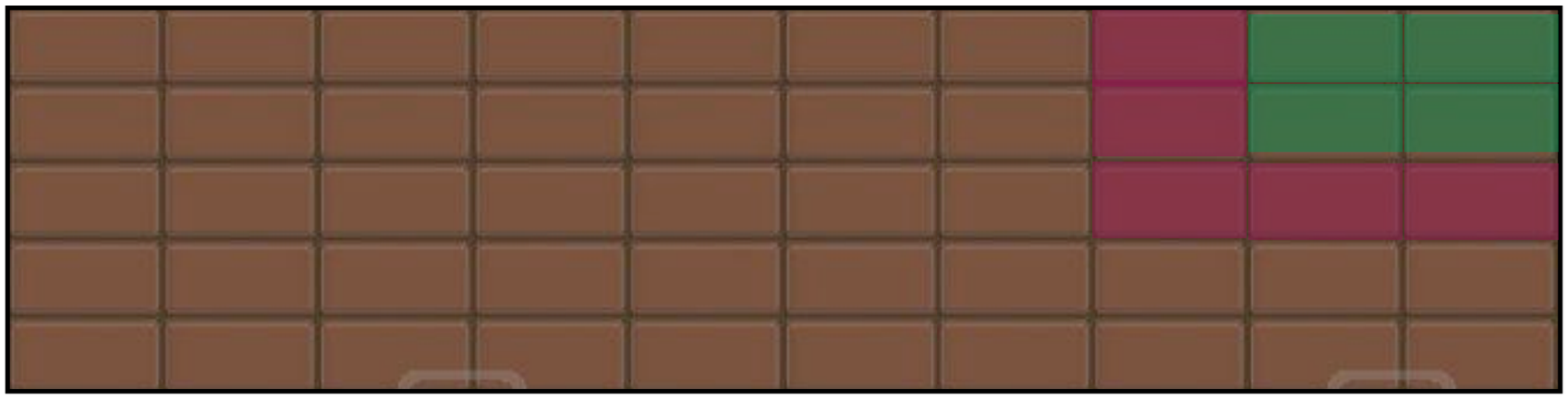
@ Alice plays against Bob



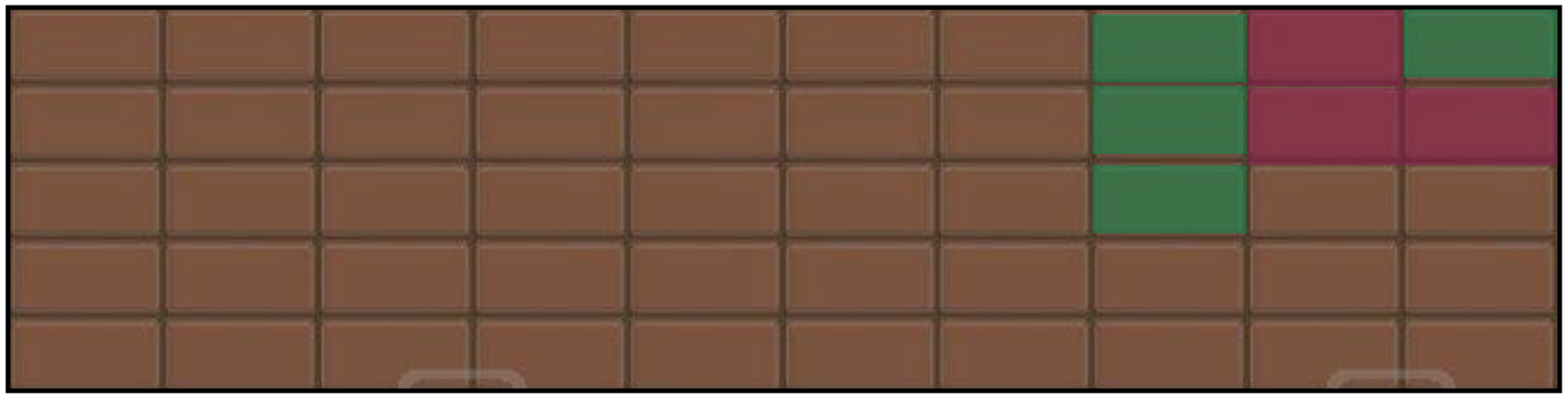
@ Bob applies his winning strategy



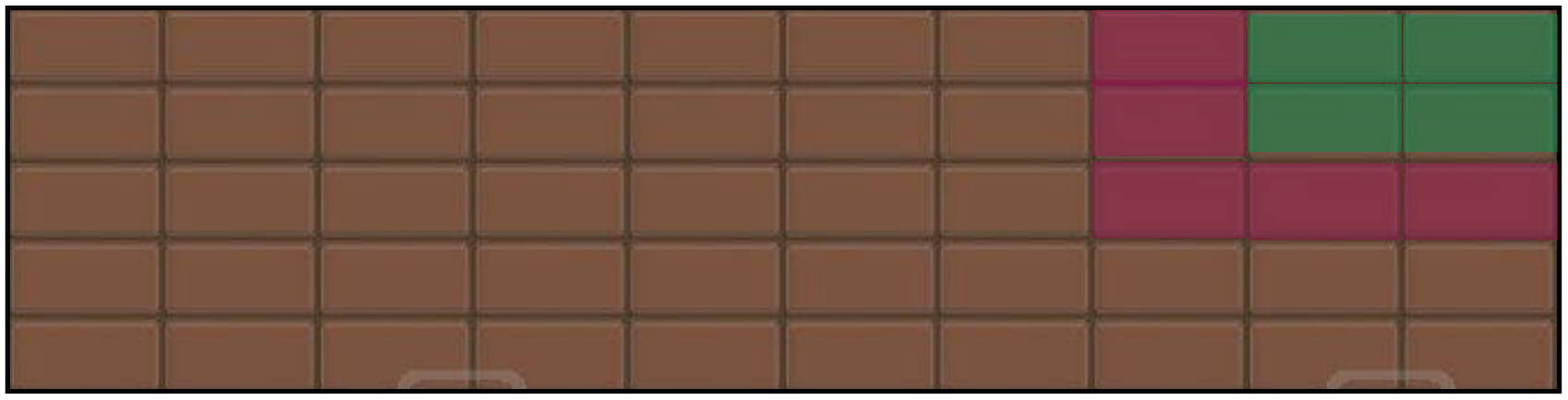
@ Alice plays against Bob



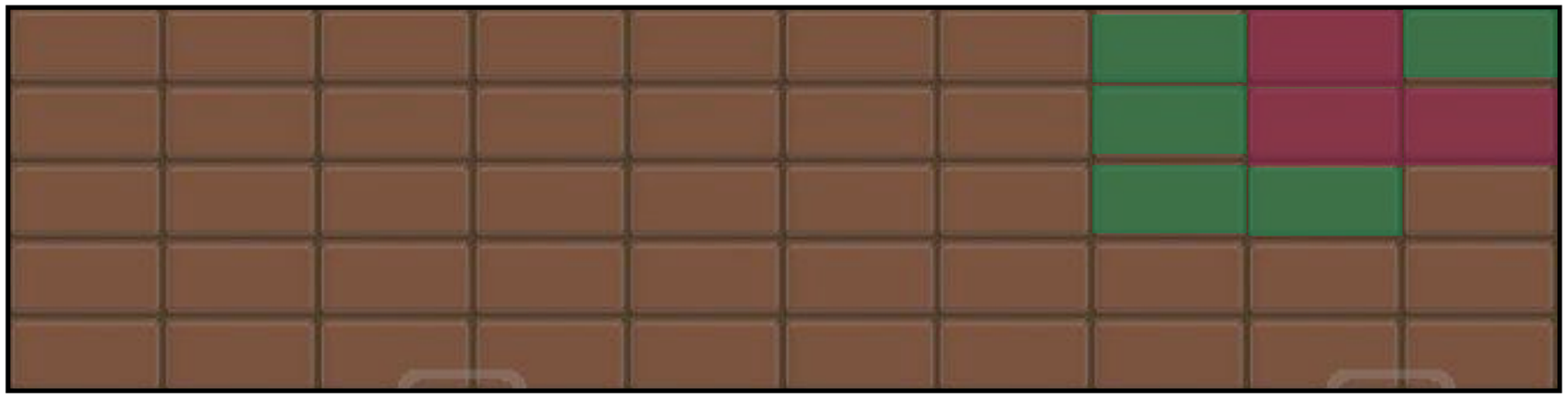
@ Bob applies his winning strategy



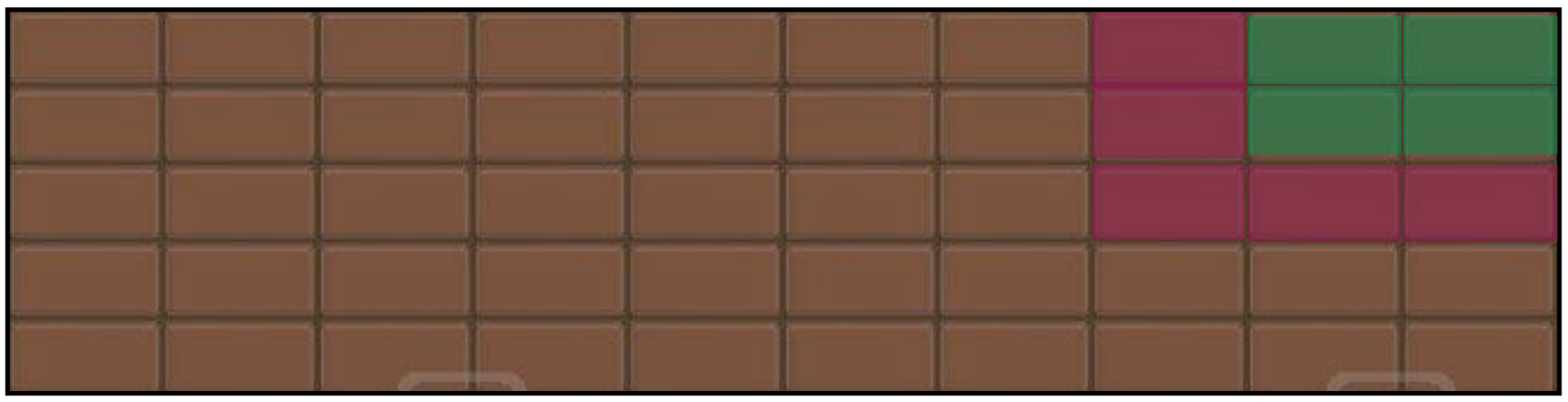
@ Alice plays against Bob



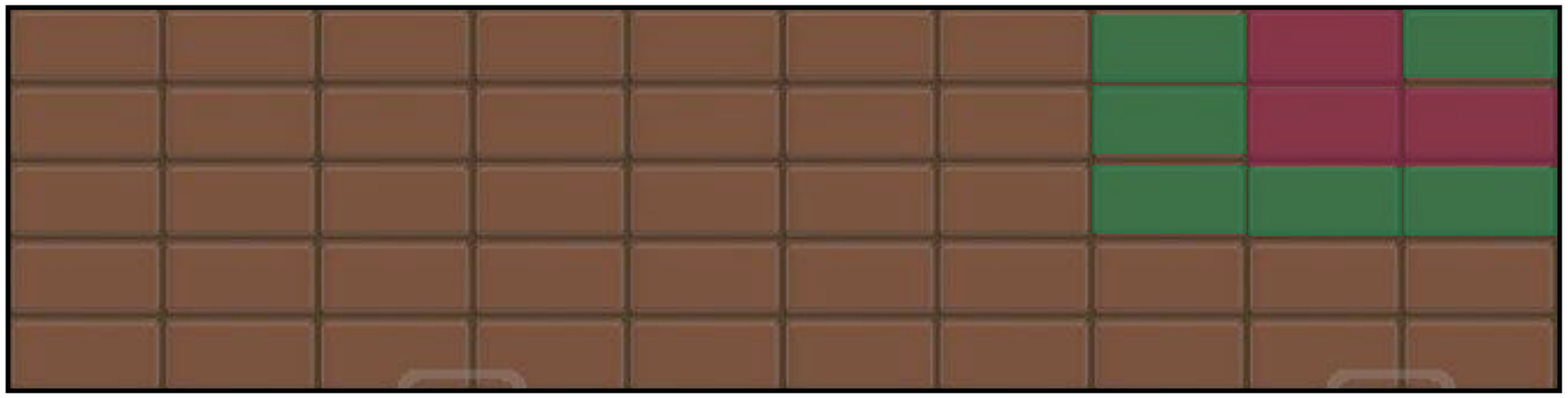
@ Bob applies his winning strategy



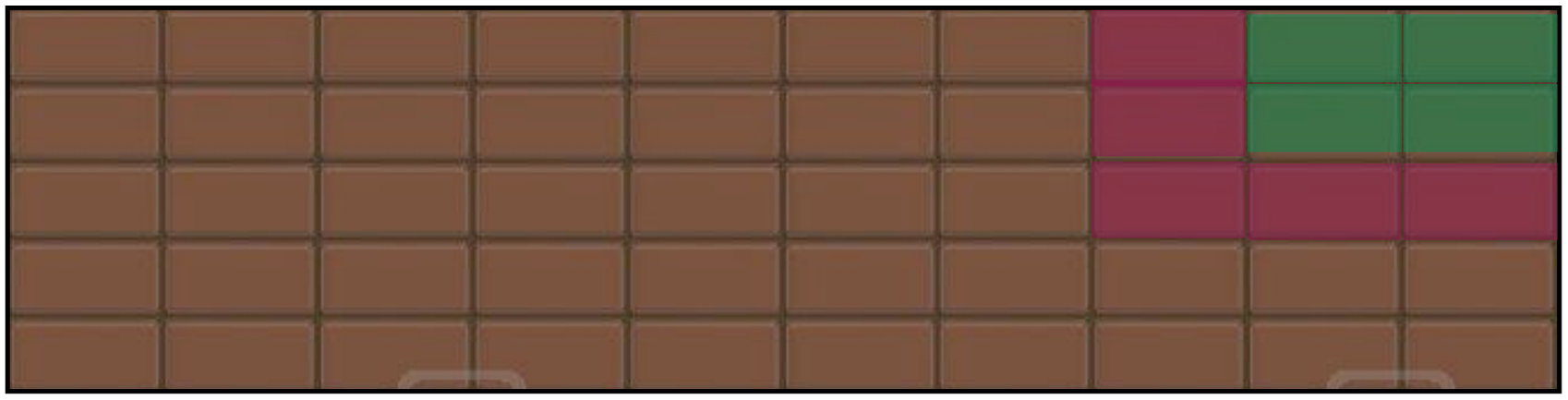
@ Alice plays against Bob



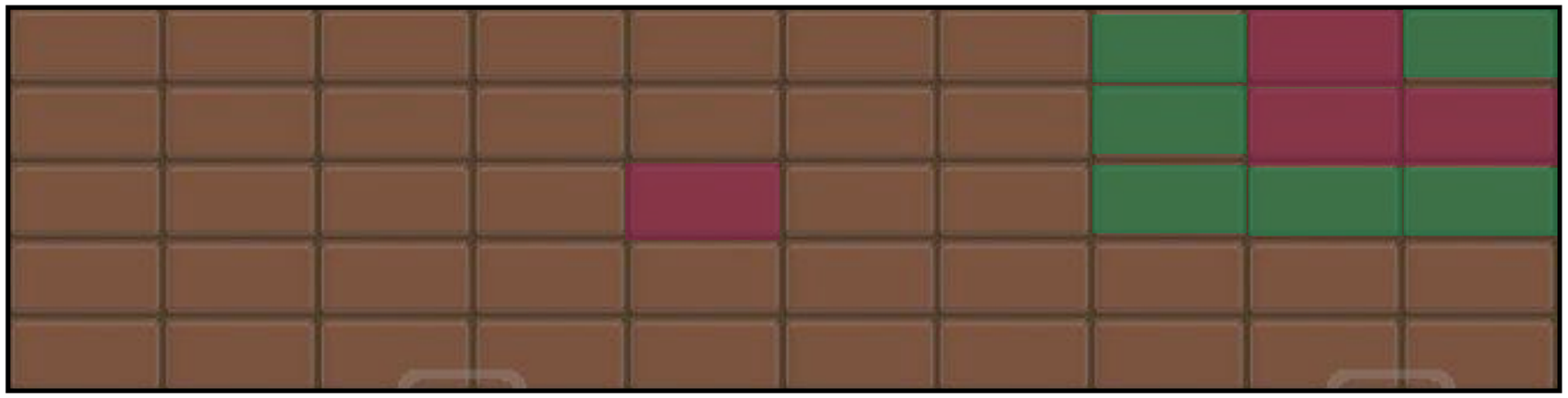
@ Bob applies his winning strategy



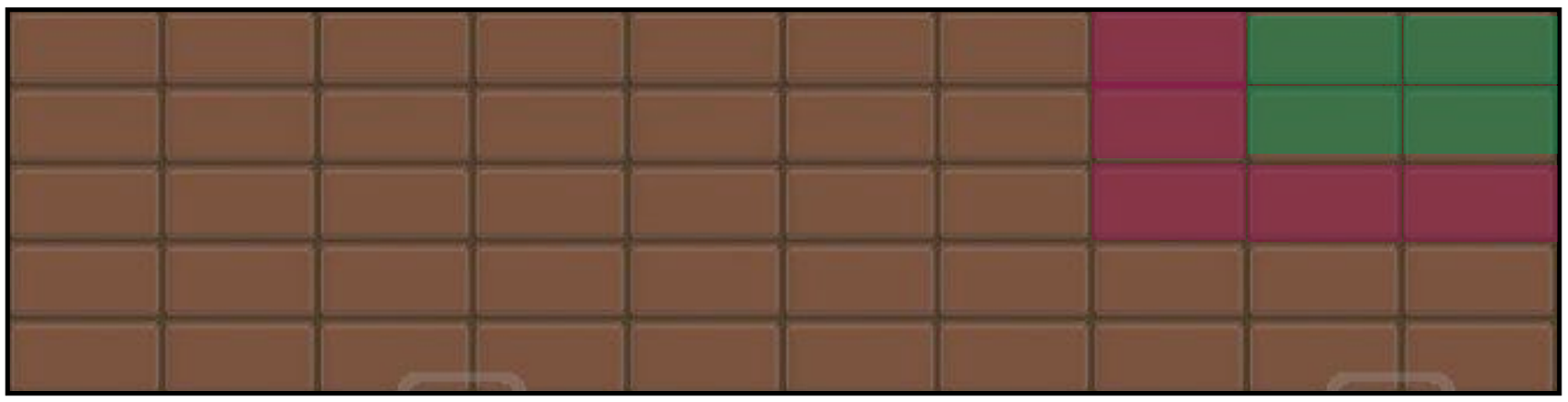
@ Alice plays against Bob



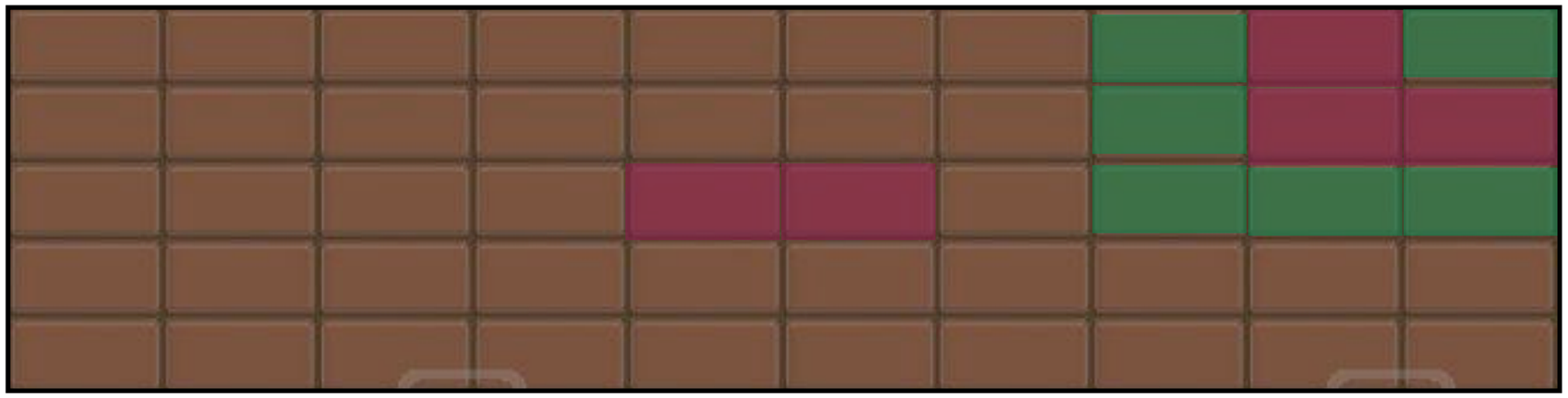
@ Bob applies his winning strategy



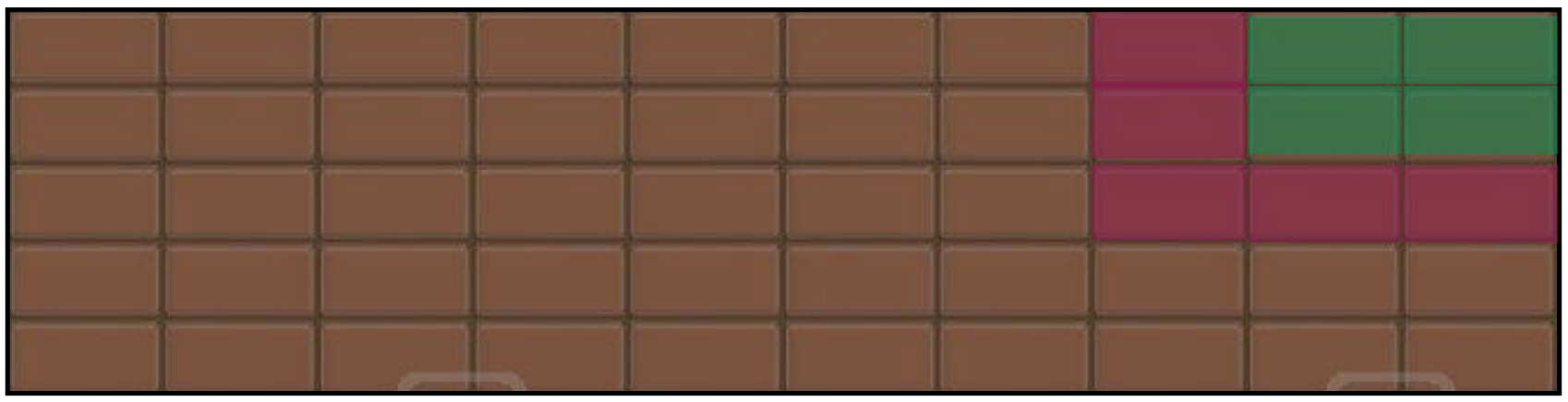
@ Alice plays against Bob



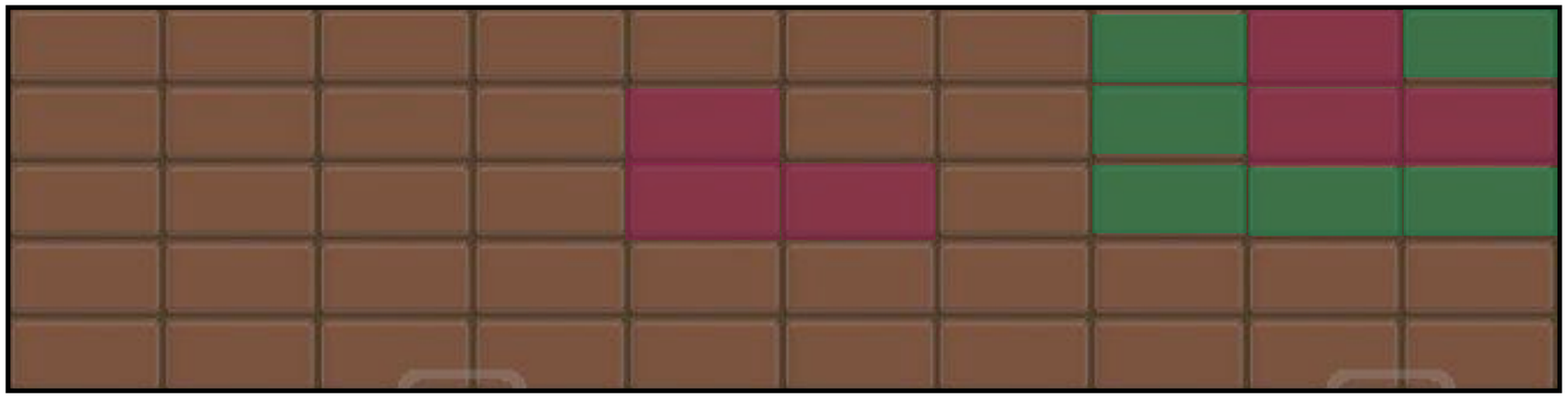
@ Bob applies his winning strategy



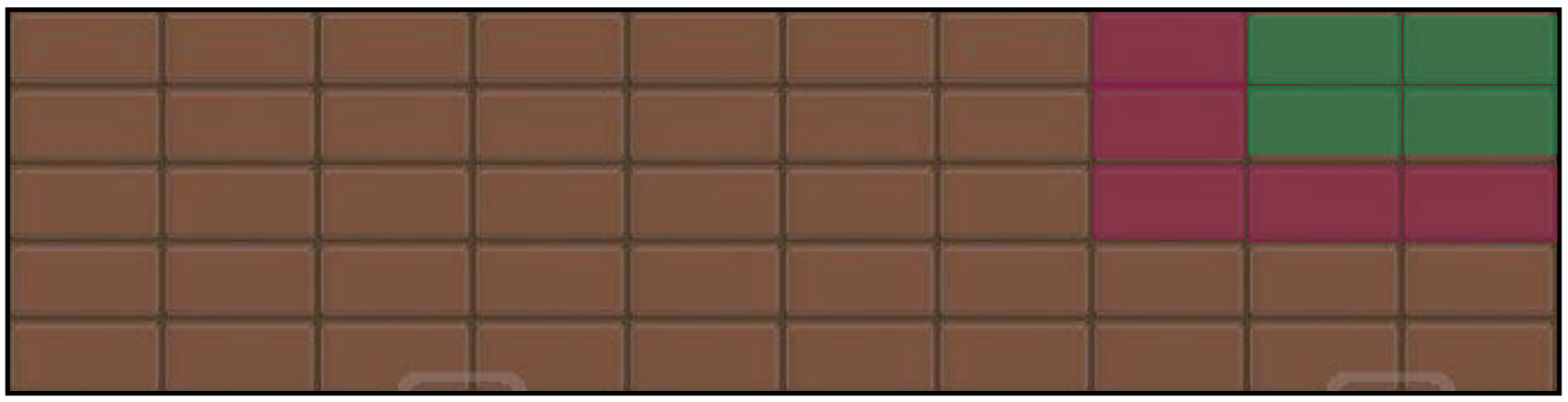
@ Alice plays against Bob



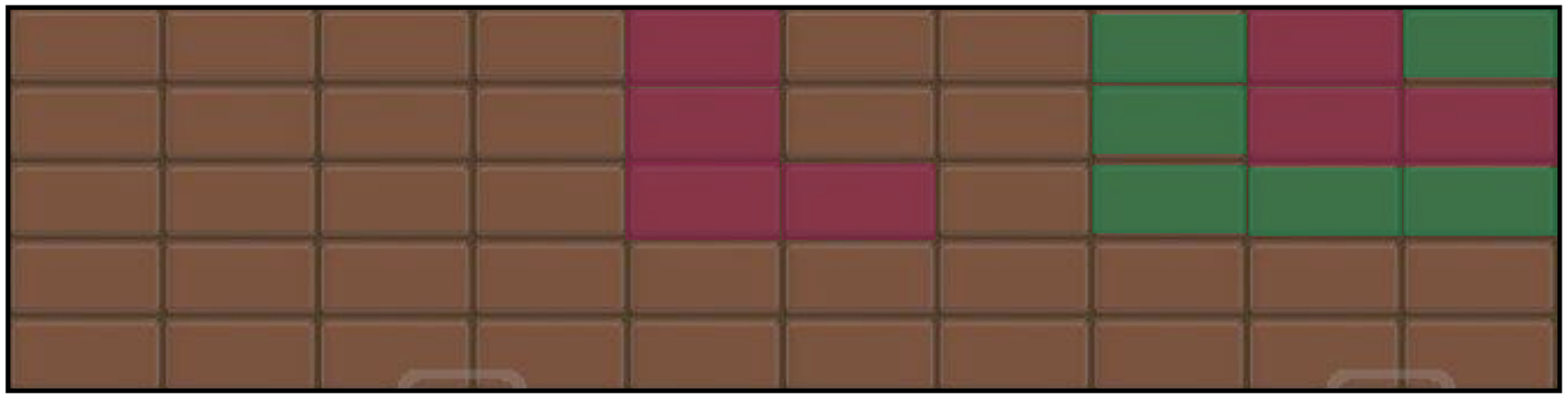
@ Bob applies his winning strategy



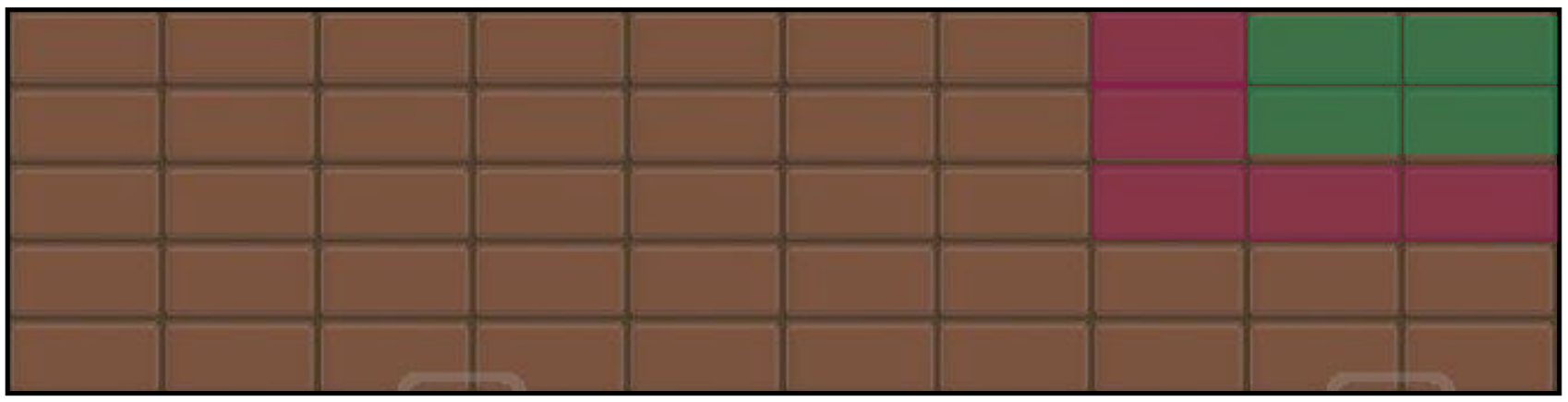
@ Alice plays against Bob



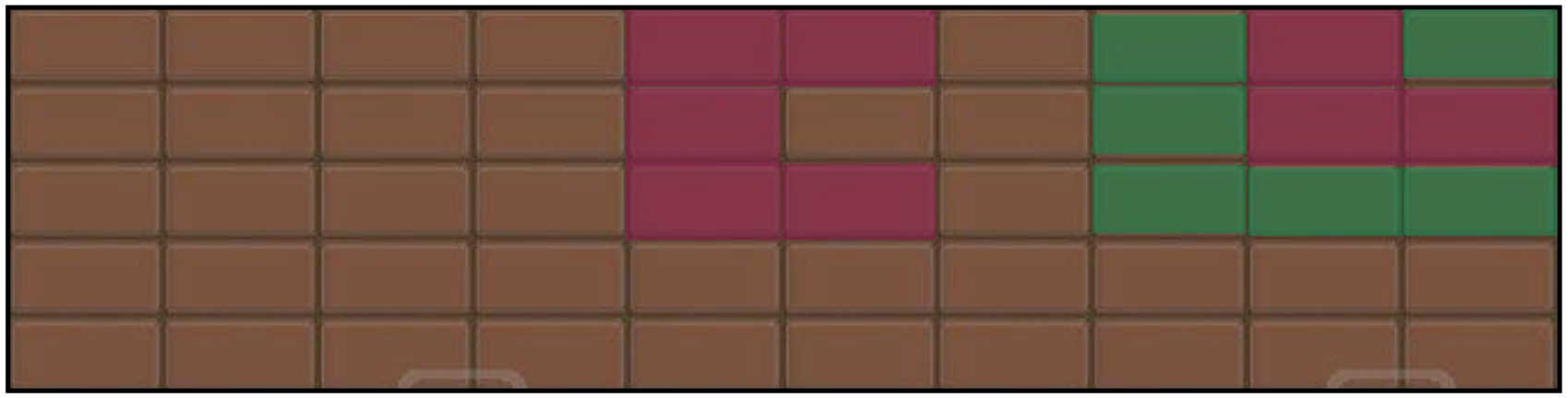
@ Bob applies his winning strategy



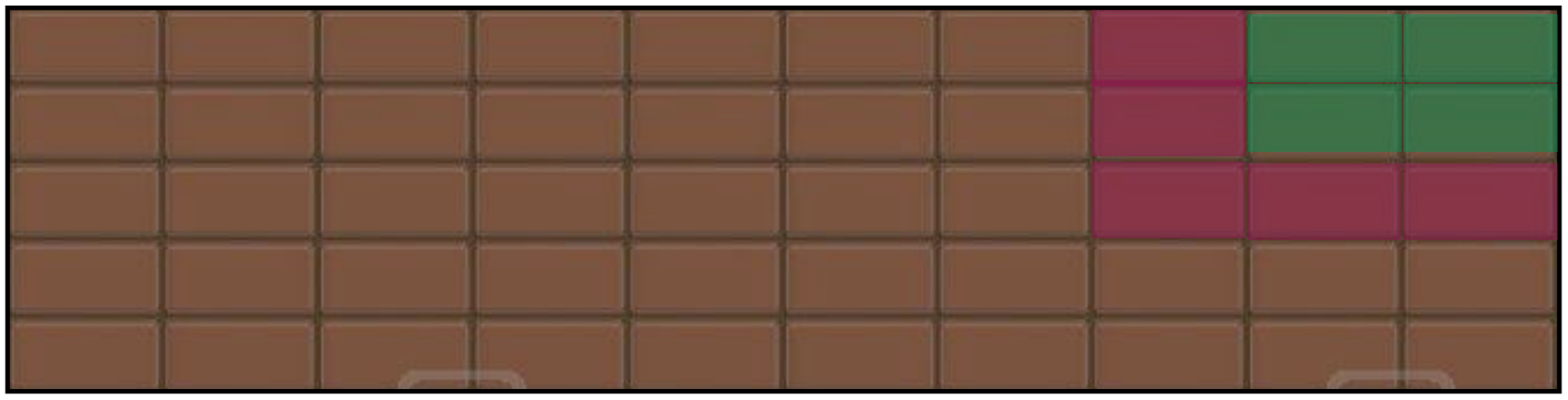
@ Alice plays against Bob



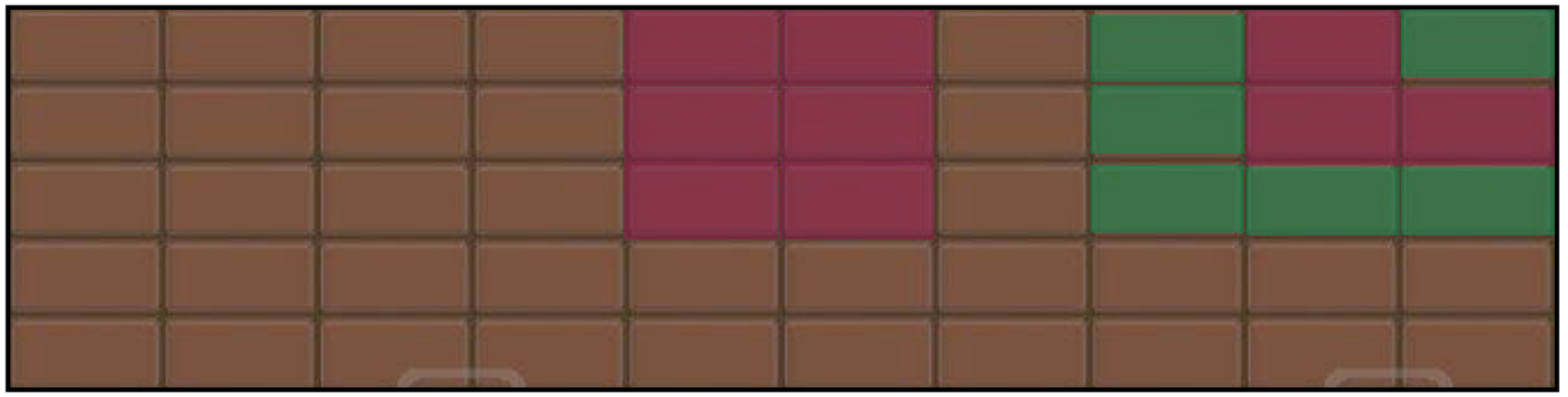
@ Bob applies his winning strategy



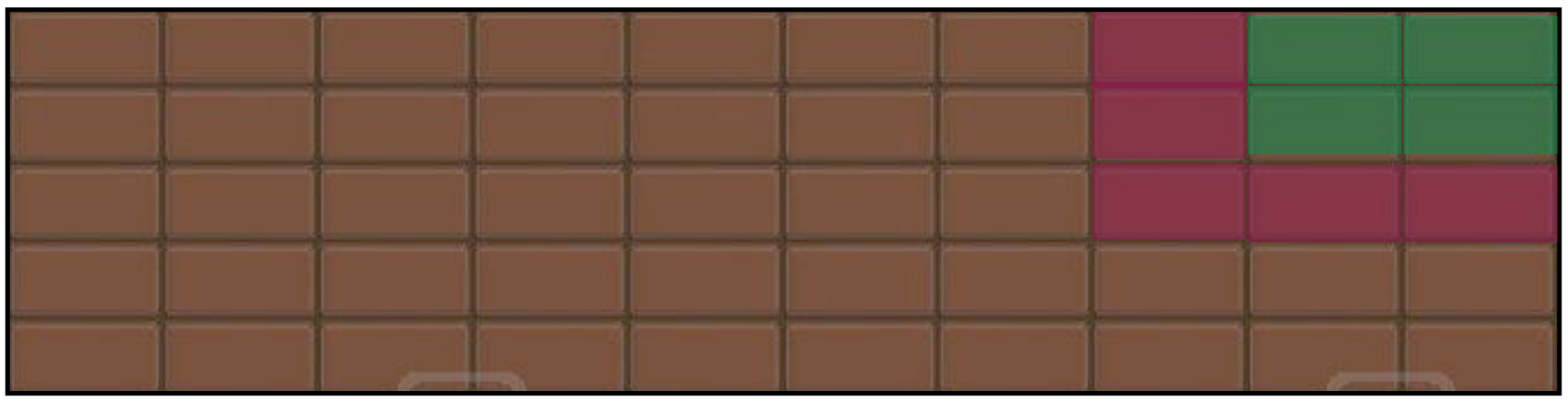
@ Alice plays against Bob



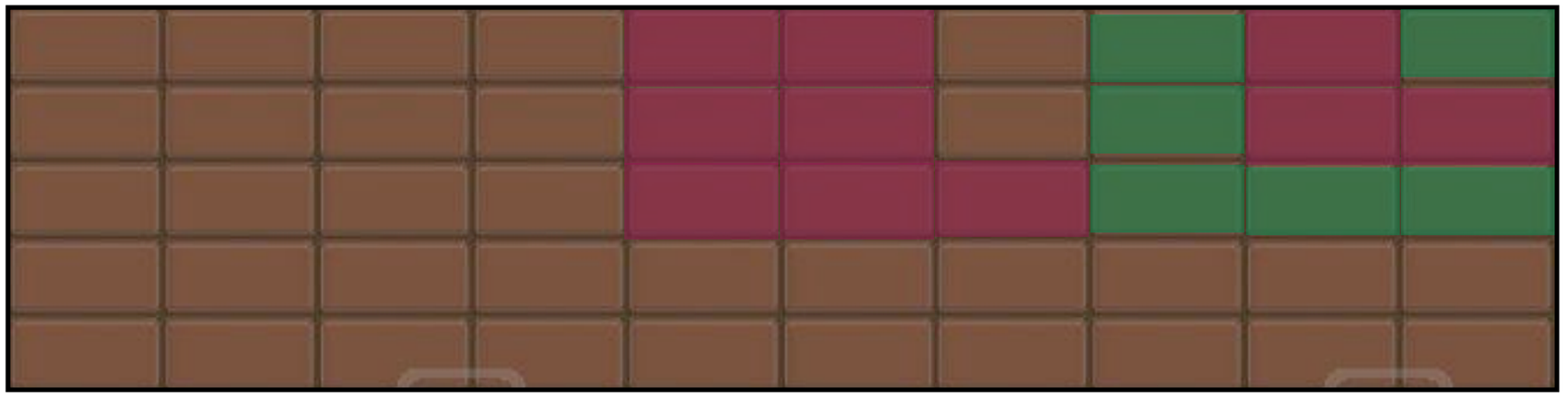
@ Bob applies his winning strategy



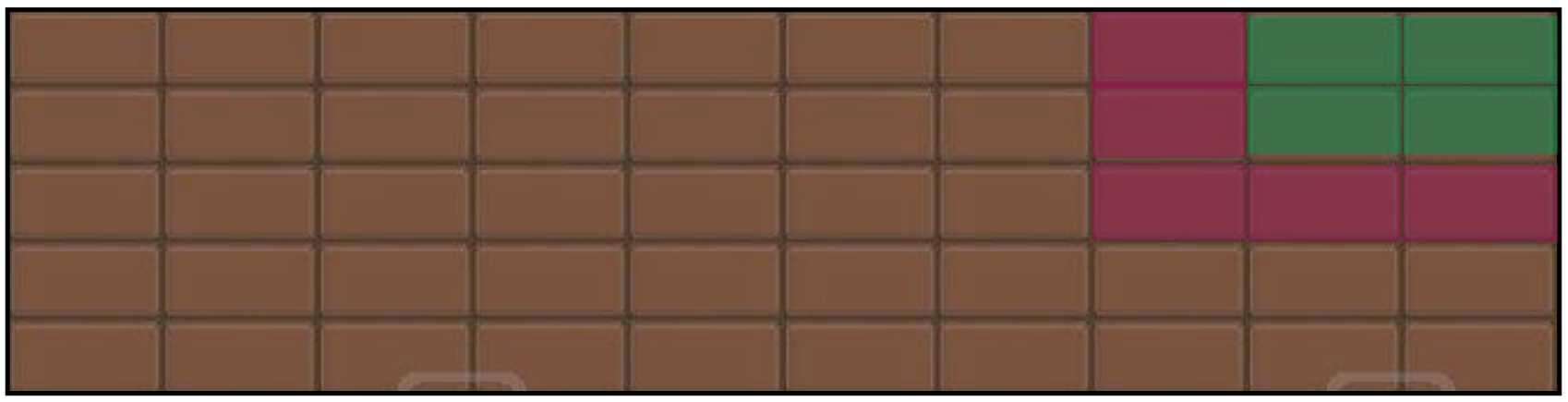
@ Alice plays against Bob



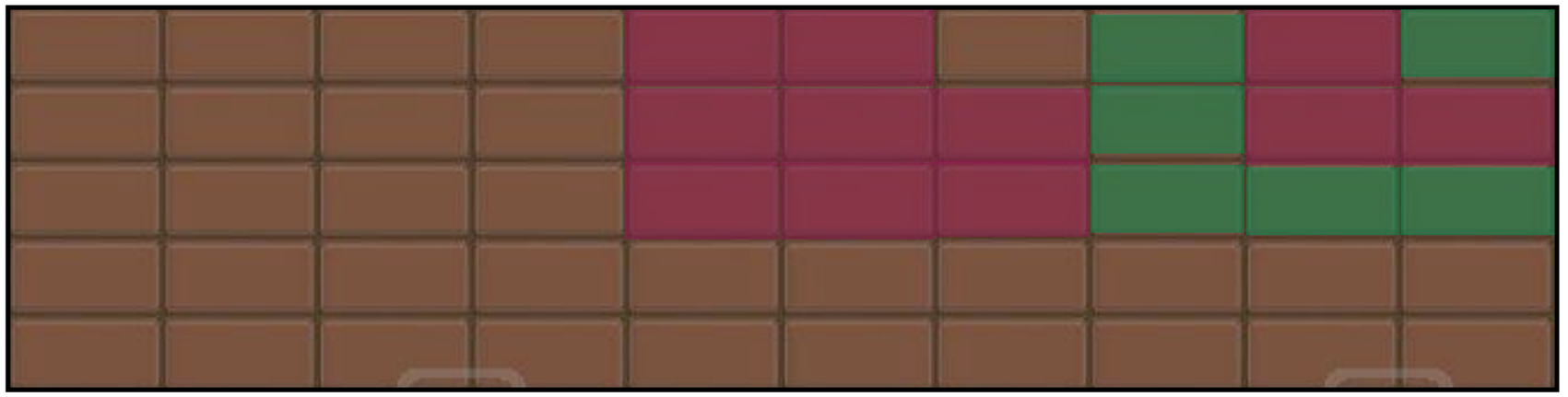
@ Bob applies his winning strategy



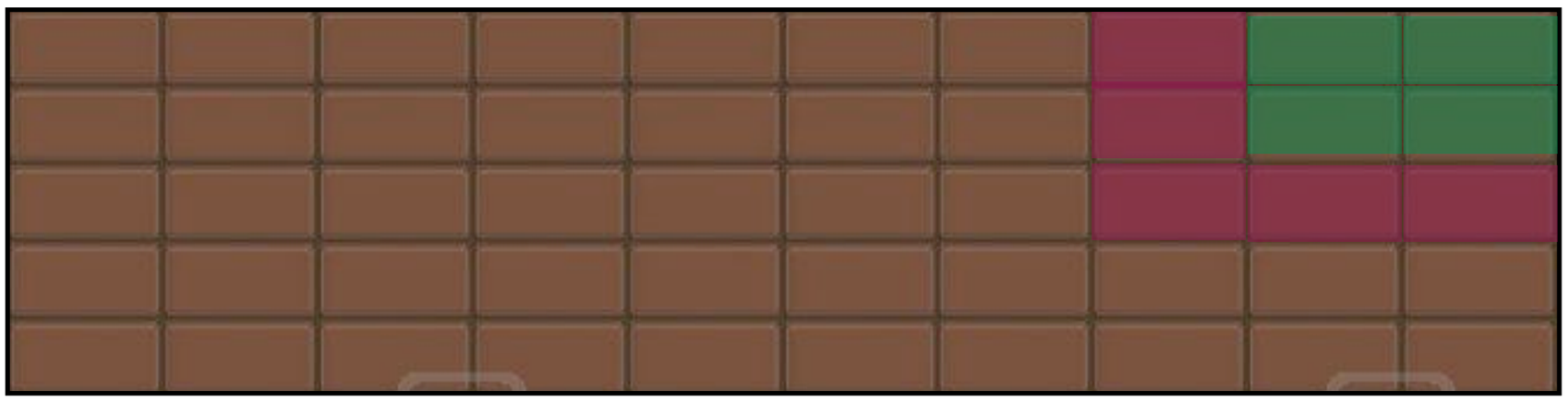
@ Alice plays against Bob



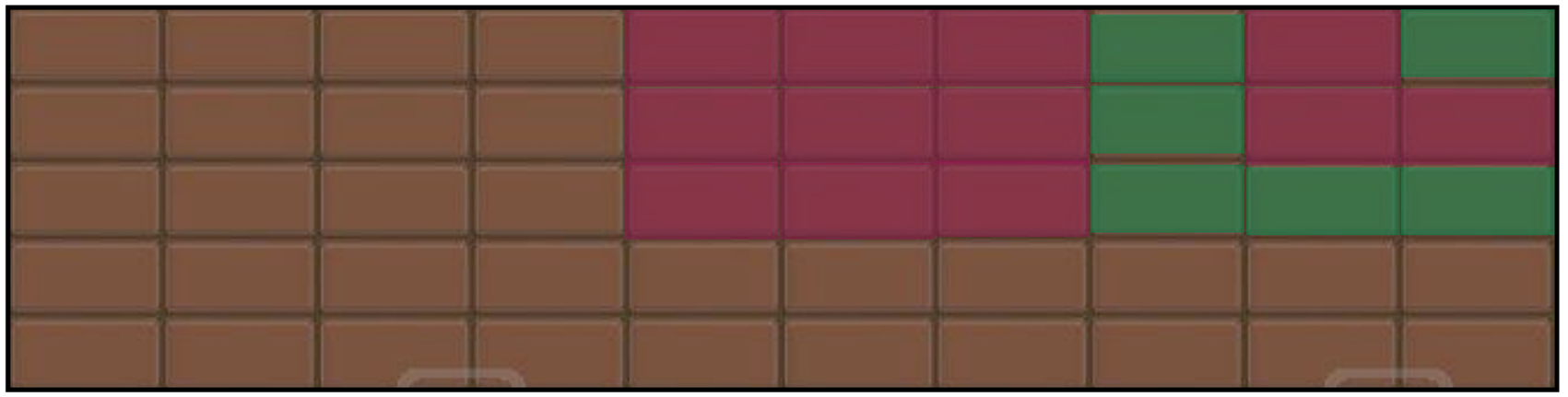
@ Bob applies his winning strategy



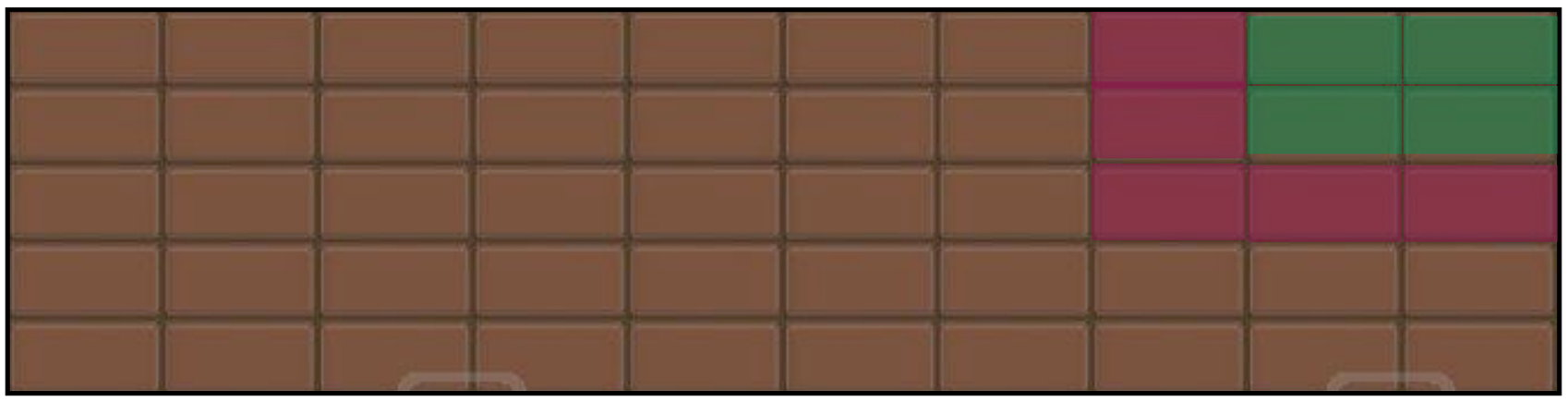
@ Alice plays against Bob



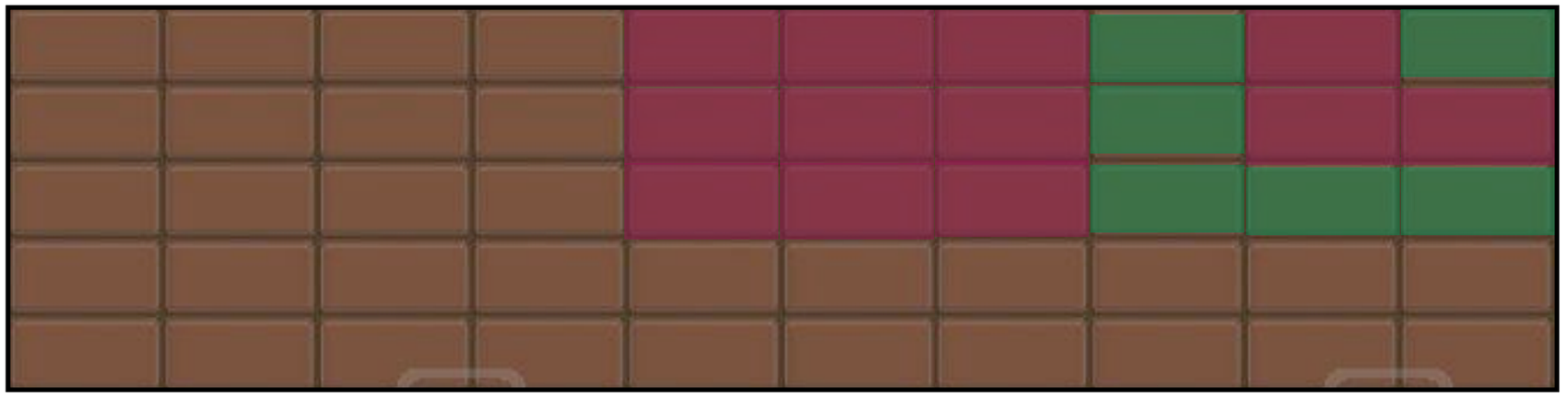
@ Bob applies his winning strategy



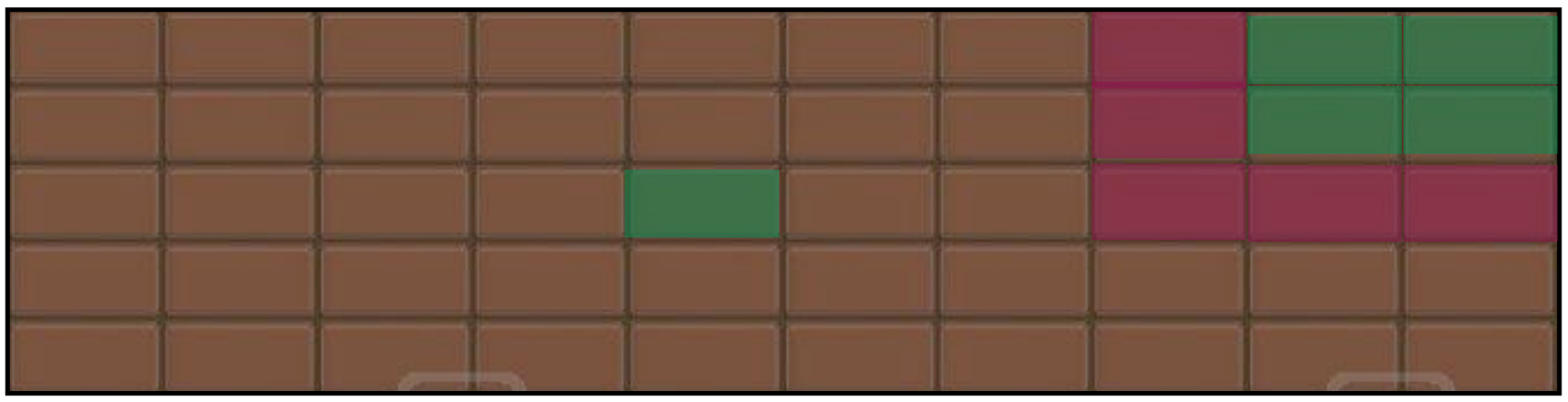
@ Alice plays against Bob



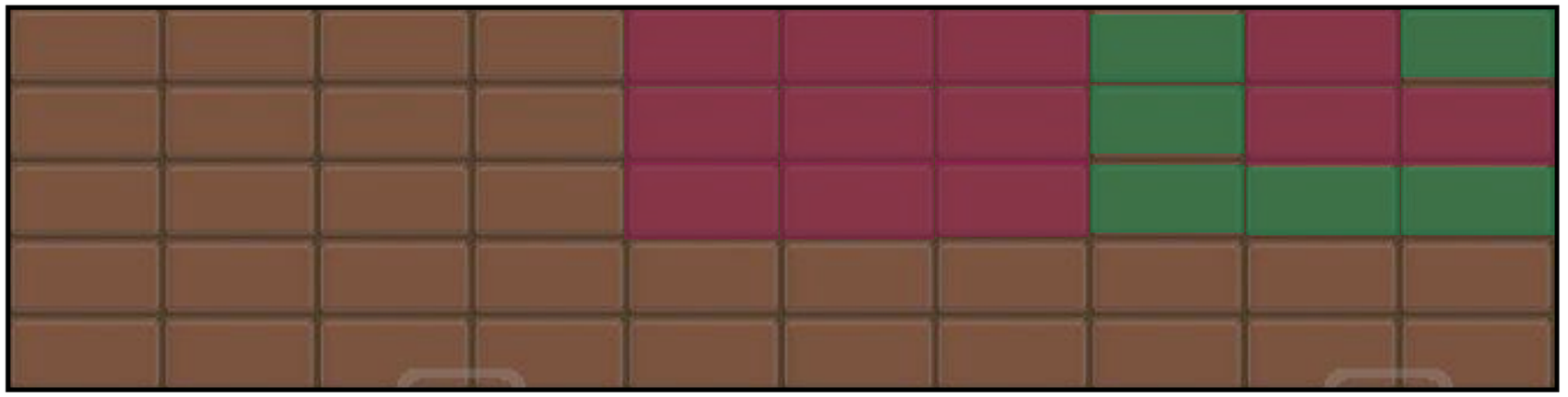
@ Bob applies his winning strategy



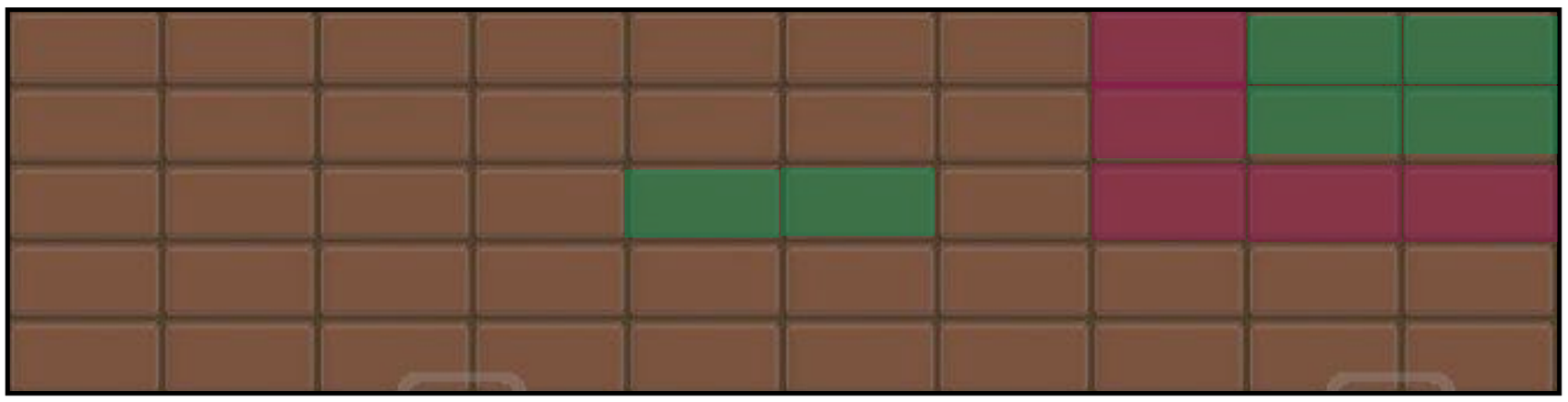
@ Alice plays against Bob



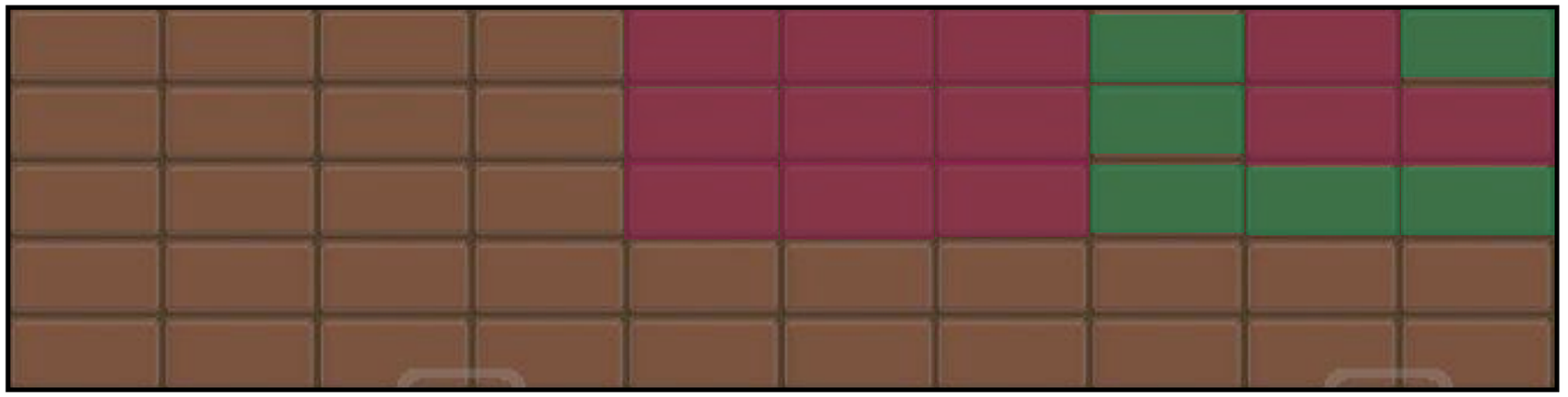
@ Bob applies his winning strategy



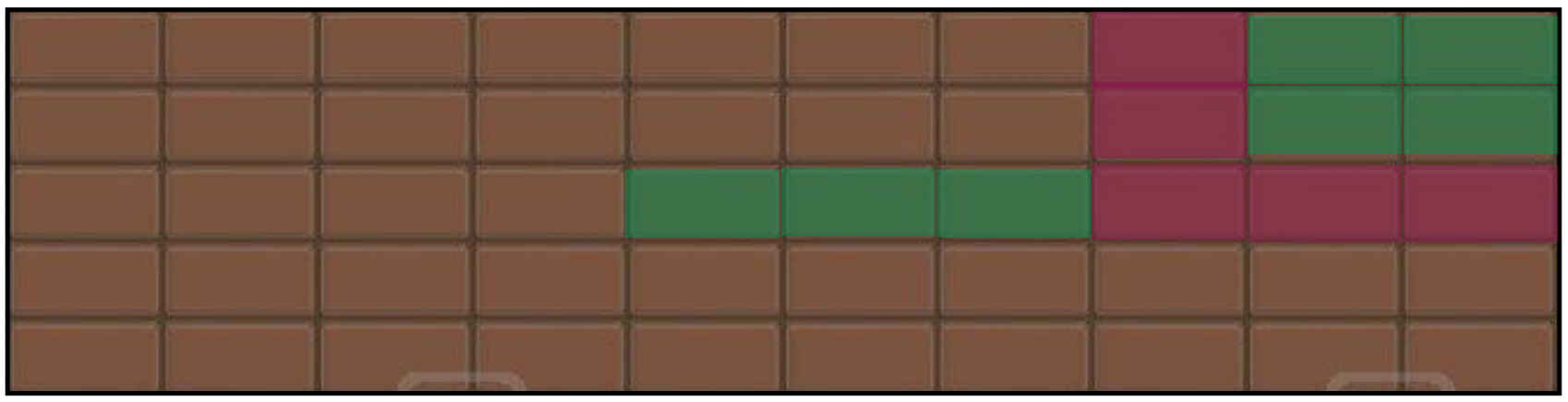
@ Alice plays against Bob



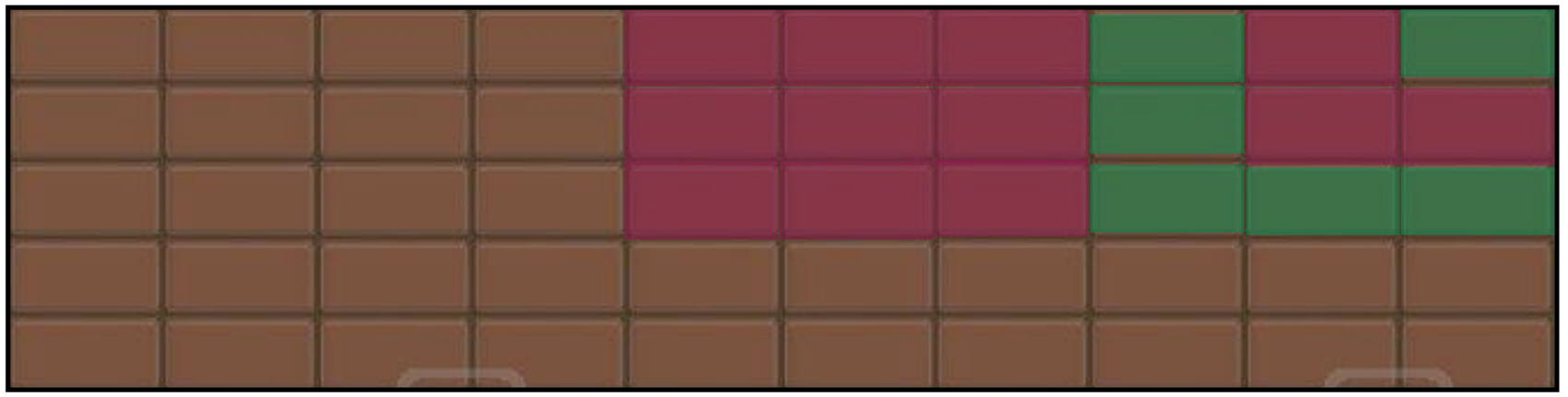
@ Bob applies his winning strategy



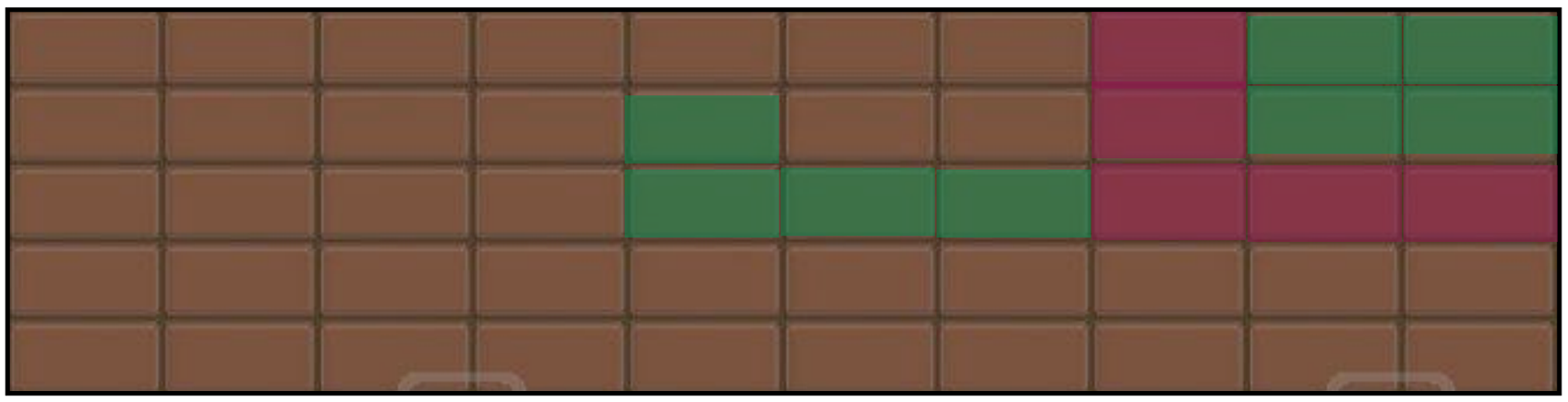
@ Alice plays against Bob



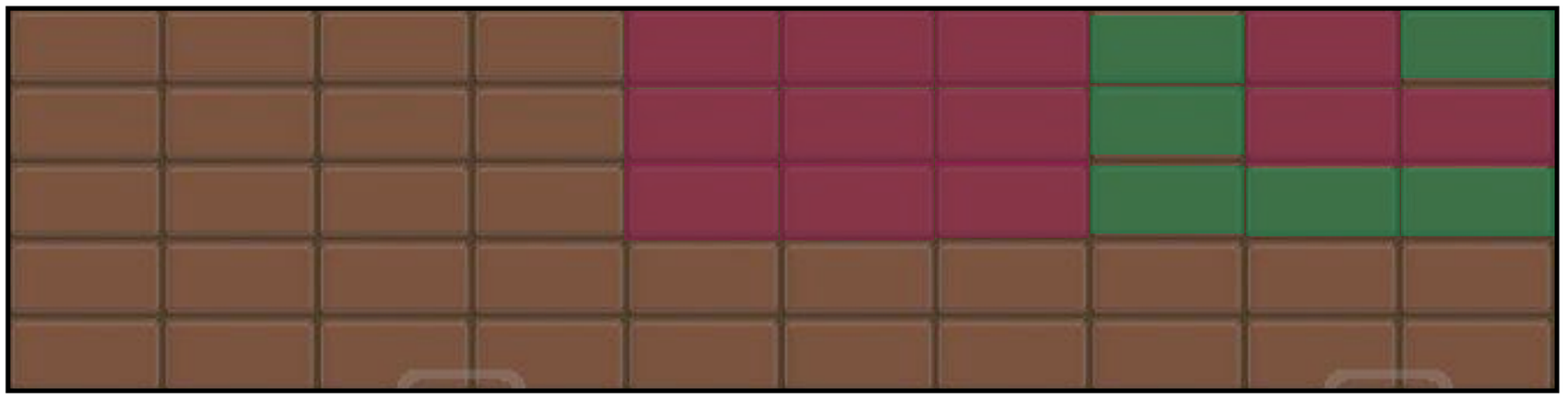
@ Bob applies his winning strategy



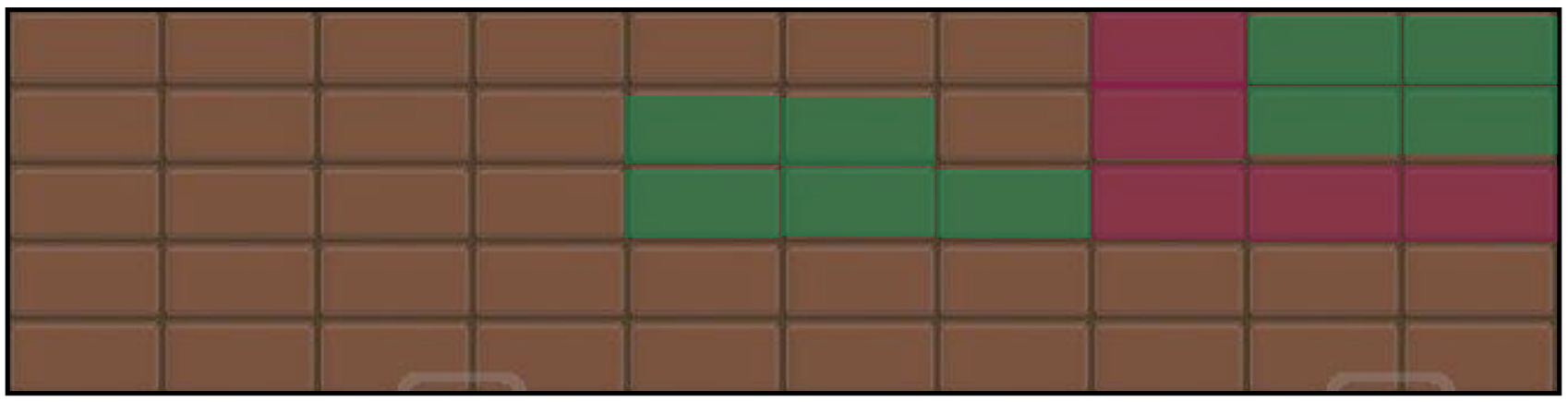
@ Alice plays against Bob



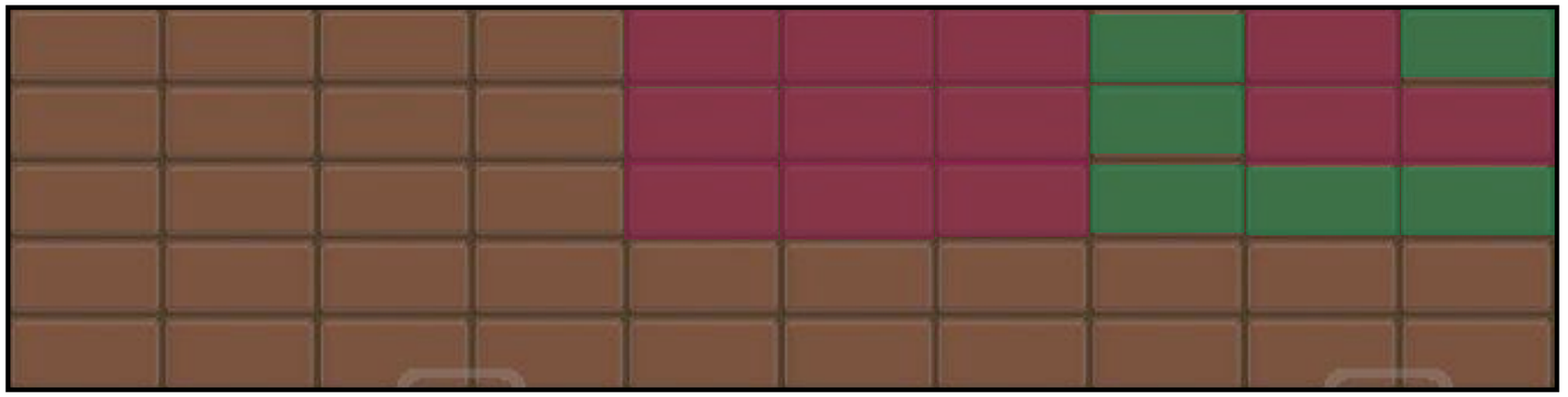
@ Bob applies his winning strategy



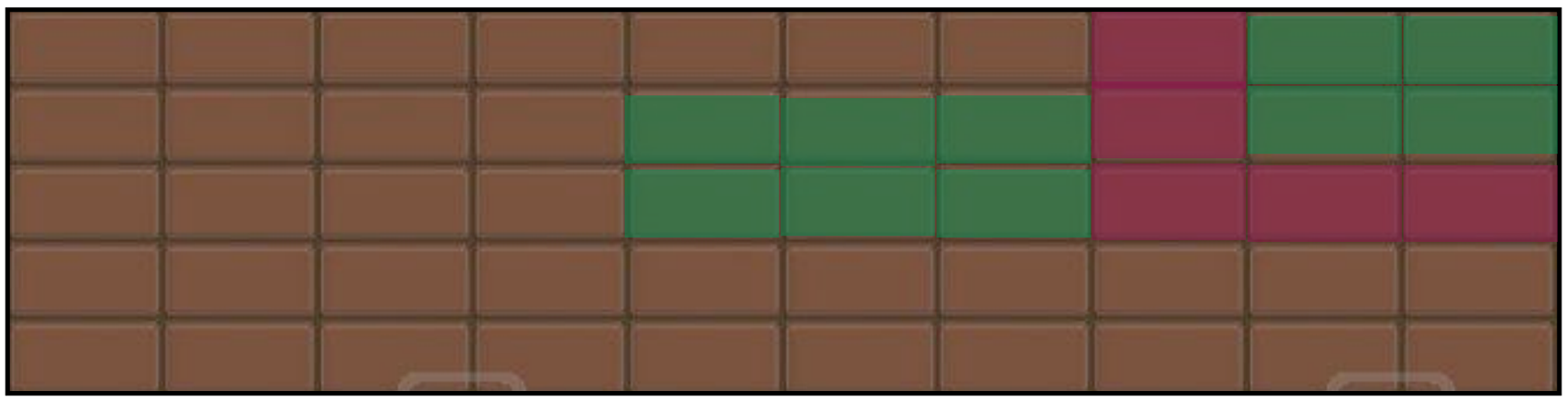
@ Alice plays against Bob



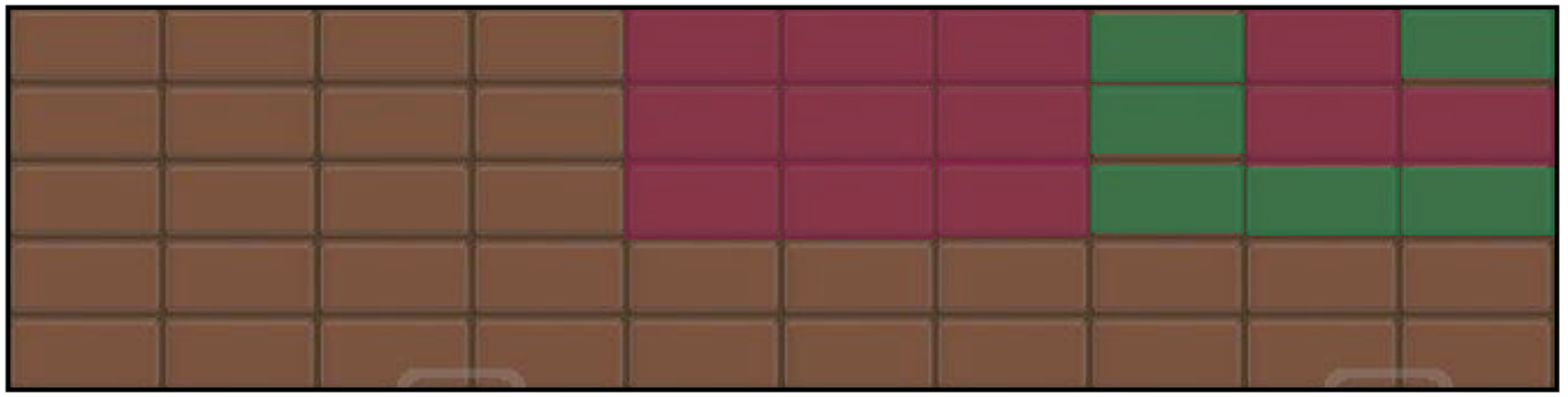
@ Bob applies his winning strategy



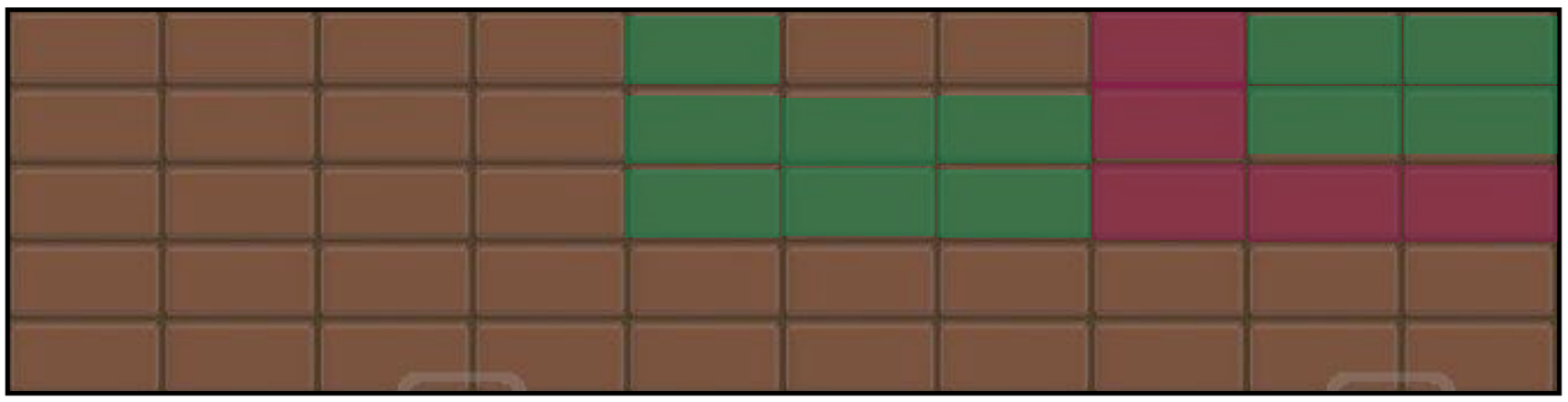
@ Alice plays against Bob



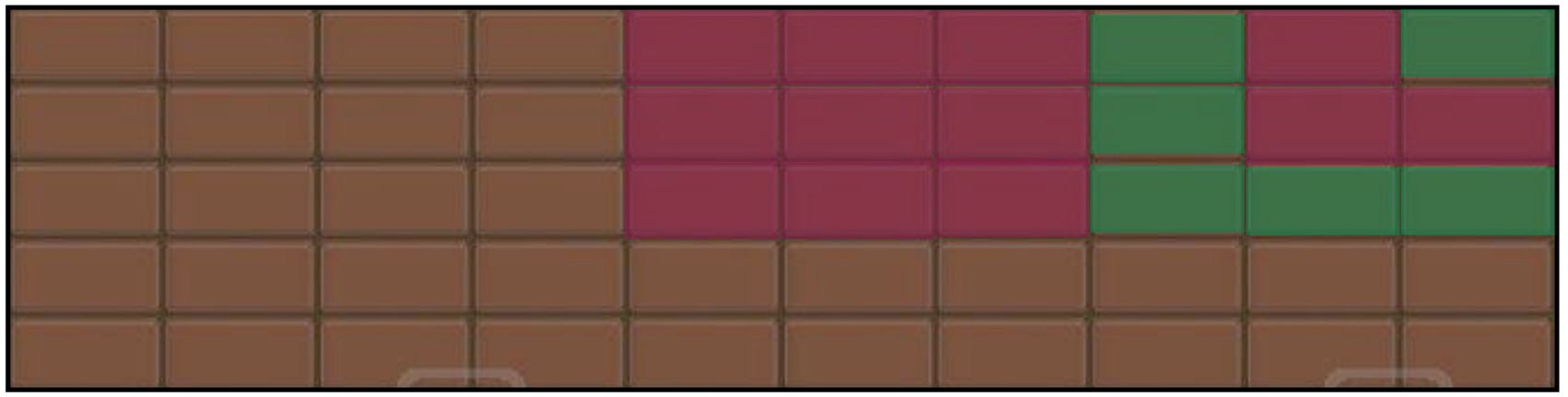
@ Bob applies his winning strategy



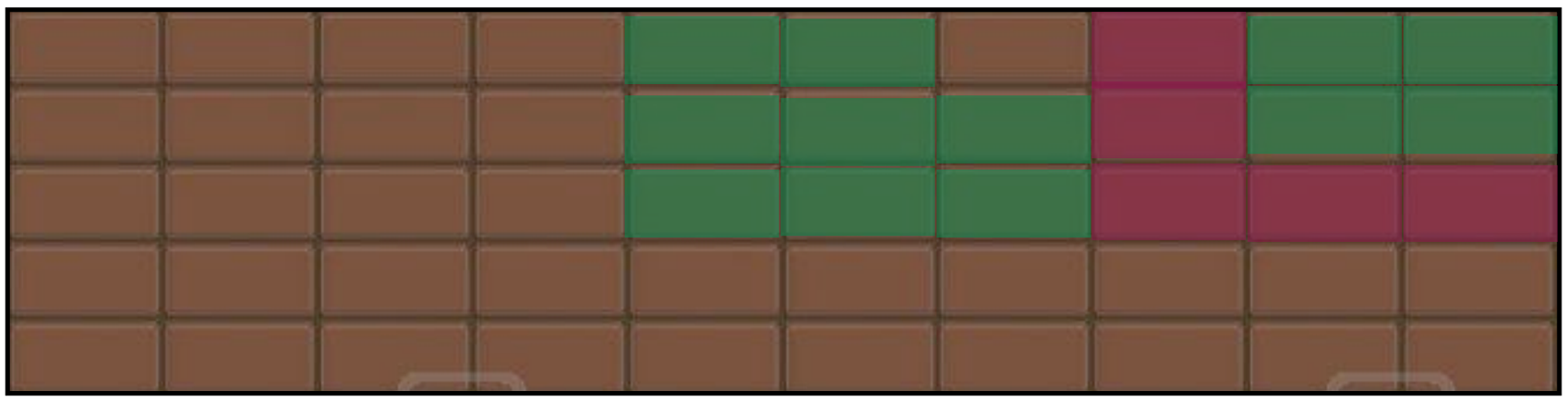
@ Alice plays against Bob



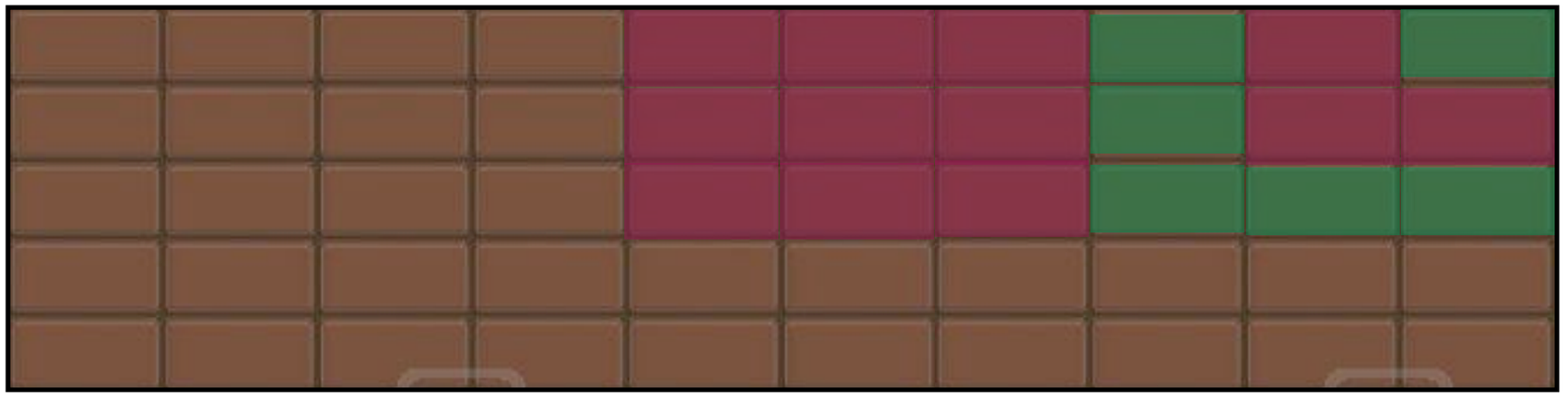
@ Bob applies his winning strategy



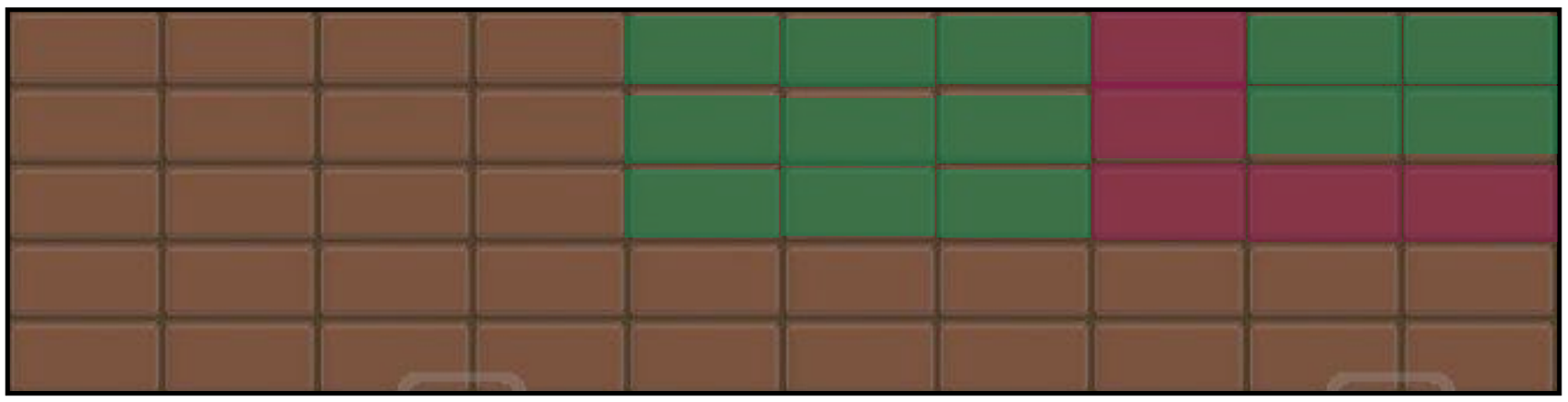
@ Alice plays against Bob



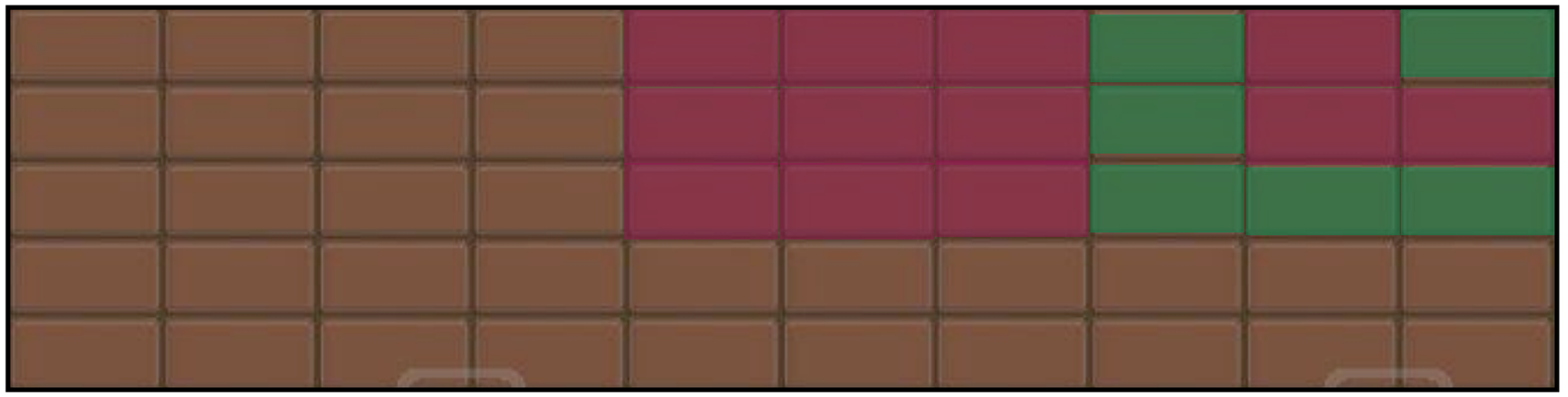
@ Bob applies his winning strategy



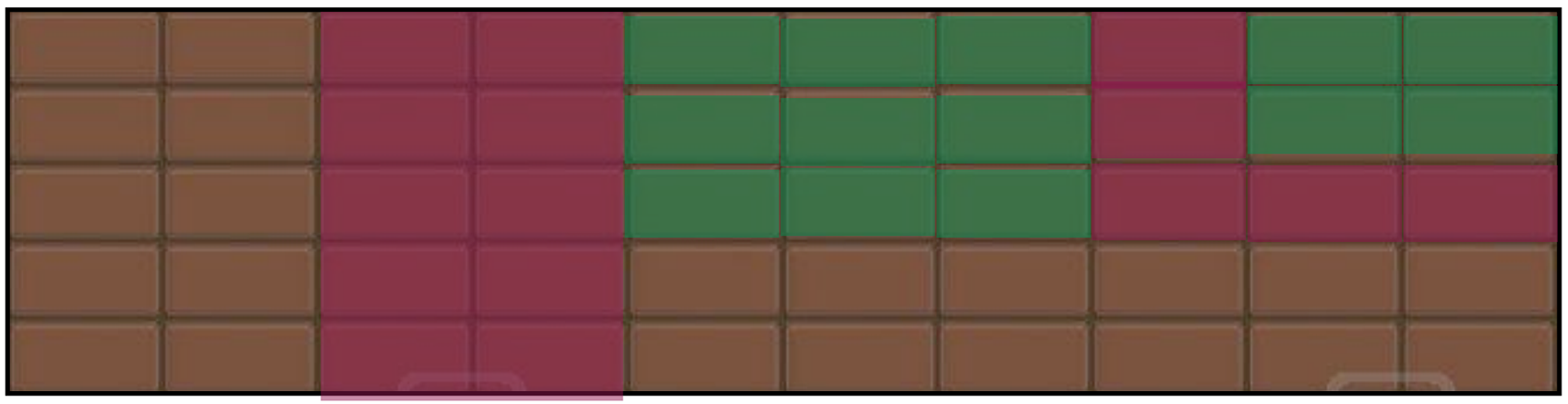
@ Alice plays against Bob



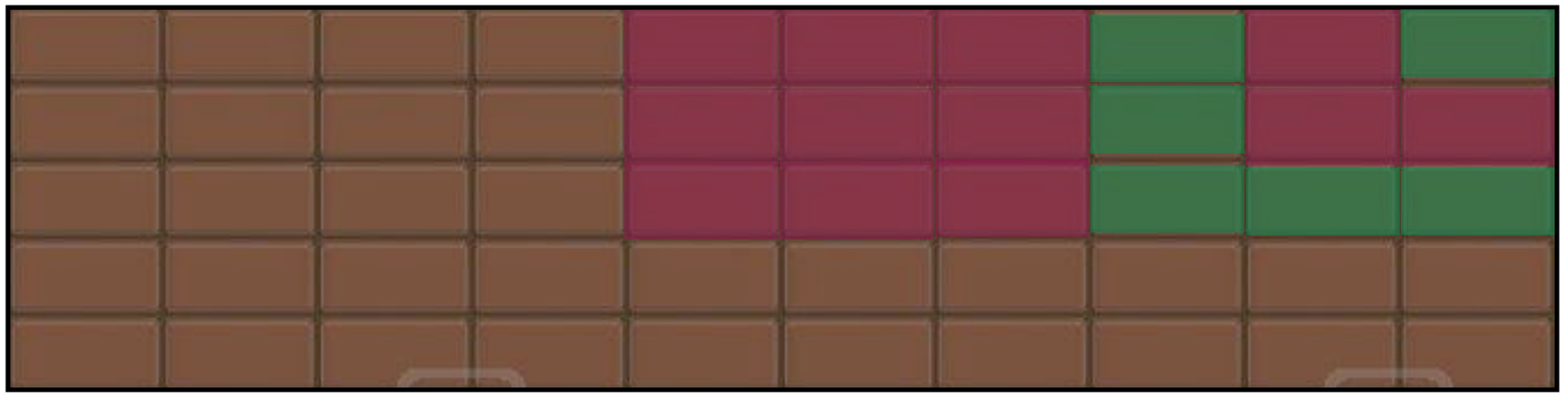
@ Bob applies his winning strategy



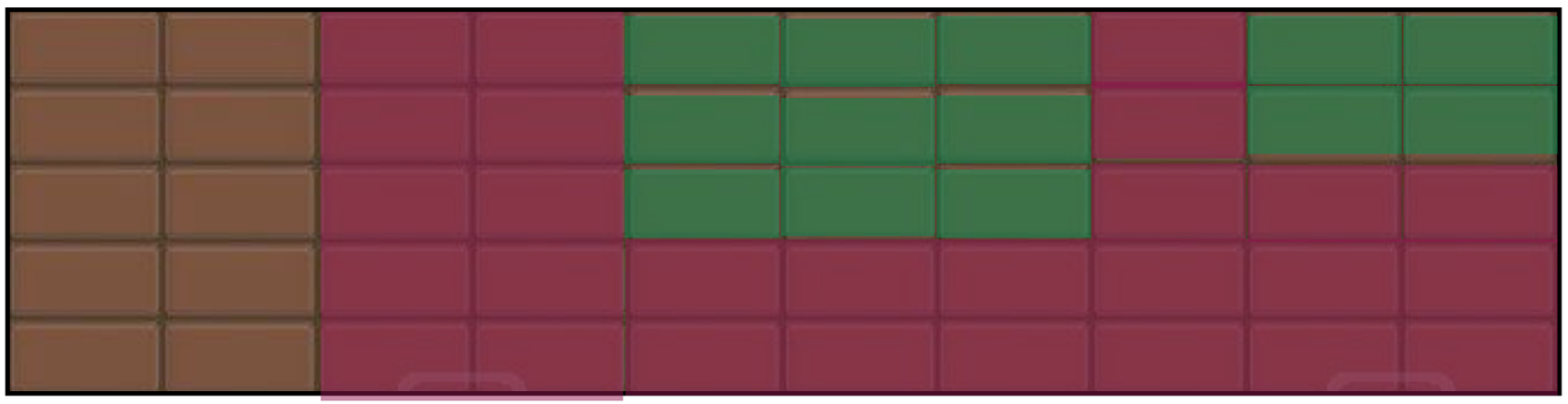
@ Alice plays against Bob



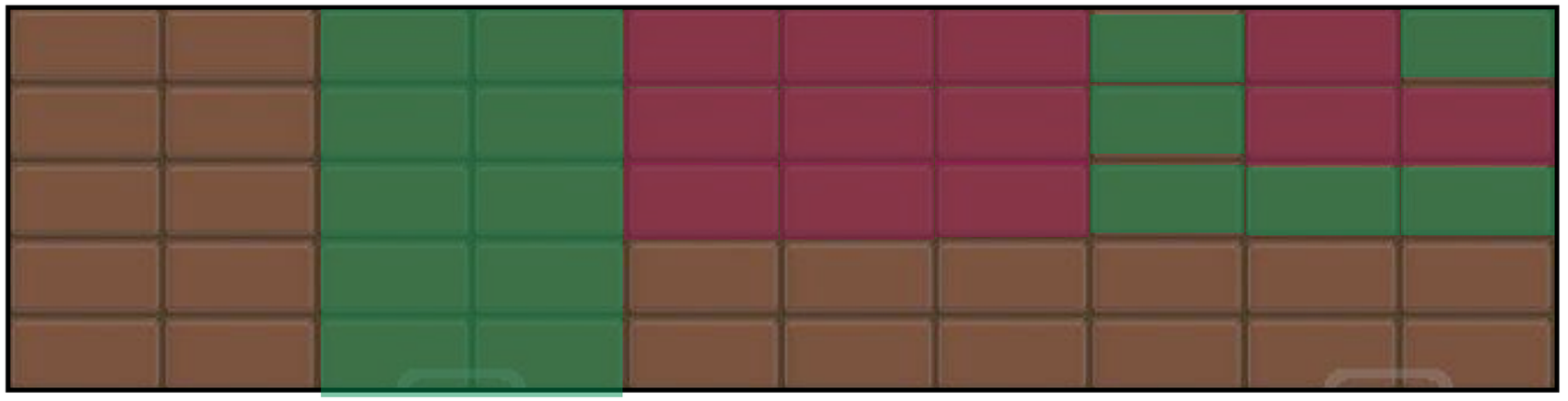
@ Bob applies his winning strategy



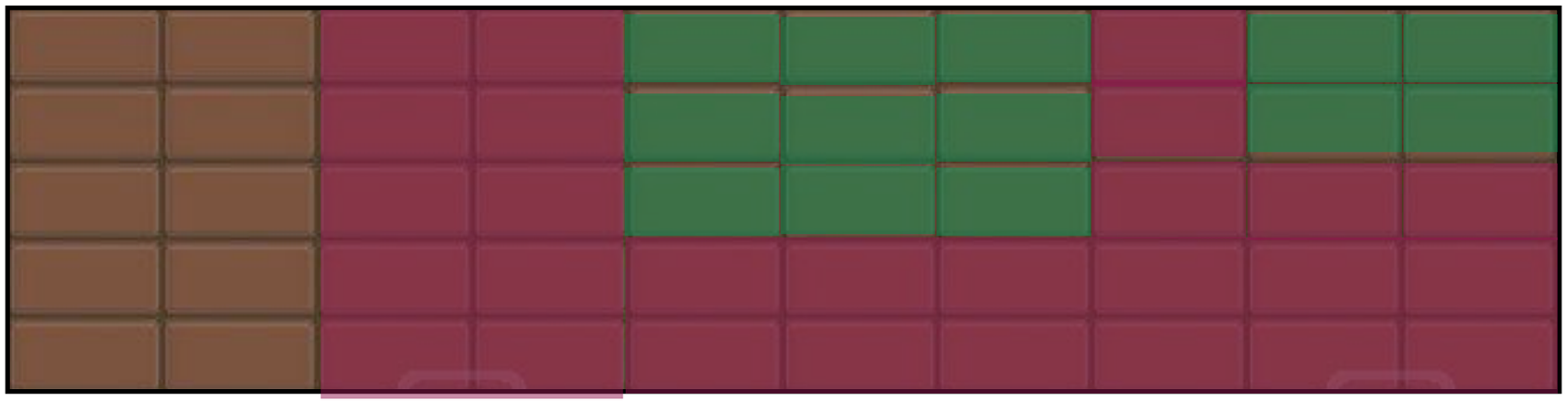
@ Alice plays against Bob



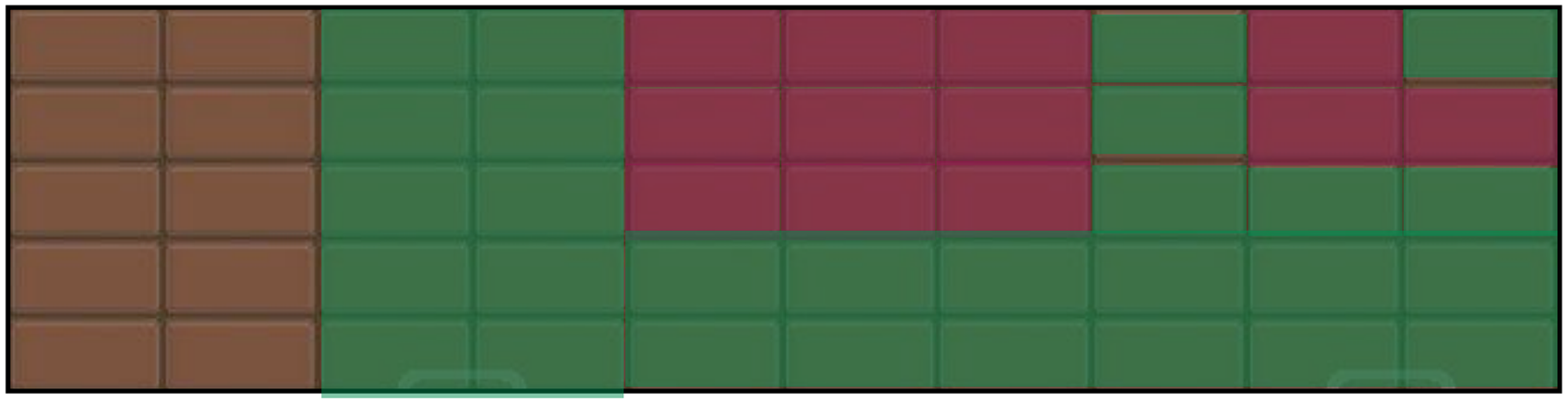
@ Bob applies his winning strategy



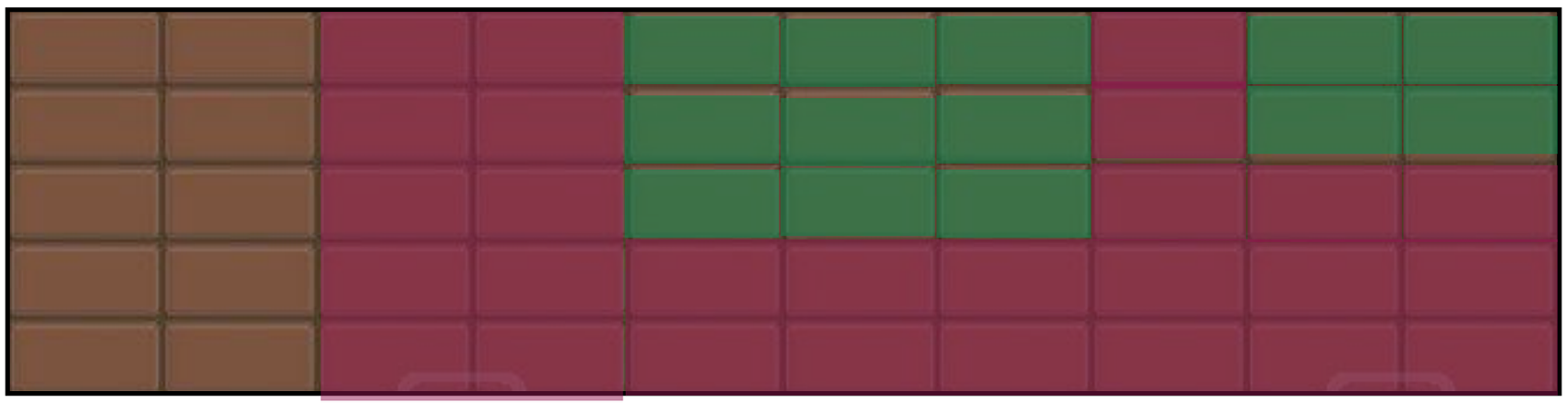
@ Alice plays against Bob



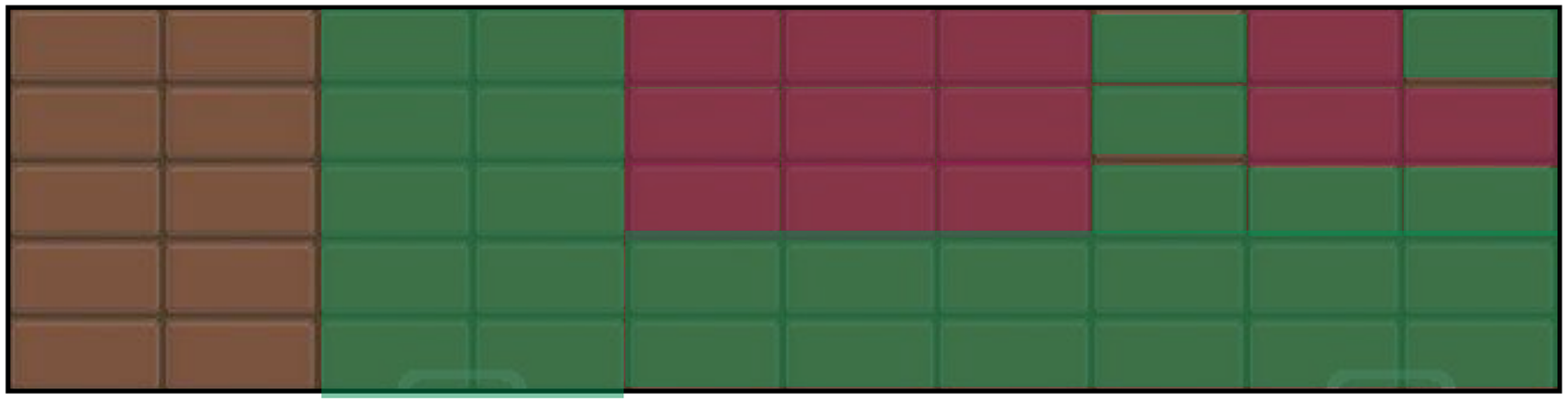
@ Bob applies his winning strategy



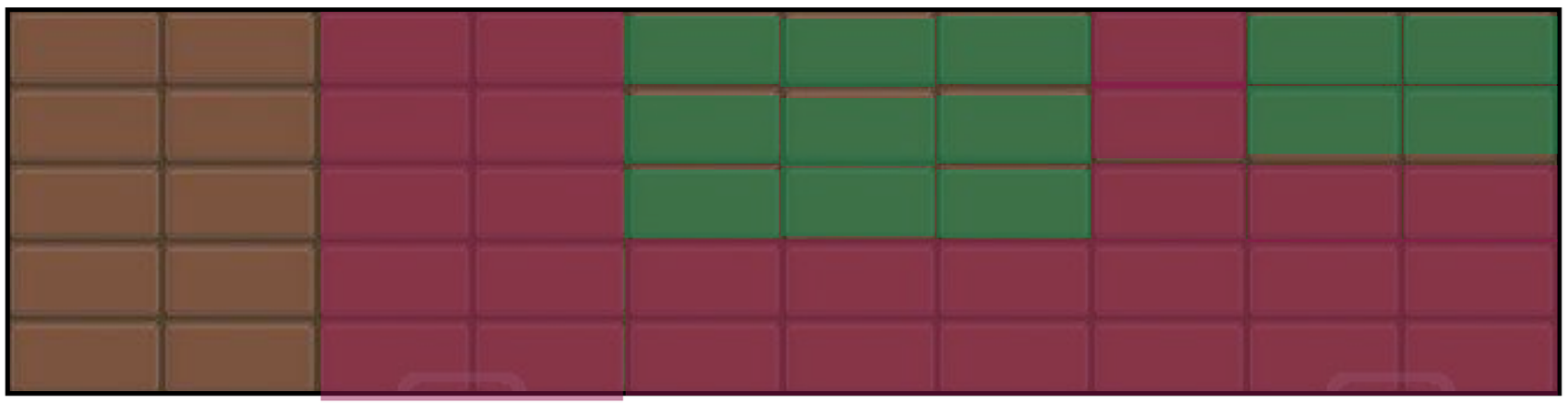
@ Alice plays against Bob



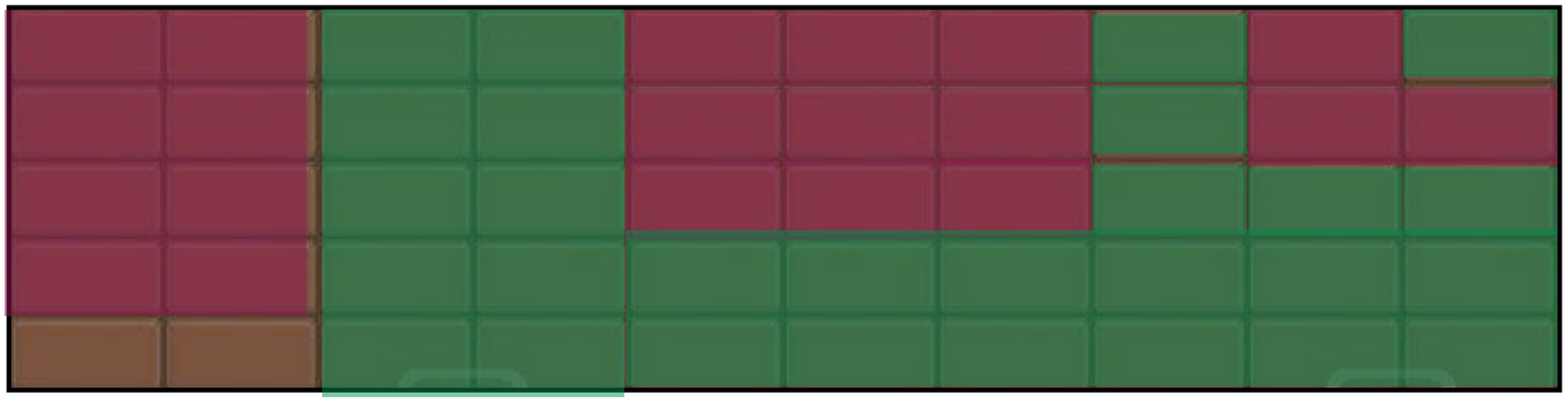
@ Bob applies his winning strategy



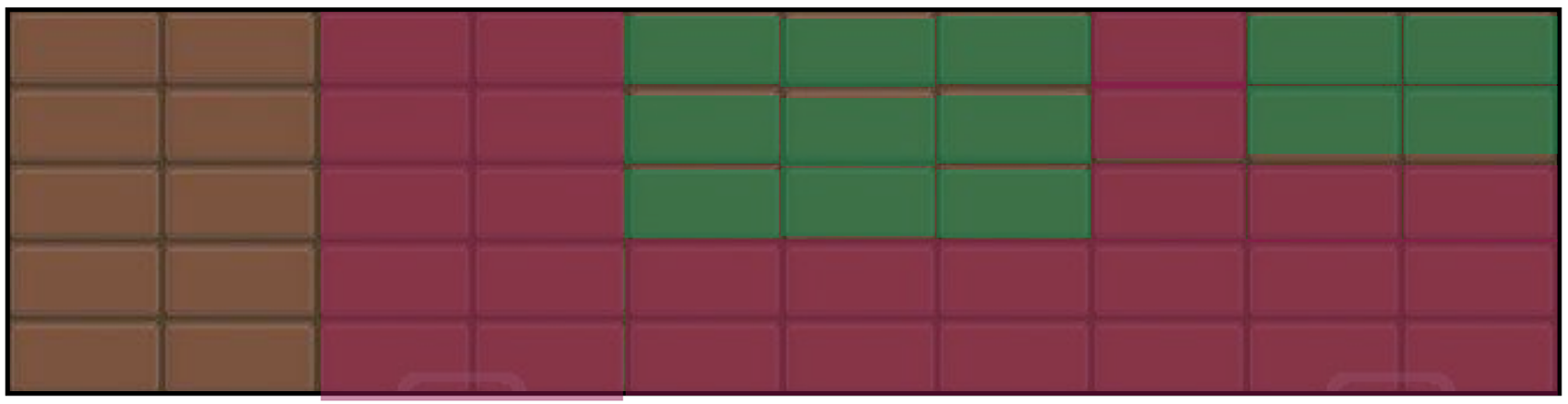
@ Alice plays against Bob



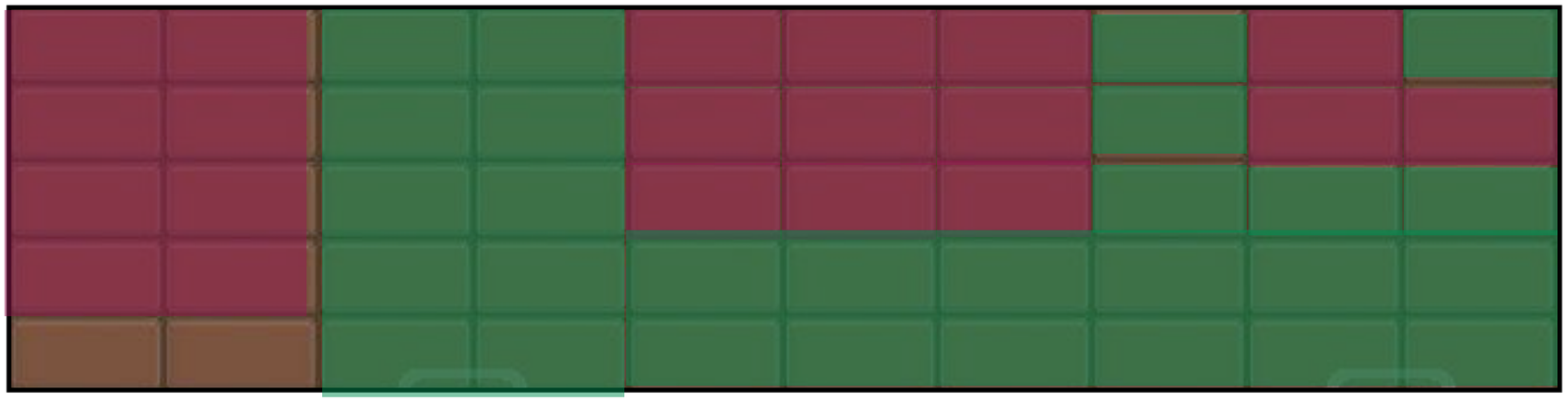
@ Bob applies his winning strategy



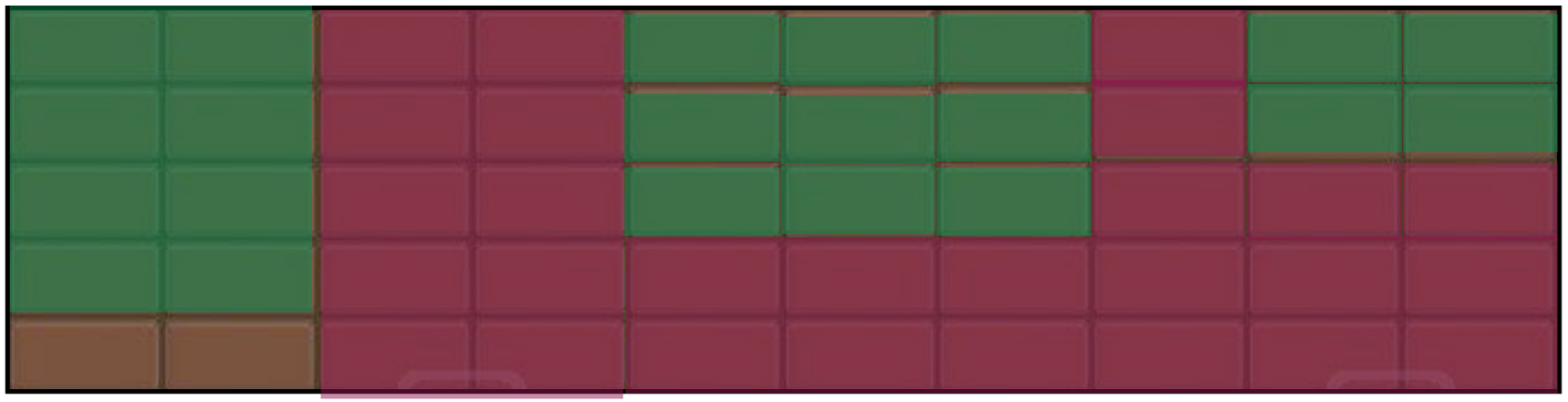
@ Alice plays against Bob



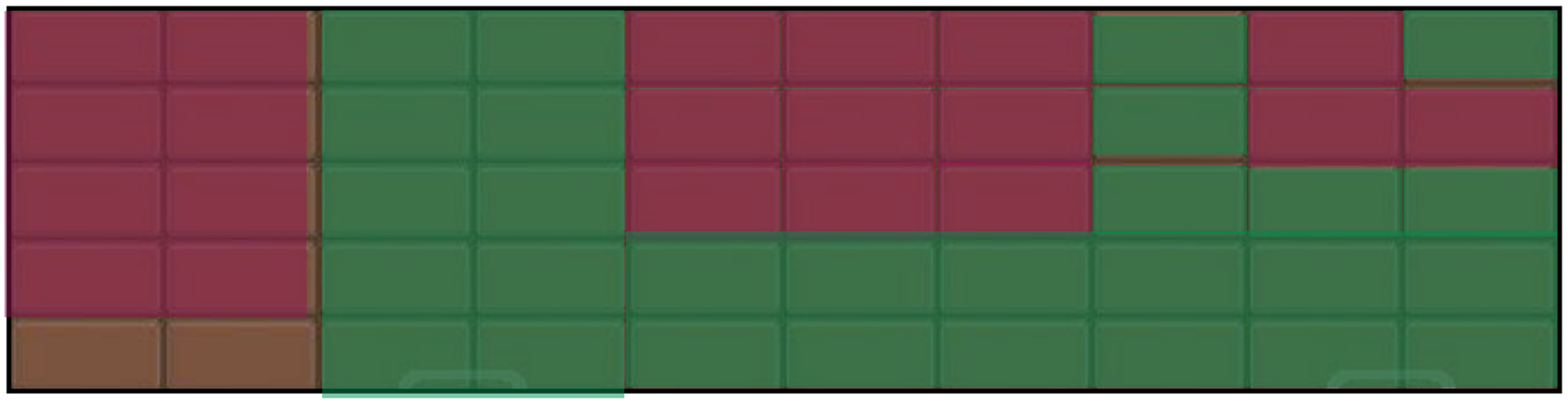
@ Bob applies his winning strategy



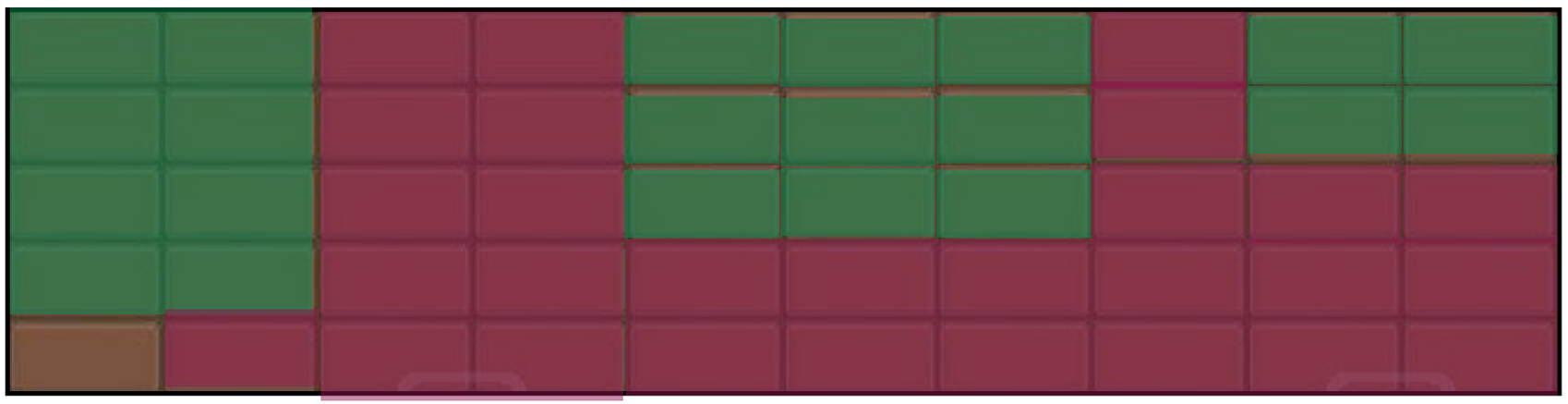
@ Alice plays against Bob



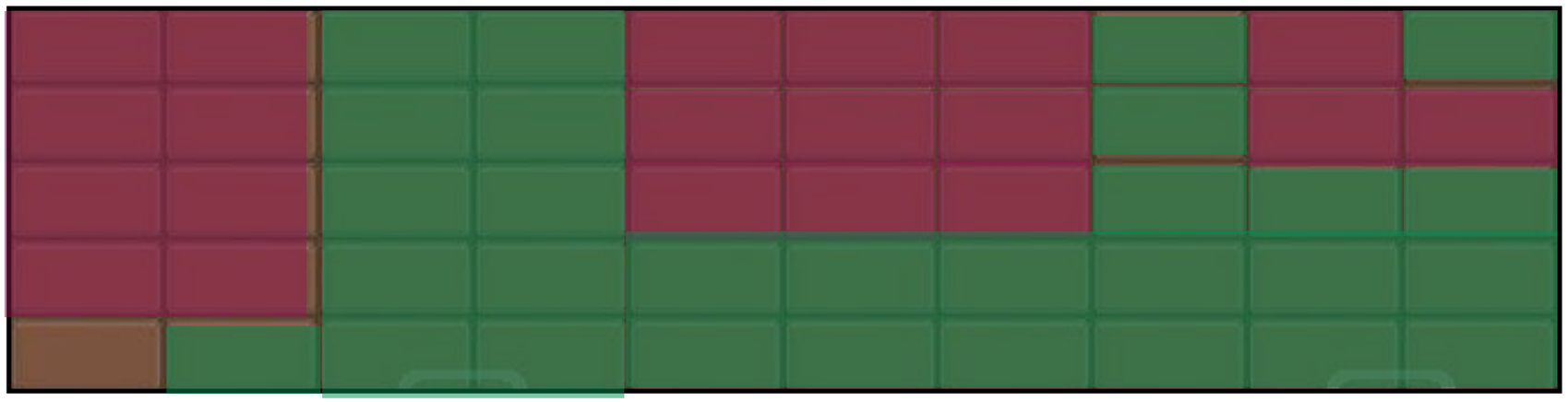
@ Bob applies his winning strategy



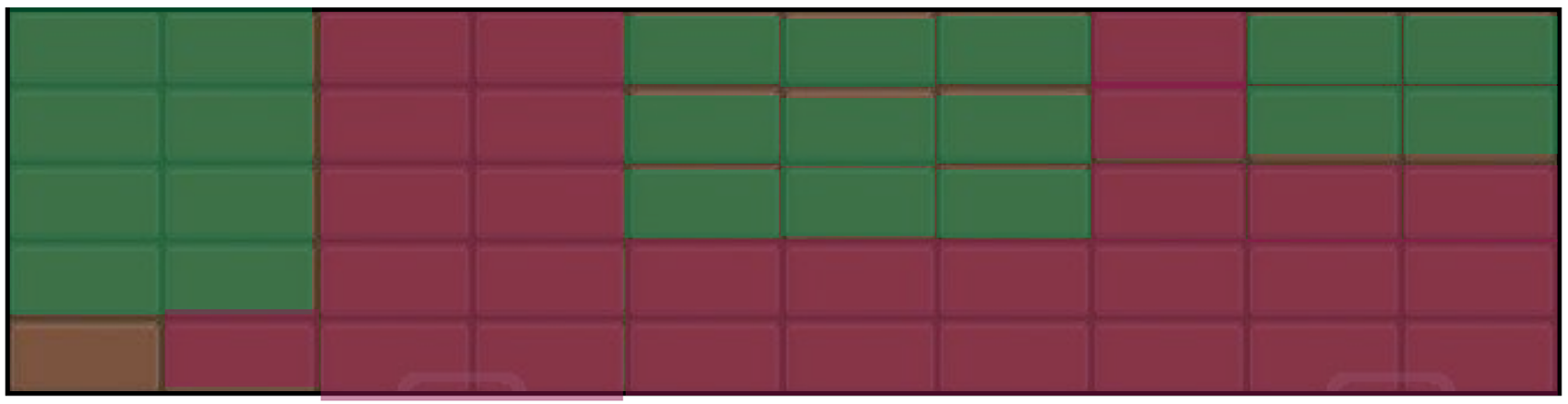
@ Alice plays against Bob



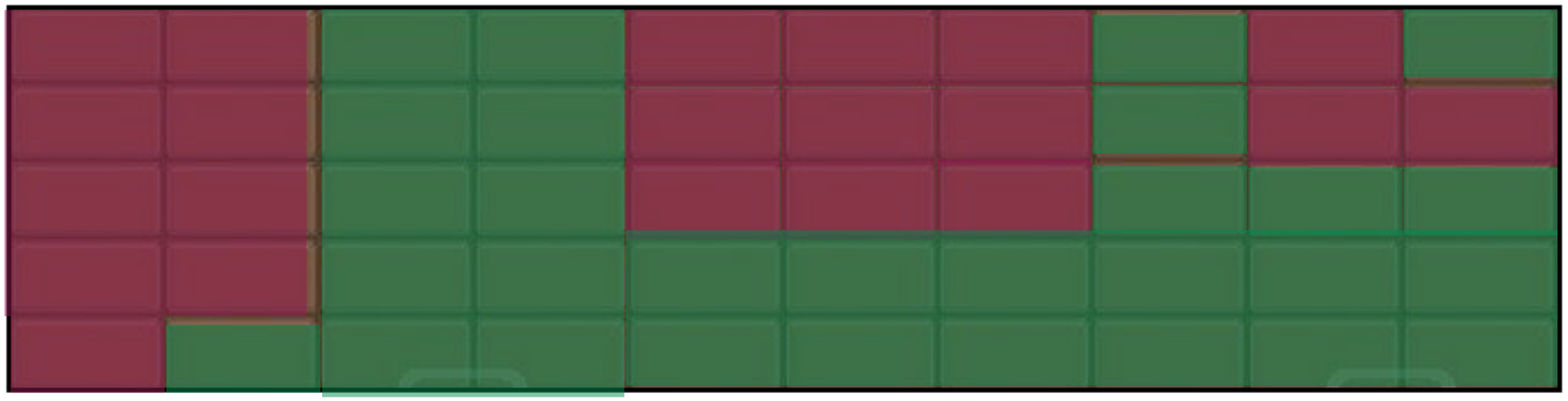
@ Bob applies his winning strategy



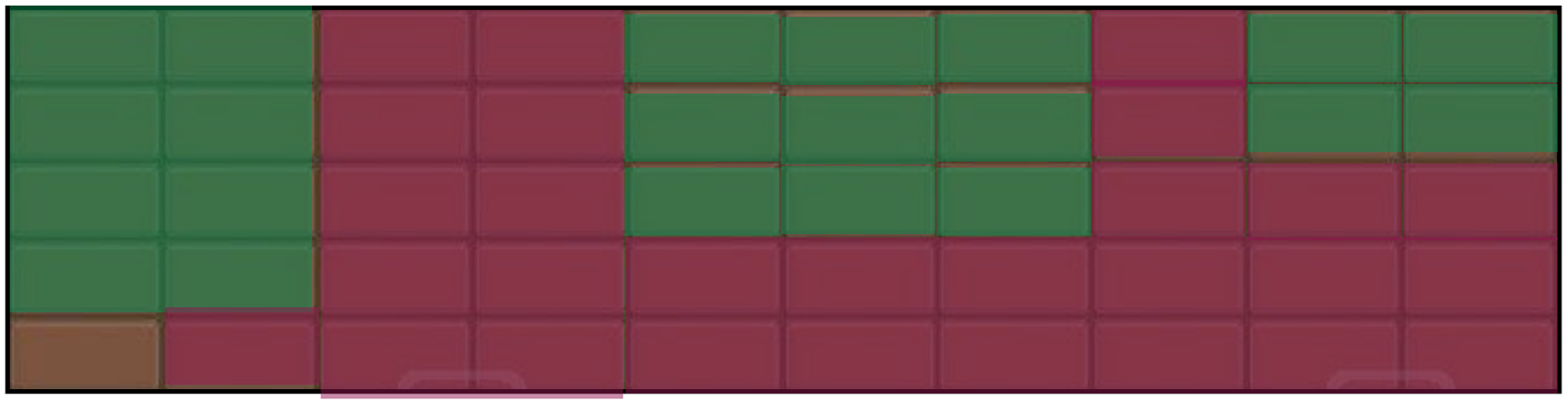
@ Alice plays against Bob



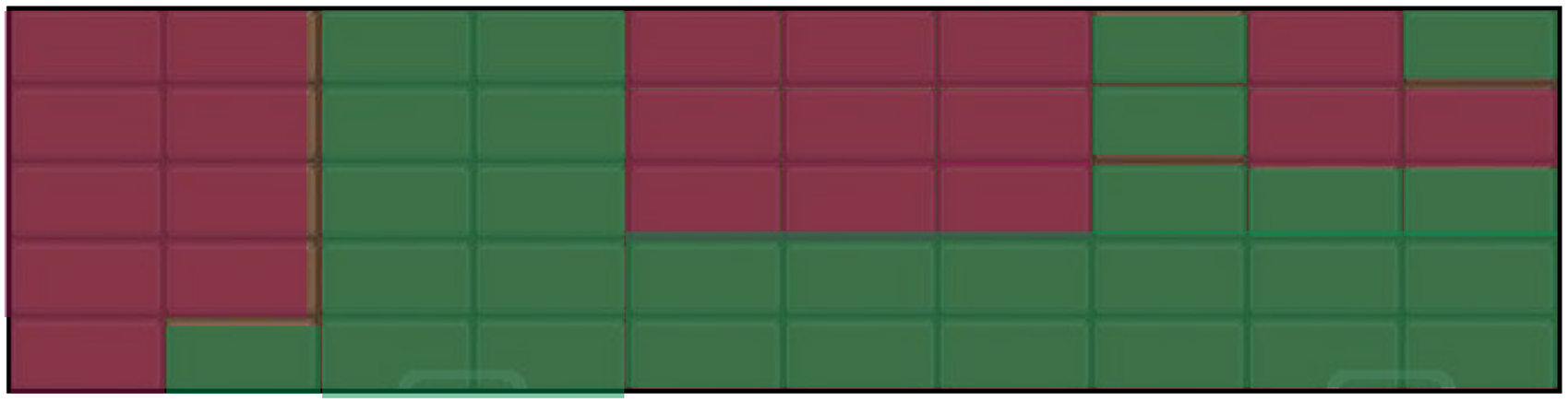
@ Bob applies his winning strategy



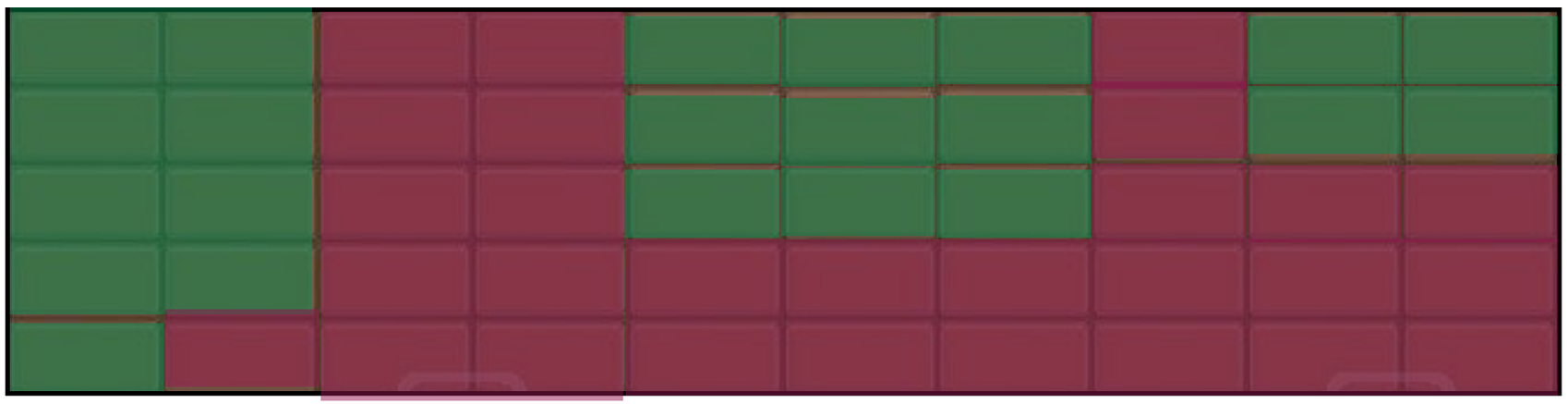
@ Alice plays against Bob



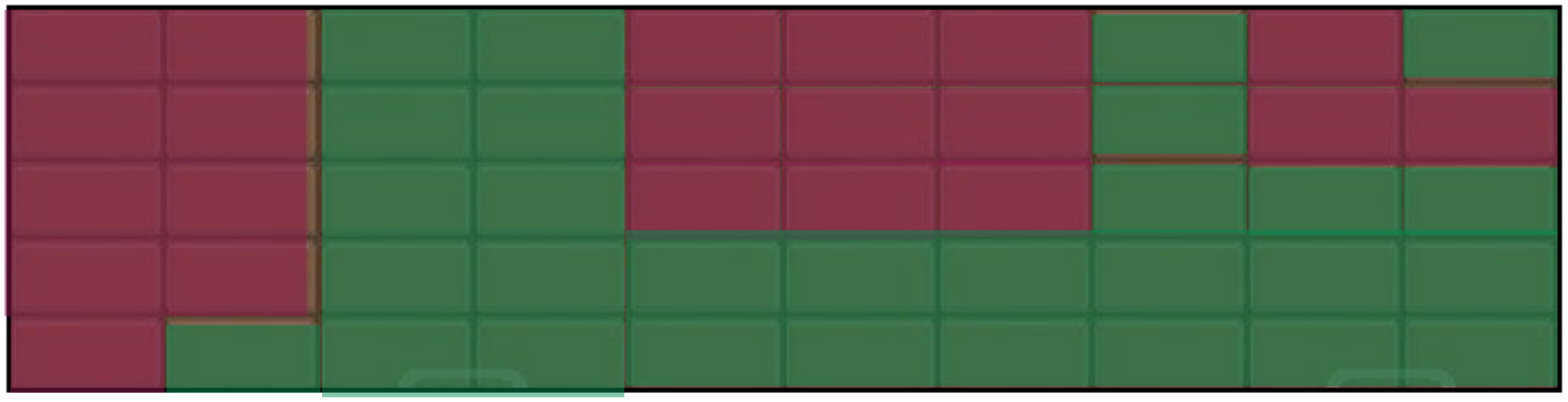
@ Bob applies his winning strategy



@ Alice plays against Bob



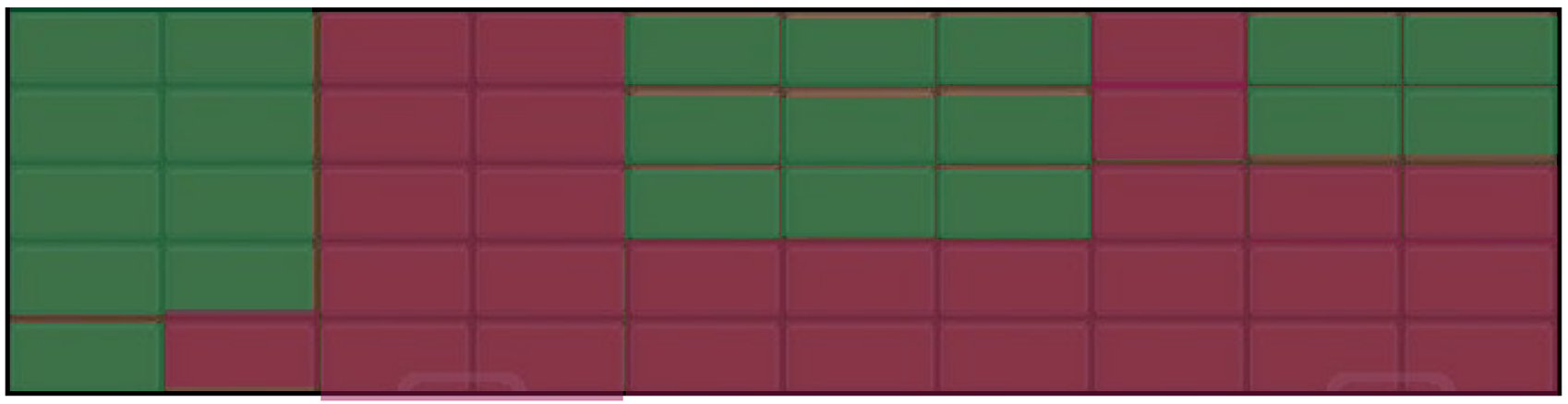
@ Bob applies his winning strategy



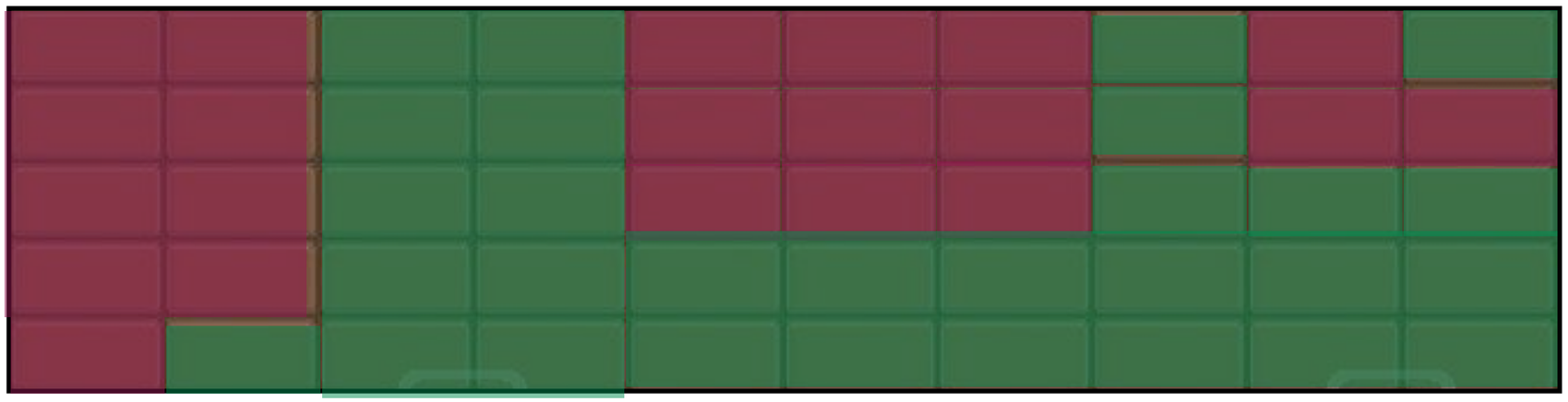
@ Except for the first piece of chocolate:

Green ↑ = Red ↓ and Green ↓ = Red ↑ therefore

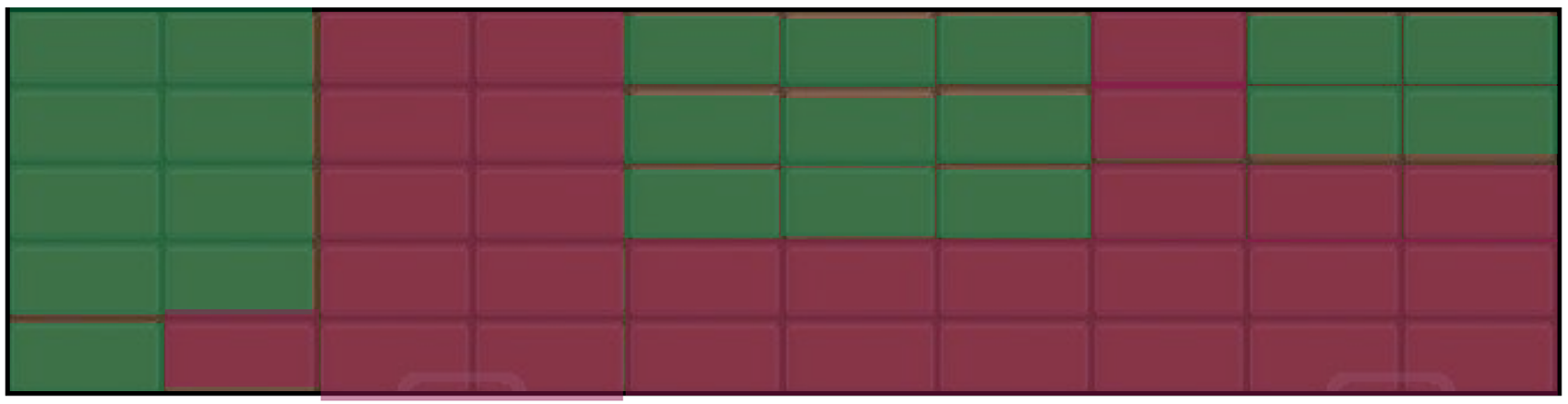
Alice munch the last one ↓ ⇒ Bob munch the last one ↓



UNIL | Université de Lausanne



@ Alice wins the game ↑. A contradiction



UNIL | Université de Lausanne



UNIL | Université de Lausanne

@ Determinacy for finite games yields "Alice has a winning strategy"

UNIL | Université de Lausanne

- Determinacy for finite games yields "Alice has a winning strategy"

- What about an infinite (∞,∞) chocolate bar ?

UNIL | Université de Lausanne

@ Determinacy for finite games yields "Alice has a winning strategy"

@ What about an infinite (∞,∞) chocolate bar ?

- This is still a finite 2-player game with perfect information, hence determined

@ Determinacy for finite games yields "Alice has a winning strategy"

@ What about an infinite (∞,∞) chocolate bar ?

- This is still a finite 2-player game with perfect information, hence determined

- Alice has a winning strategy

- same as with (n,n) -

Unil
UNIL | Université de Lausanne



UNIL | Université de Lausanne

@ What about a three dimension infinite (∞,∞,∞) chocolate bar ?



@ What about a three dimension infinite (∞,∞,∞) chocolate bar ?

- This is still a finite 2-player game with perfect information, hence determined



@ What about a three dimension infinite (∞,∞,∞) chocolate bar ?

- This is still a finite 2-player game with perfect information, hence determined

- *Unknown* who has a winning strategy

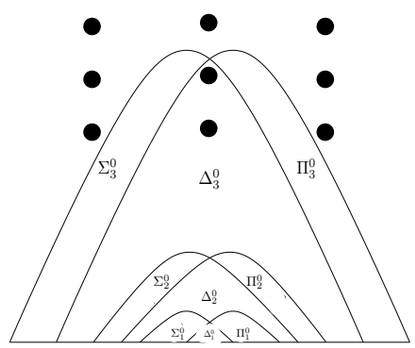


UNIL | Université de Lausanne

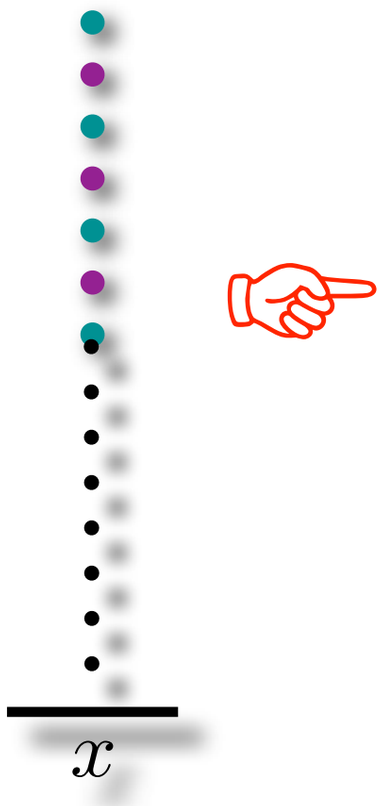


An example: evaluation game for 1st order logic

non determined game

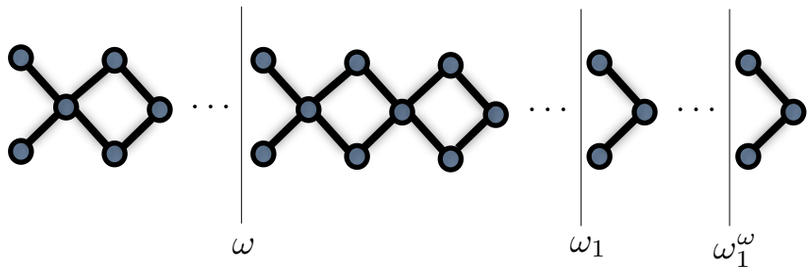


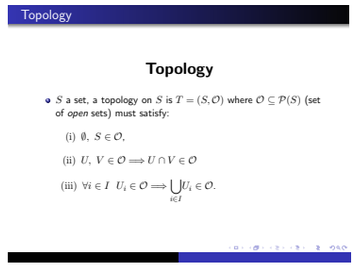
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

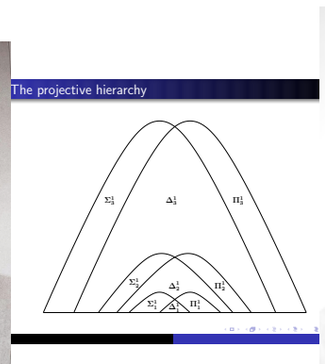
Games that construct mappings



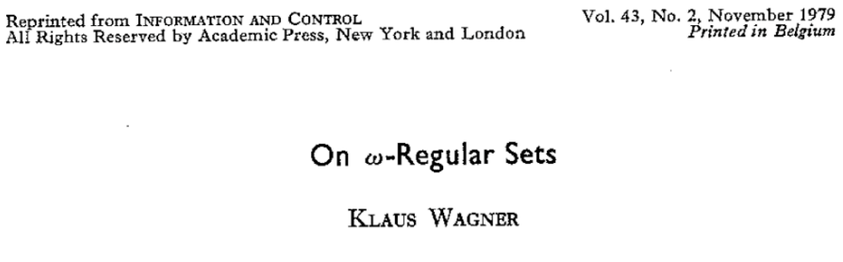










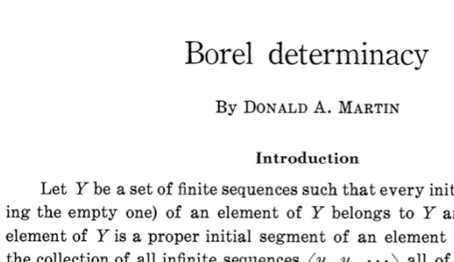


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

An example: evaluation game for 1st order logic

UNIL | Université de Lausanne

An example: evaluation game for 1st order logic

@ Given a 1st order language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$, and a formula ϕ:

An example: evaluation game for 1st order logic

@ Given a 1st order language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$, and a formula ϕ:

- Assume ϕ contains only connectors among $\{\wedge, \vee, \neg\}$ -

An example: evaluation game for 1st order logic

@ Given a 1st order language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$, and a formula ϕ:

- Assume ϕ contains only connectors among $\{\wedge, \vee, \neg\}$ -

- Verifier has a winning strategy in $EV(\phi, \mathcal{M})$

$\Leftrightarrow$

ϕ holds true in $\mathcal{M}$

UNIL | Université de Lausanne



UNIL | Université de Lausanne

$$\mathcal{EV}(\phi, m)$$

UNIL | Université de Lausanne

$EV(\phi, m)$

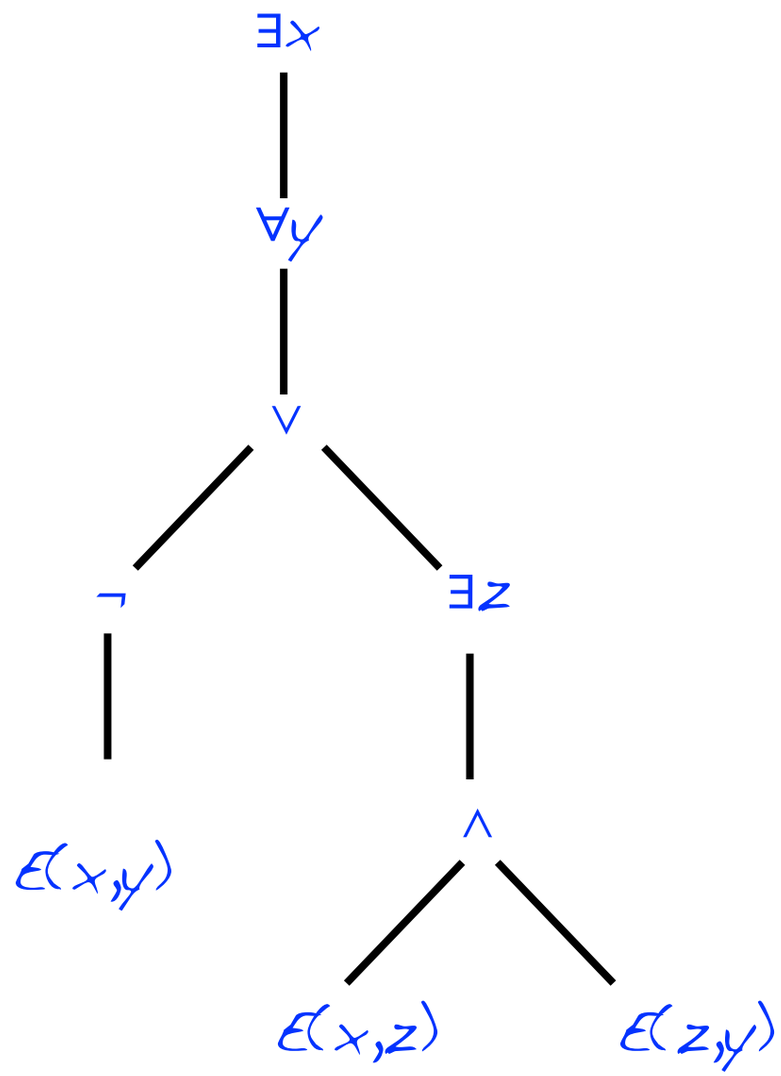
- Play goes down the tree decomposition of ϕ

$EV(\phi, \mathcal{M})$

- Play goes down the tree decomposition of ϕ
- $\phi = \exists x \forall y \, (\neg E(x,y) \vee \exists z \, (E(x,z) \wedge E(z,y)))$

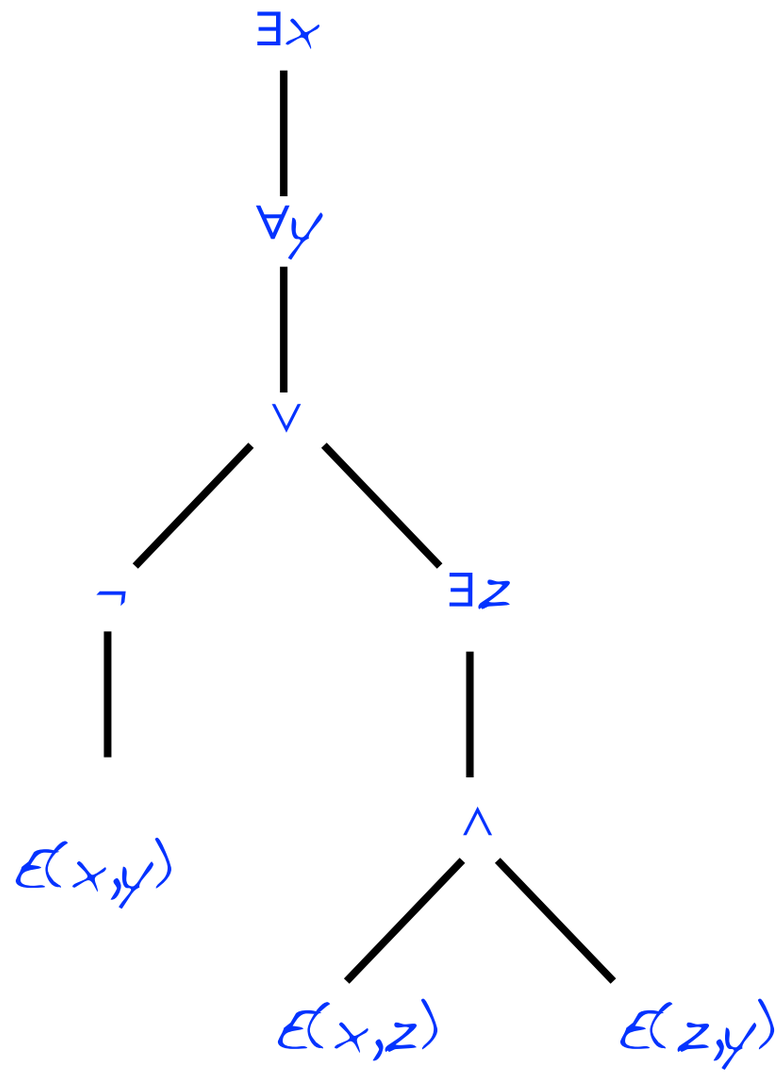
$EV(\phi, \mathcal{M})$

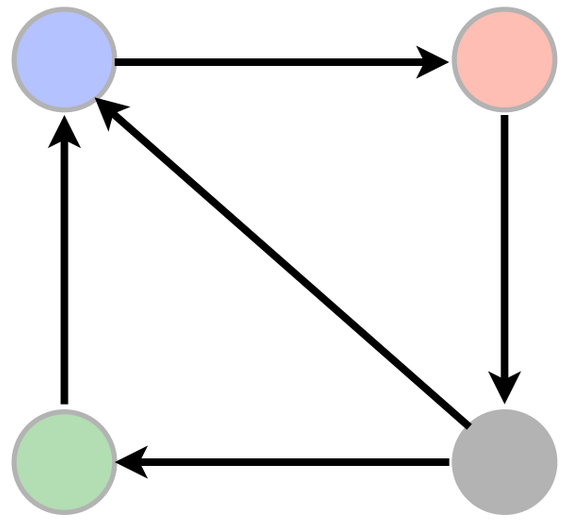
- Play goes down the tree decomposition of ϕ
- $\phi = \exists x \forall y \, (\neg E(x,y) \vee \exists z \, (E(x,z) \wedge E(z,y)))$



$EV(\phi, \mathcal{M})$

- Play goes down the tree decomposition of ϕ
- $\phi = \exists x \forall y \, (\neg E(x,y) \vee \exists z \, (E(x,z) \wedge E(z,y)))$

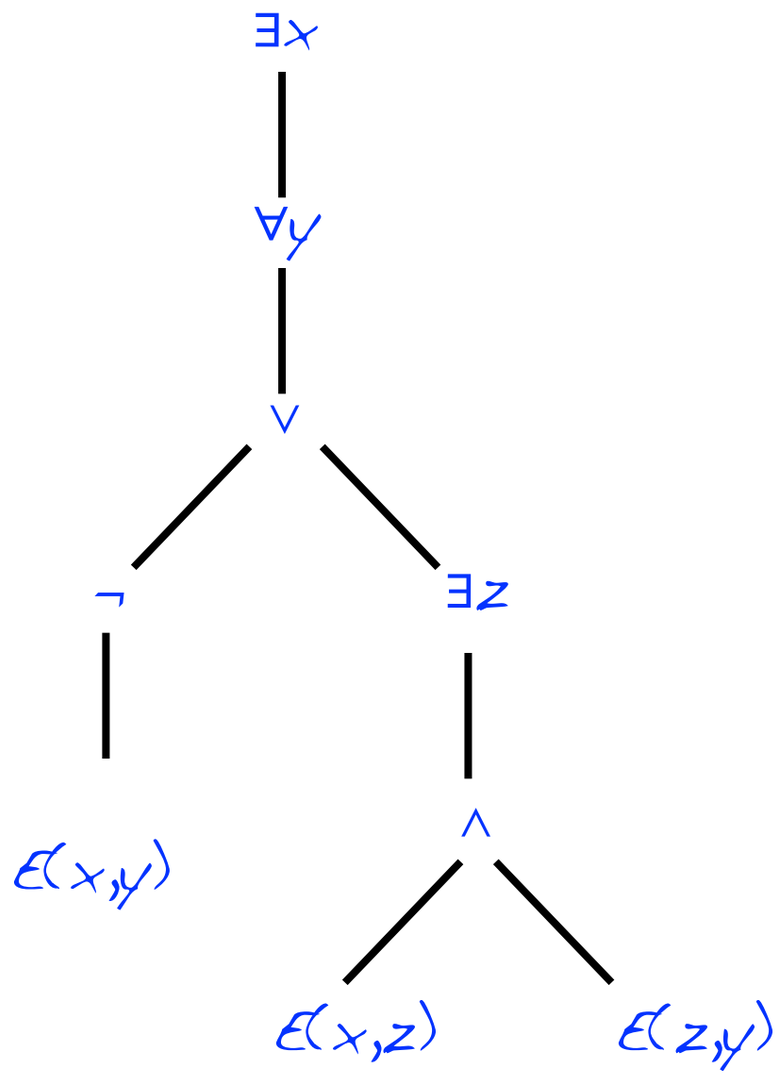


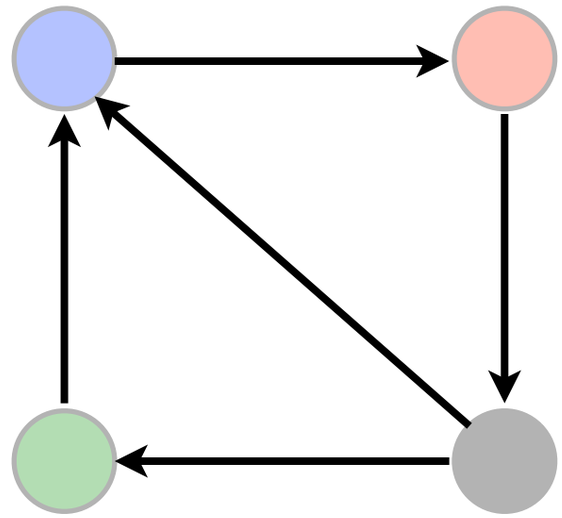


An example: evaluation game for 1st order logic

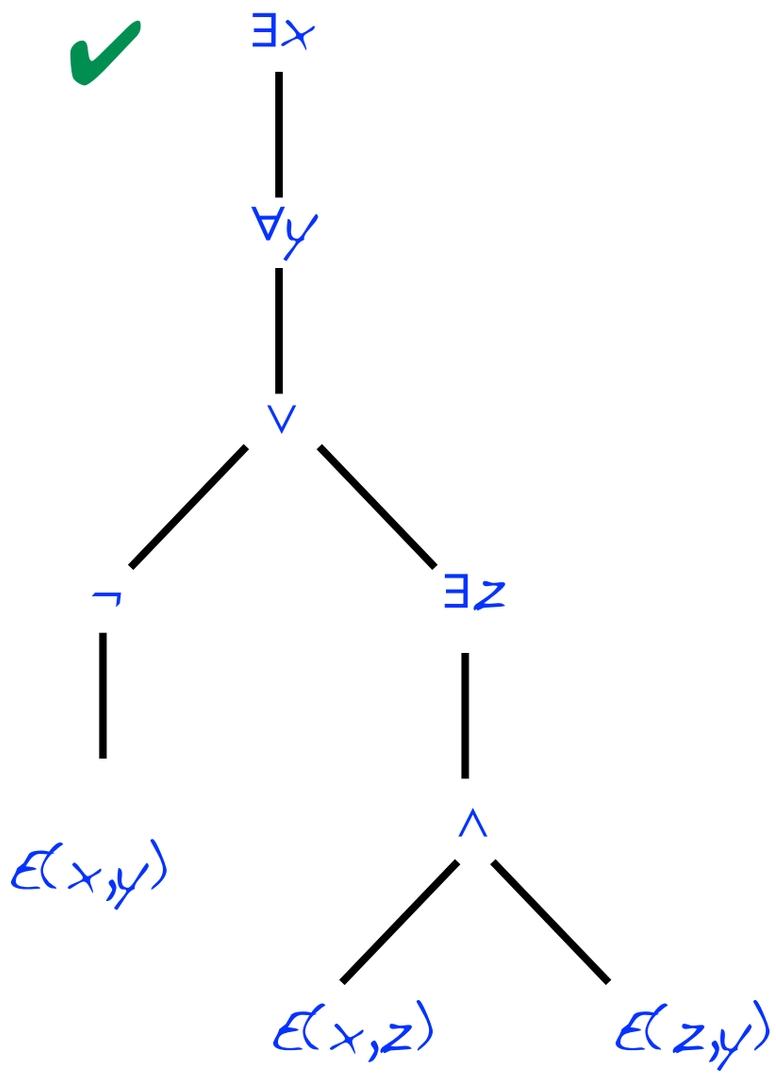
UNIL | Université de Lausanne

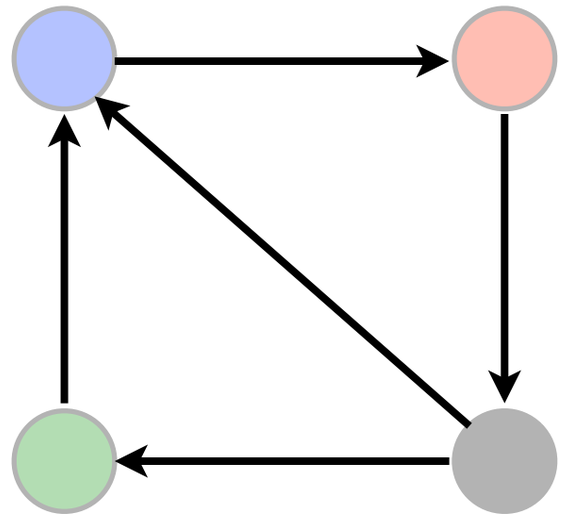
An example: evaluation game for 1st order logic



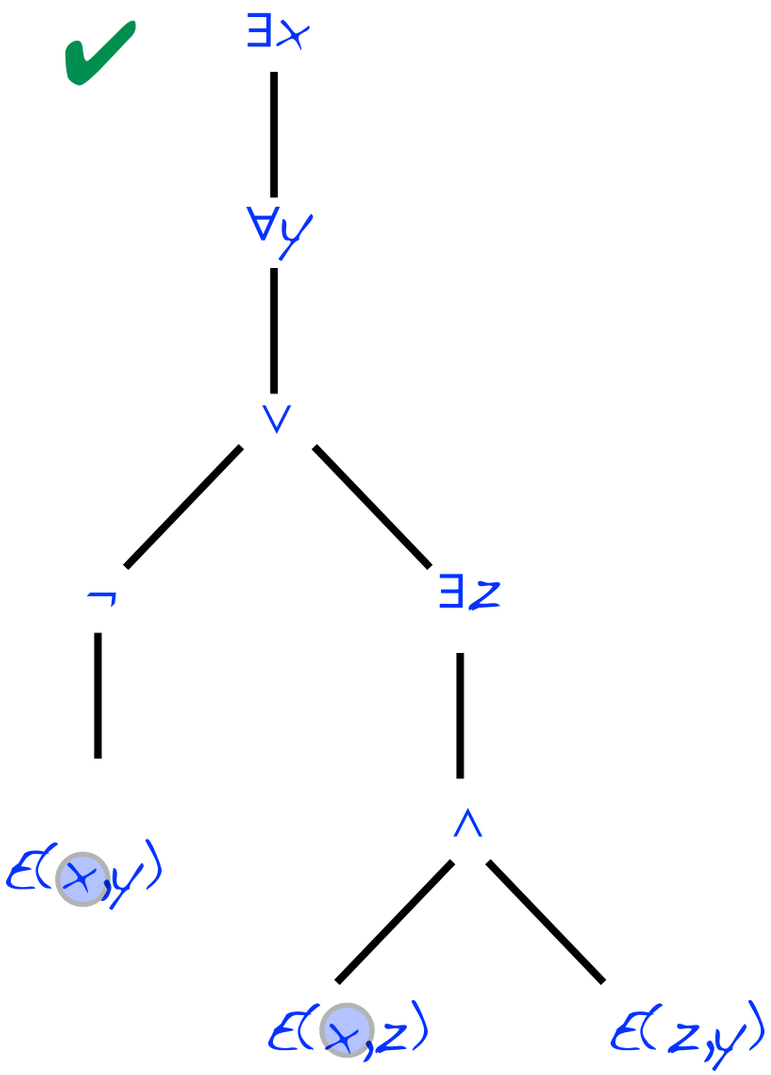


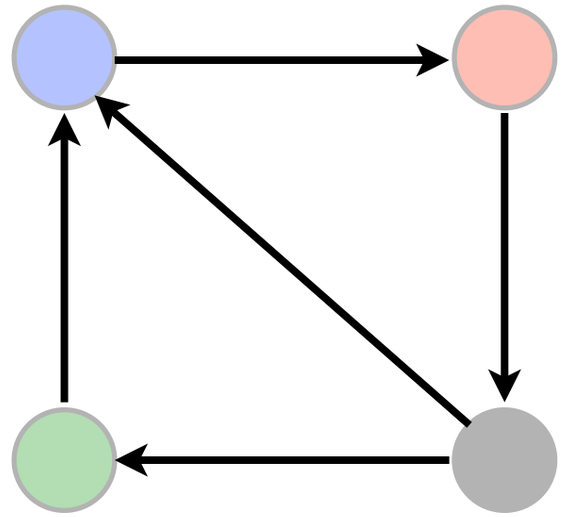
An example: evaluation game for 1st order logic



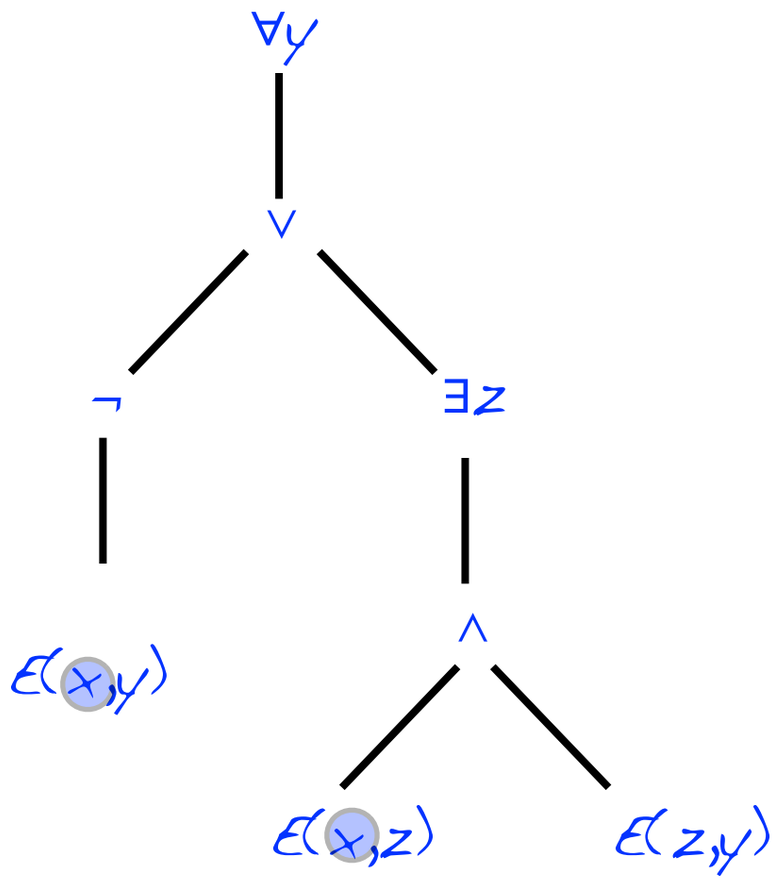


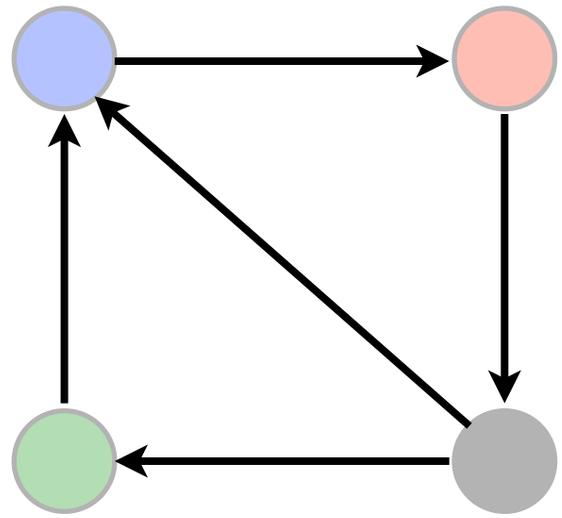
An example: evaluation game for 1st order logic



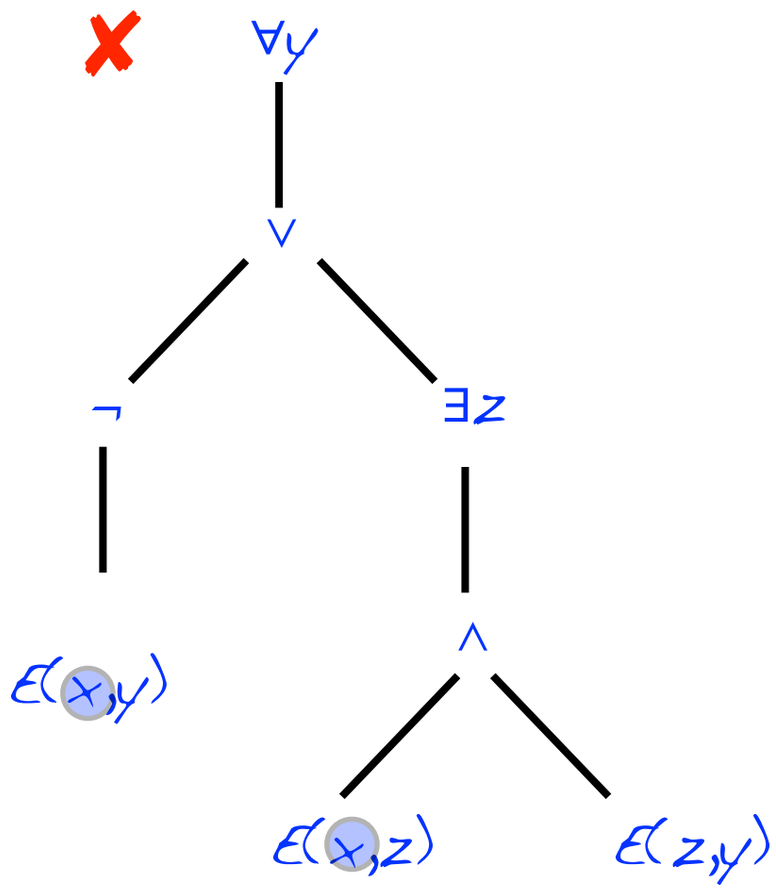


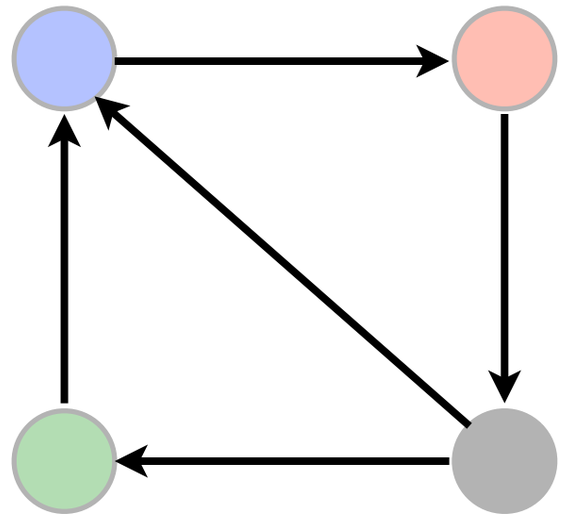
An example: evaluation game for 1st order logic



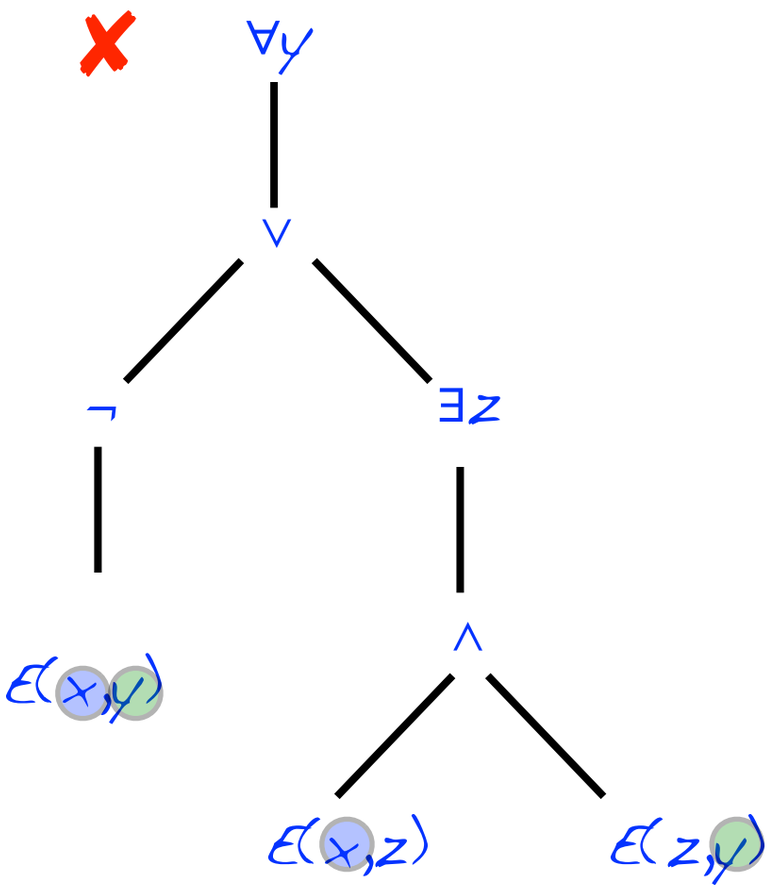


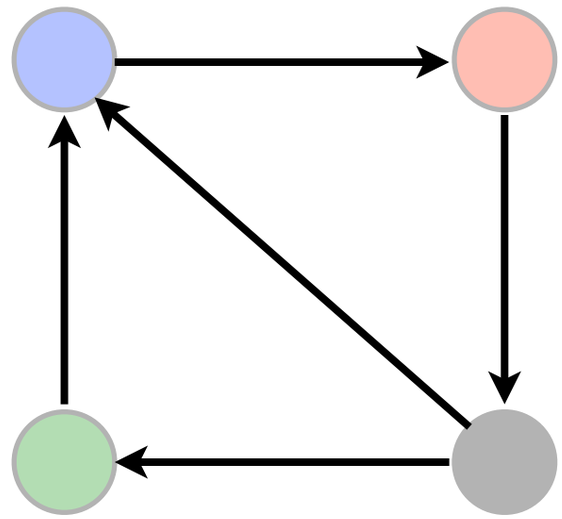
An example: evaluation game for 1st order logic



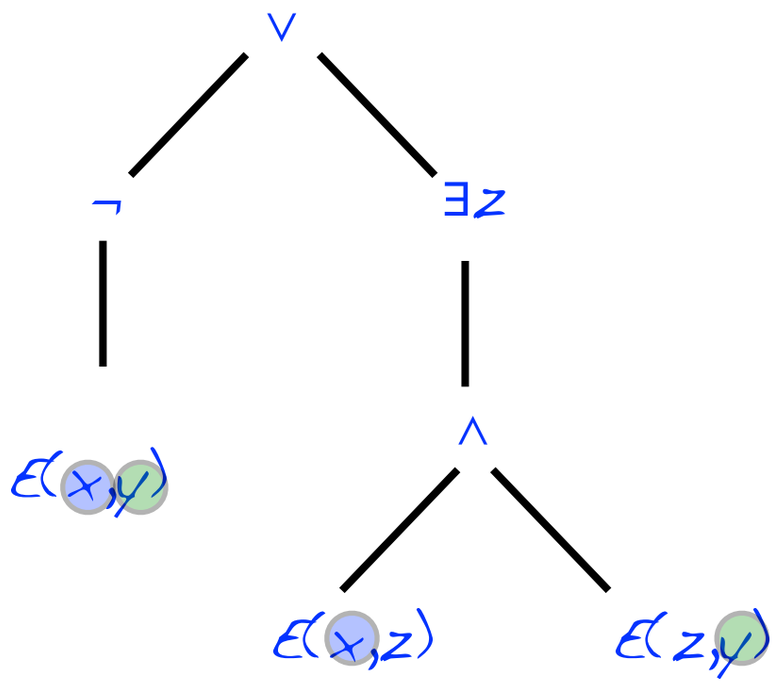


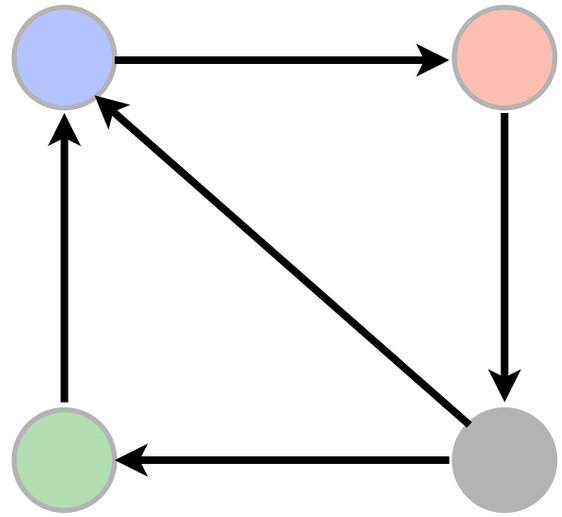
An example: evaluation game for 1st order logic



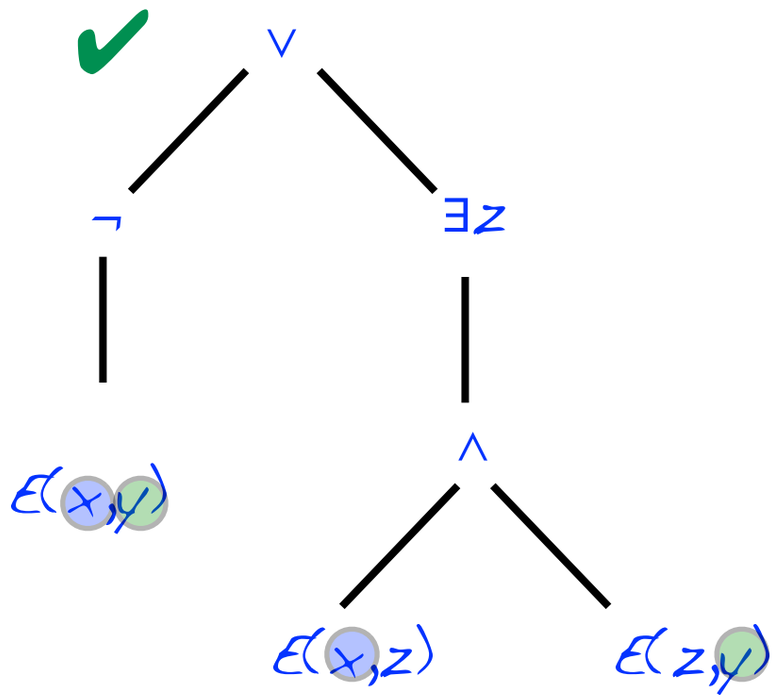


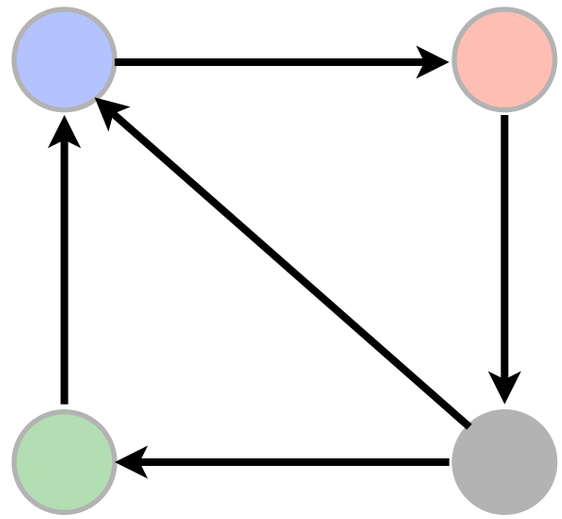
An example: evaluation game for 1st order logic



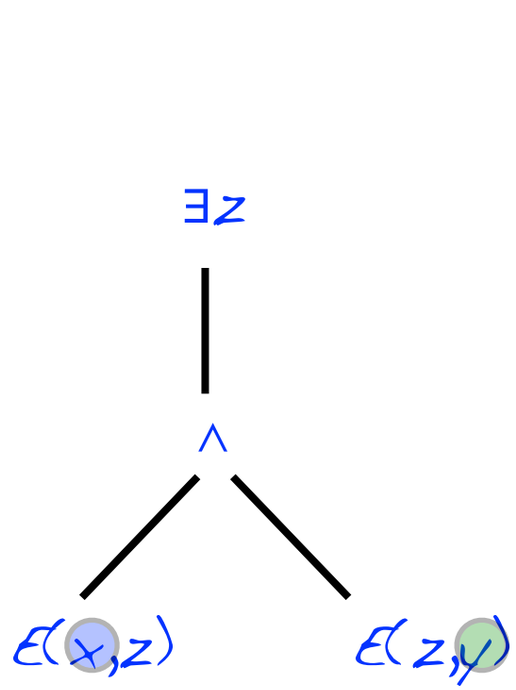


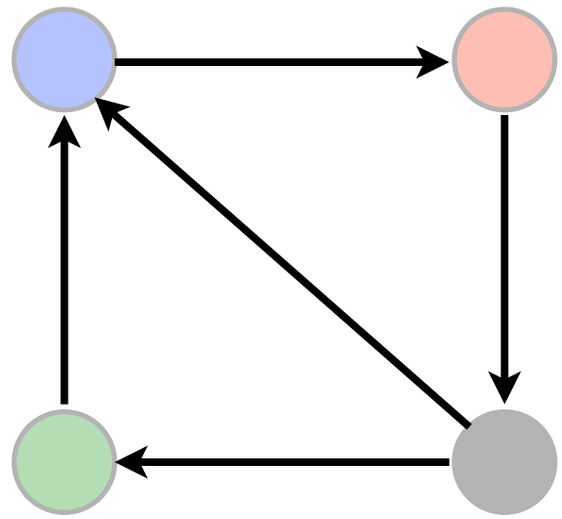
An example: evaluation game for 1st order logic





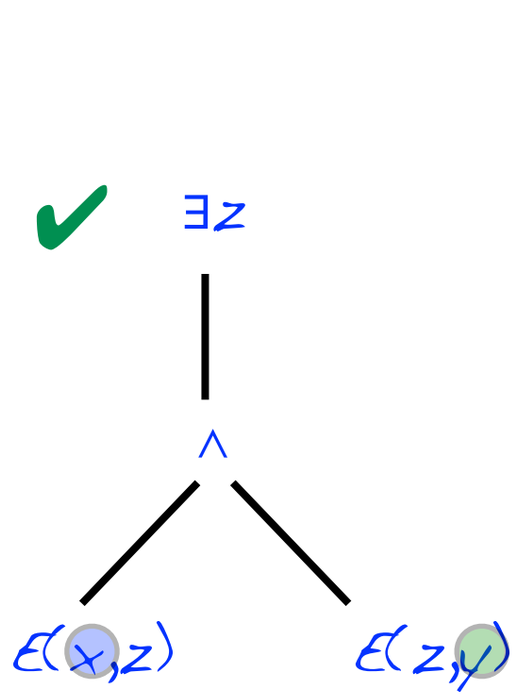
An example: evaluation game for 1st order logic

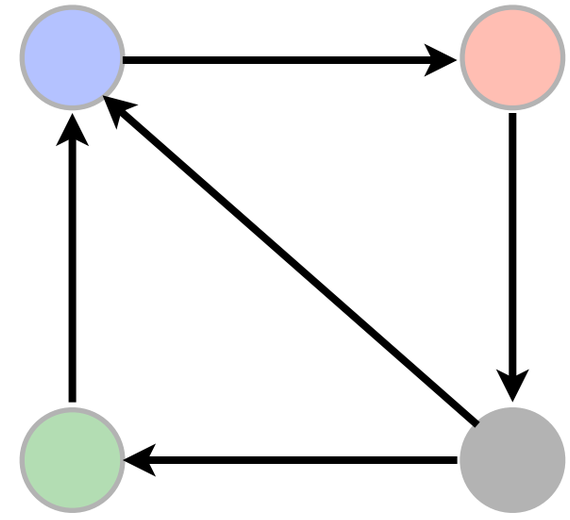




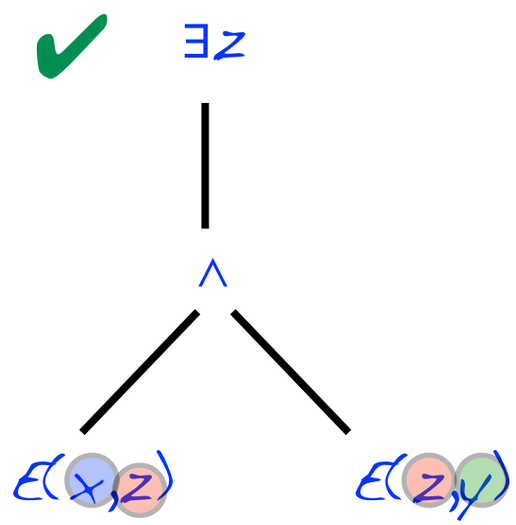
UNIL | Université de Lausanne

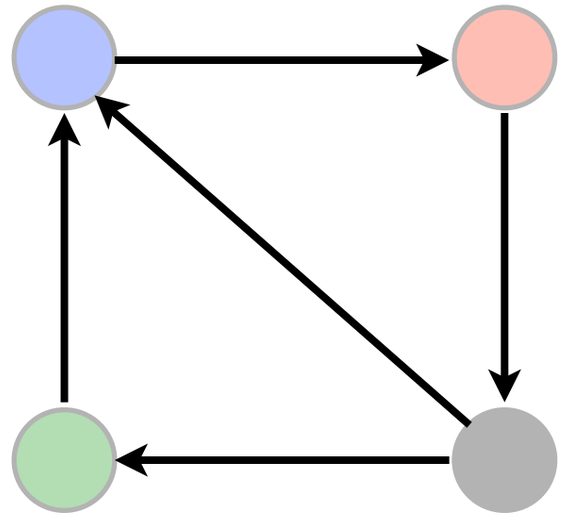
An example: evaluation game for 1st order logic



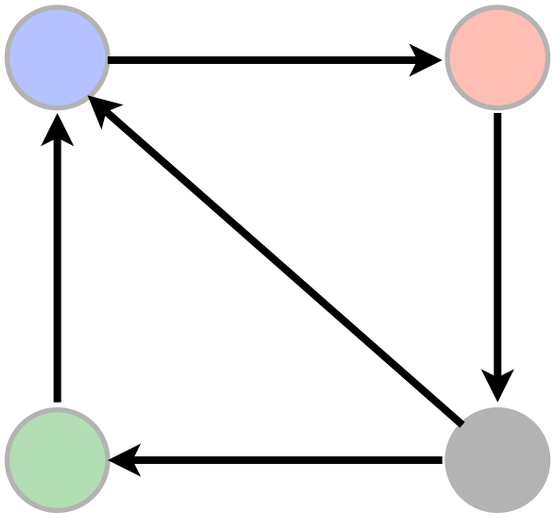


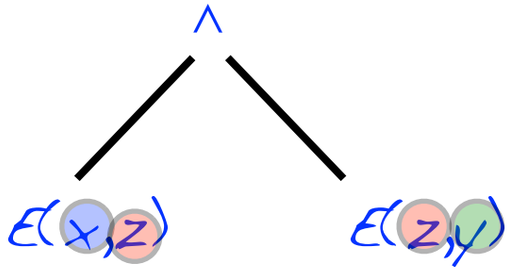
An example: evaluation game for 1st order logic





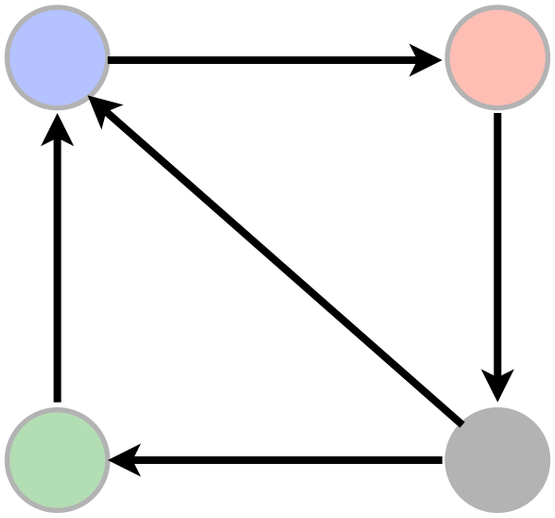
An example: evaluation game for 1st order logic

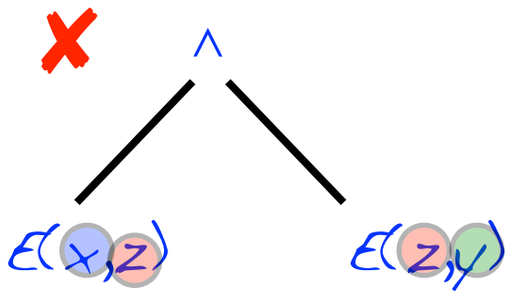




UNIL | Université de Lausanne

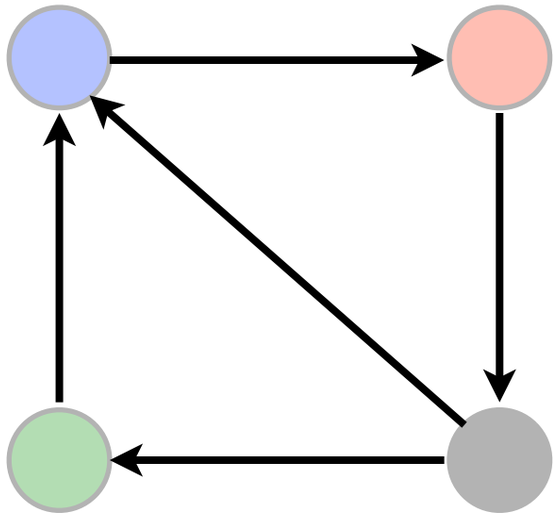
An example: evaluation game for 1st order logic





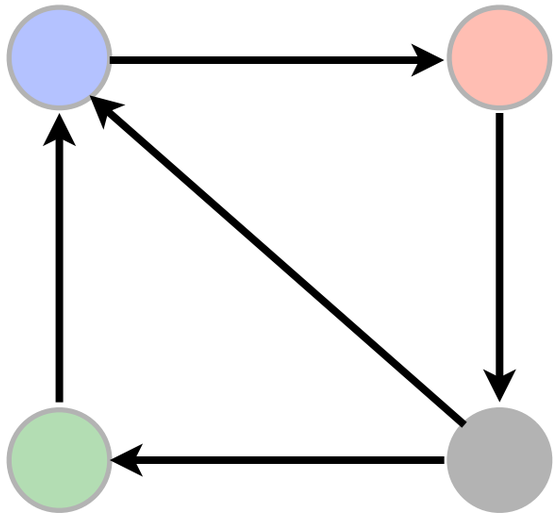
UNIL | Université de Lausanne

An example: evaluation game for 1st order logic



$E(z,y)$

An example: evaluation game for 1st order logic



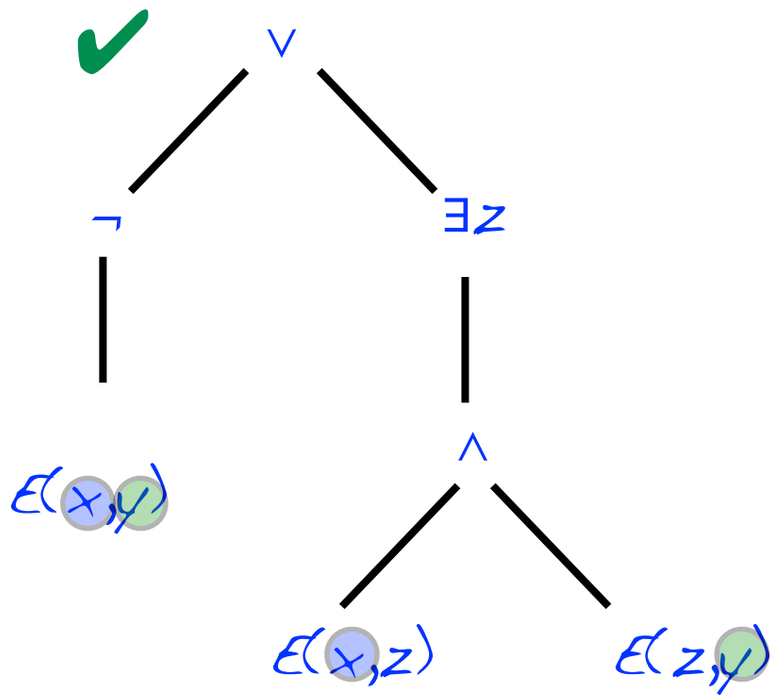
$E(z,y)$

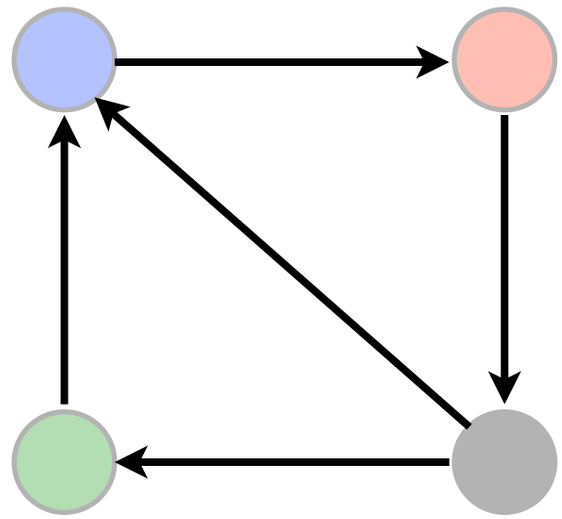
✘ wins !

An example: evaluation game for 1st order logic

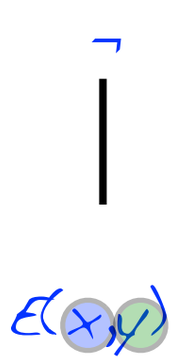
UNIL | Université de Lausanne

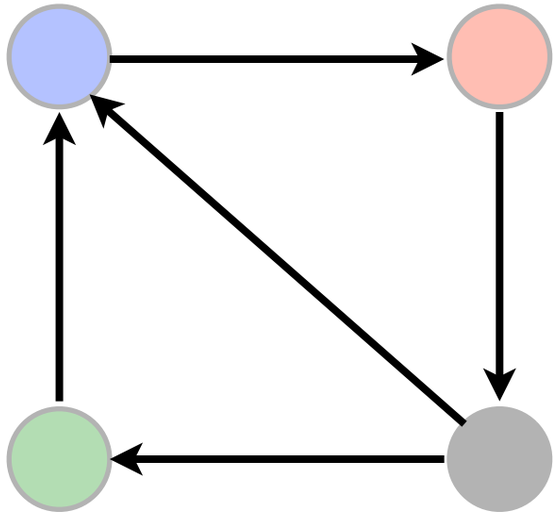
An example: evaluation game for 1st order logic





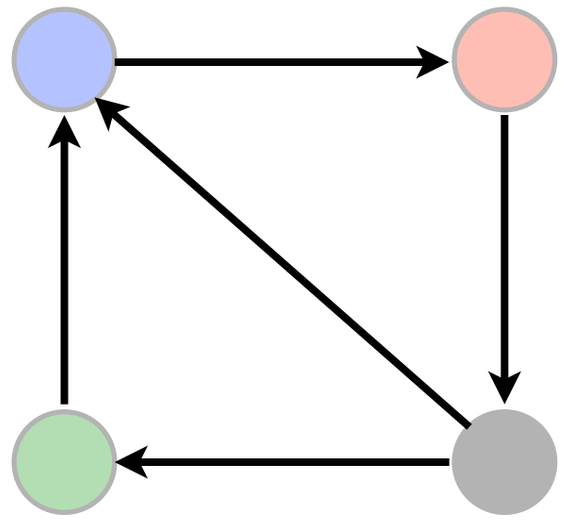
An example: evaluation game for 1st order logic



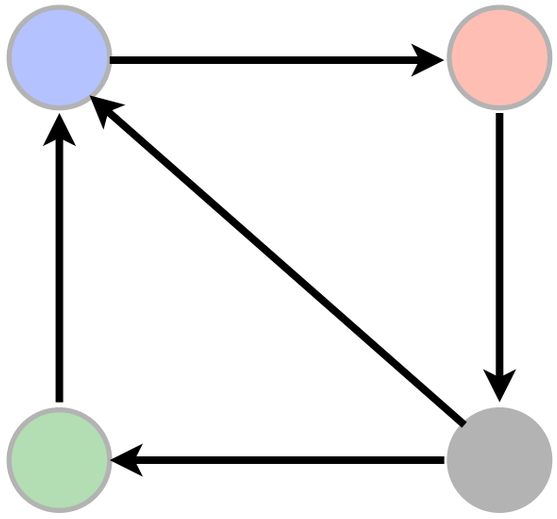


An example: evaluation game for 1st order logic





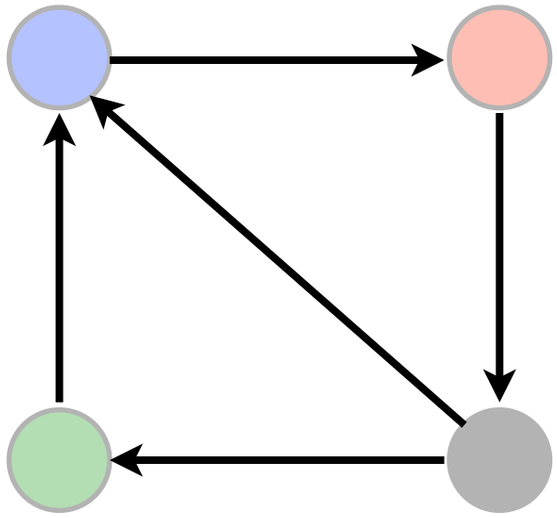
An example: evaluation game for 1st order logic



$E(x,y)$

UNIL | Université de Lausanne

An example: evaluation game for 1st order logic



$E(x,y)$

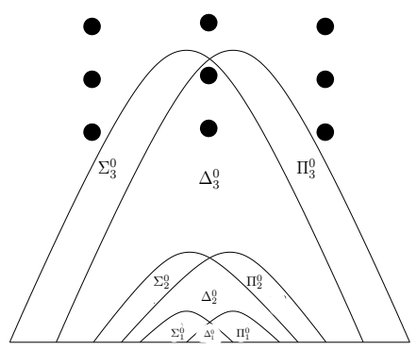
✔ wins !

UNIL | Université de Lausanne

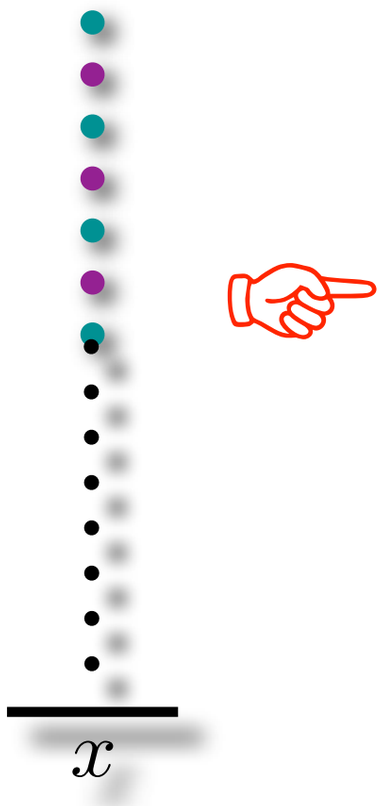


An example: evaluation game for 1st order logic

non determined game

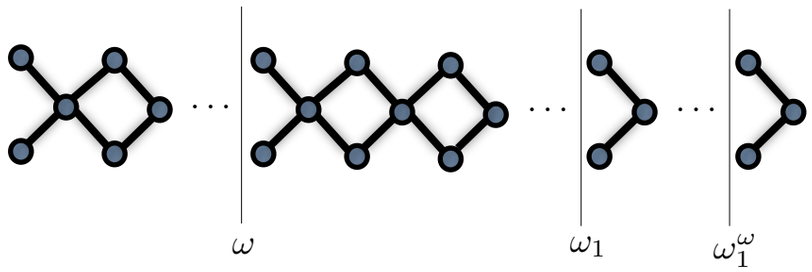


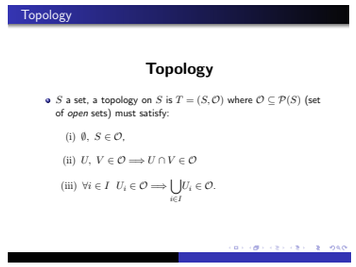
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

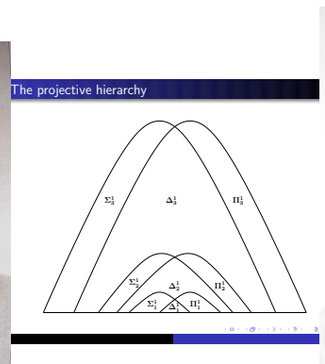
Games that construct mappings



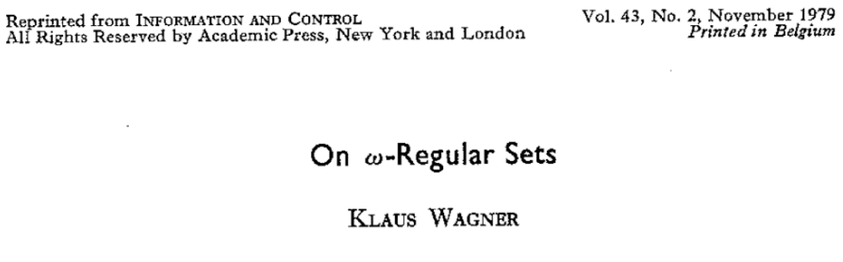










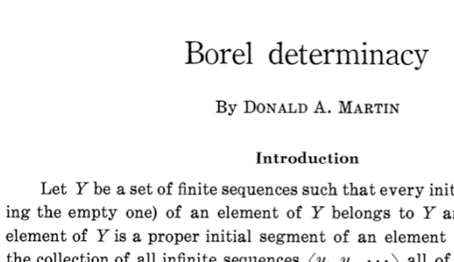


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

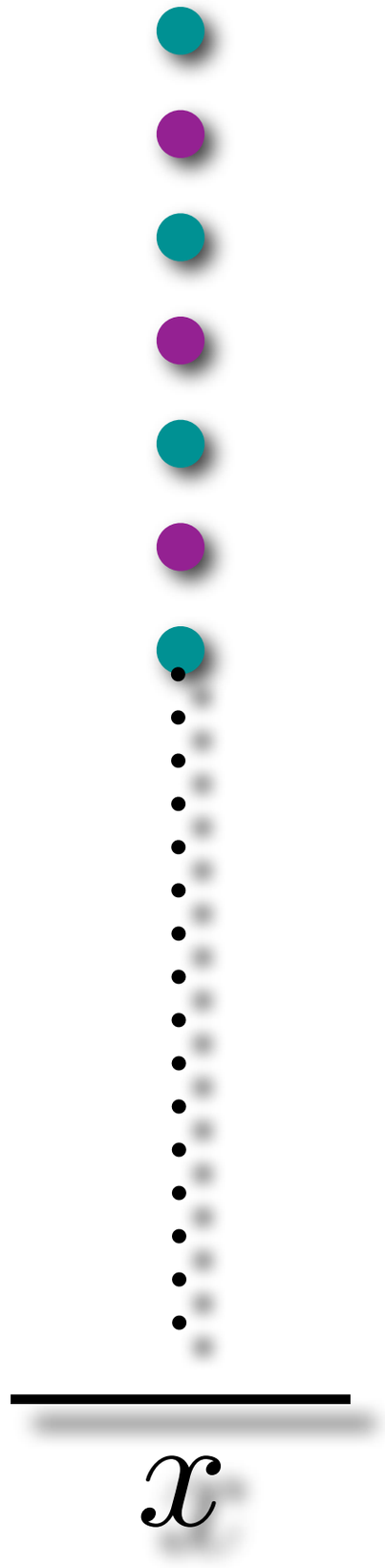
Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

$\mathcal{G}(A)$



x



UNIL | Université de Lausanne

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially ^{1} = we know the winner after finitely many moves (even if the game has infinite duration)

  - essentially ^{2} = we know the winner after finitely many moves, otherwise Bob wins

UNIL | Université de Lausanne



UNIL | Université de Lausanne

@ For simplicity let us consider games where players choose integers

(the same would hole with 0,1 or any finite or countable alphabet)

@ An integer may "code" any kind of finite object such as
a move on a graph, a path in a graph,...

UNIL | Université de Lausanne

UNIL | Université de Lausanne

Alice

Bob

Alice



Bob

Alice



Bob

Alice



Bob

Alice



Bob

Alice



Bob

Alice



Bob

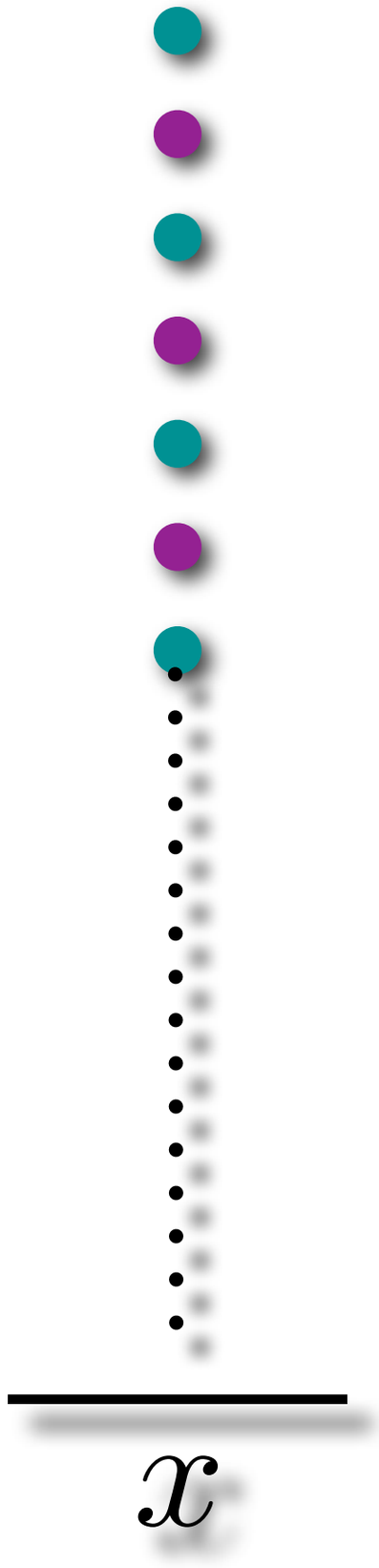
Alice

Bob



Alice

Bob

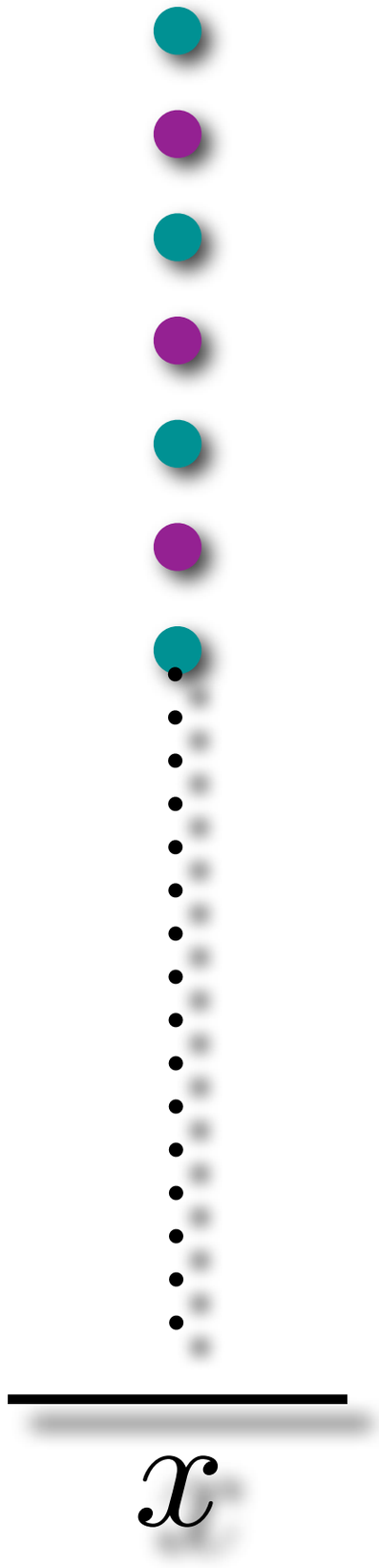


x

UNIL | Université de Lausanne

Alice

Bob



Alice wins $\iff x \in A$

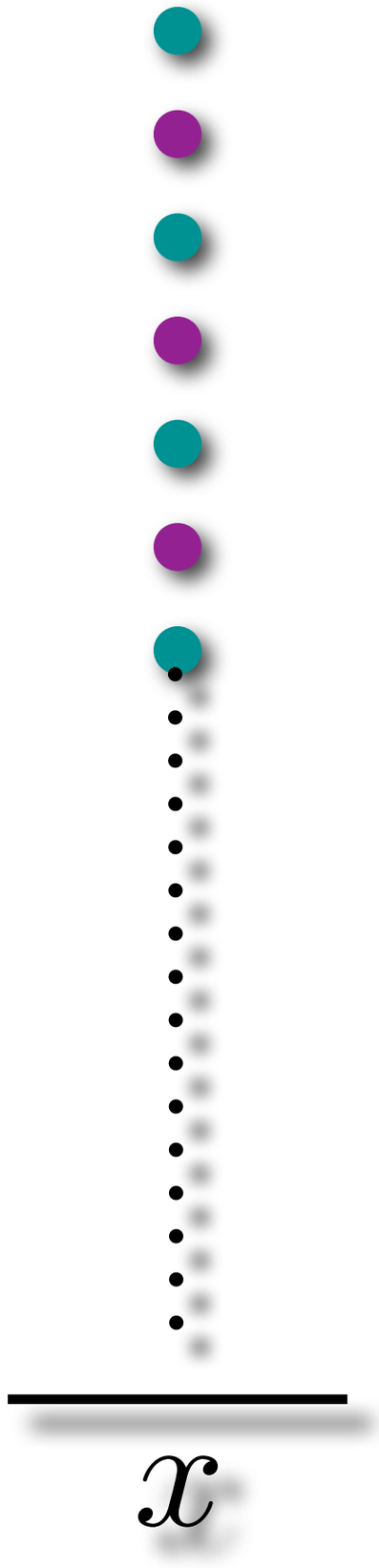
x

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



x

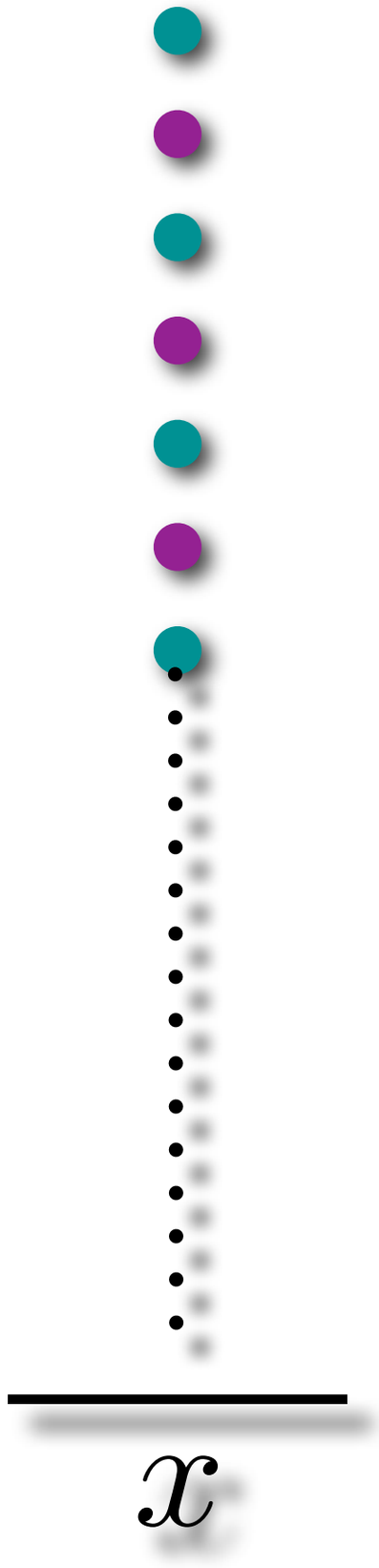
Alice wins $\iff x \in A$

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



Alice wins $\iff x \in A$

Bob wins $\iff x \in B = A^{\complement}$

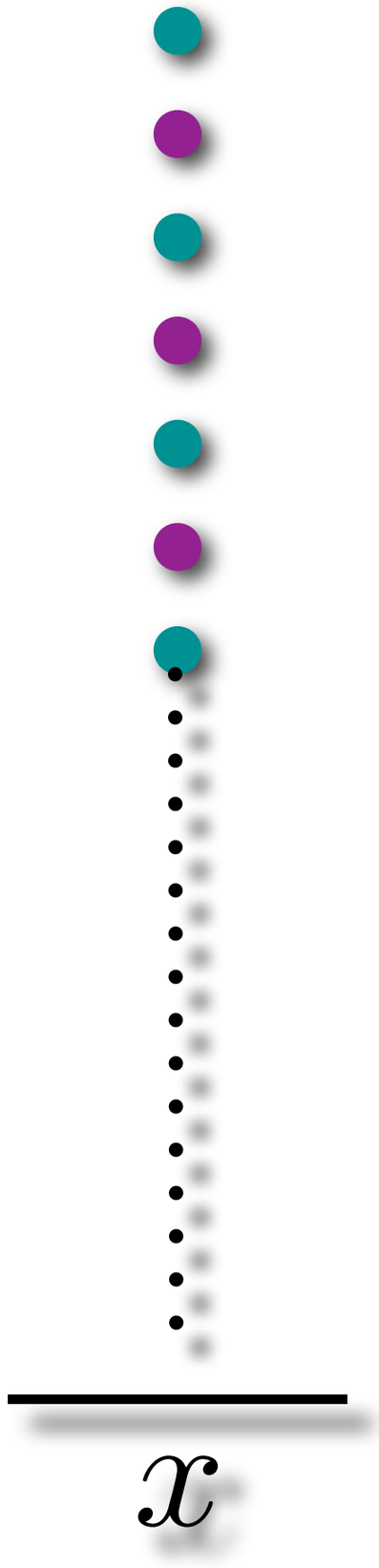
x

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



Alice wins $\iff x \in A$

Bob wins $\iff x \notin A$

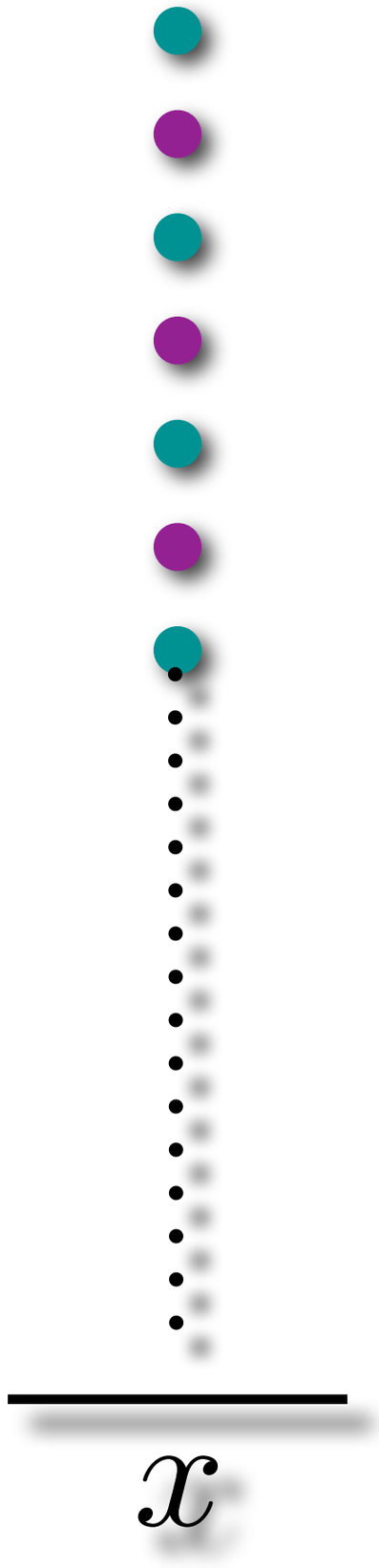
x

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



Alice wins $\iff x \in A$

Bob wins $\iff x \notin A$

x

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



@ What if this game were finite ?

Alice wins $\iff x \in A$

Bob wins $\iff x \notin A$

x

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



x

@ What if this game were finite ?

UNIL | Université de Lausanne

$$\frac{}{x}$$

UNIL | Université de Lausanne

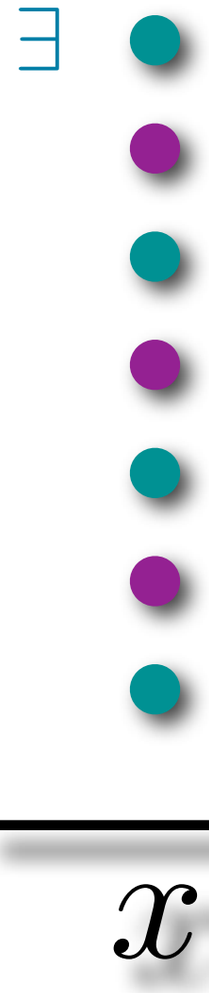
Alice has a w.s. $\iff$



$$\frac{}{x}$$

UNIL | Université de Lausanne

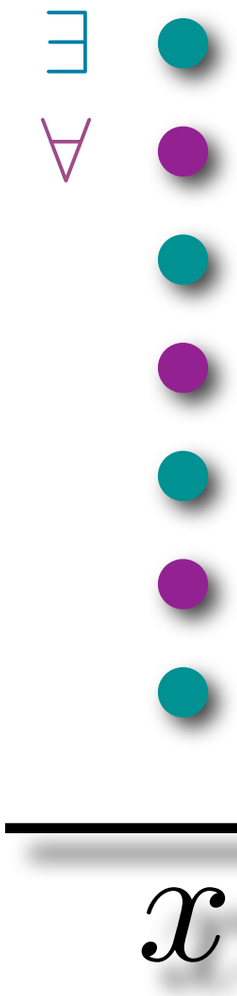
Alice has a w.s. $\iff$



$$\frac{}{x}$$

UNIL | Université de Lausanne

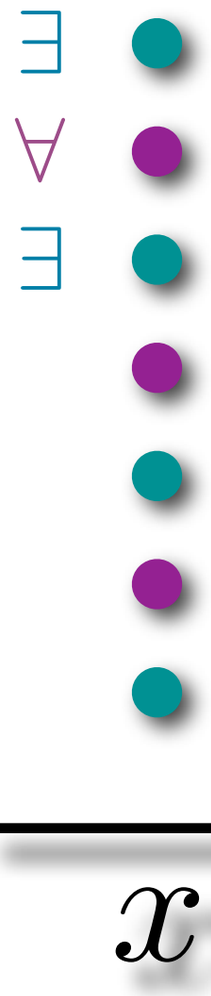
Alice has a w.s. $\iff$



$$\frac{}{x}$$

Unil
UNIL | Université de Lausanne

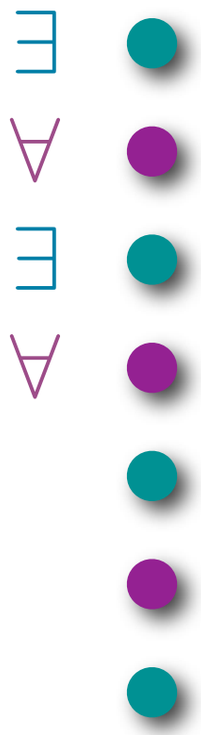
Alice has a w.s. $\iff$



$$\frac{}{x}$$

Unil
UNIL | Université de Lausanne

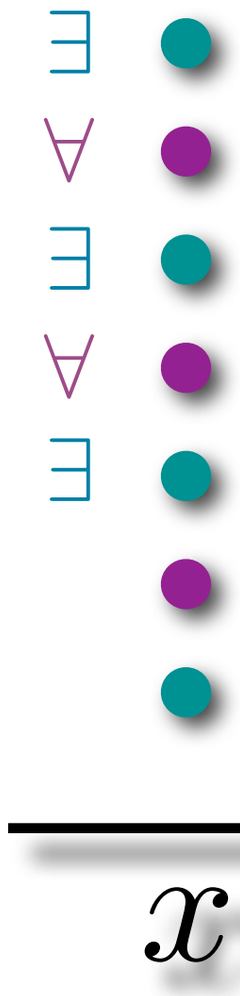
Alice has a w.s. $\iff$



$$\frac{}{x}$$

Unil
UNIL | Université de Lausanne

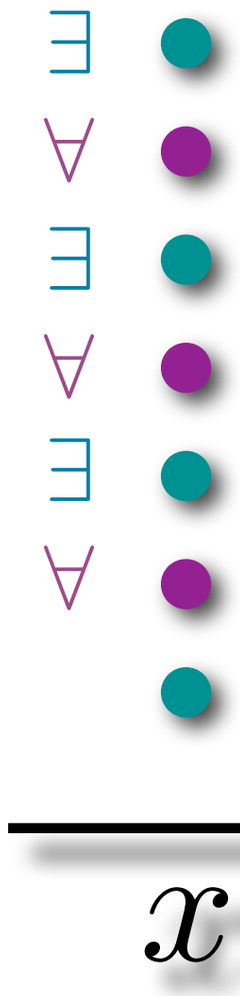
Alice has a w.s. $\iff$



$$\frac{}{x}$$

Unil
UNIL | Université de Lausanne

Alice has a w.s. $\iff$



$$\frac{}{x}$$

Unil
UNIL | Université de Lausanne

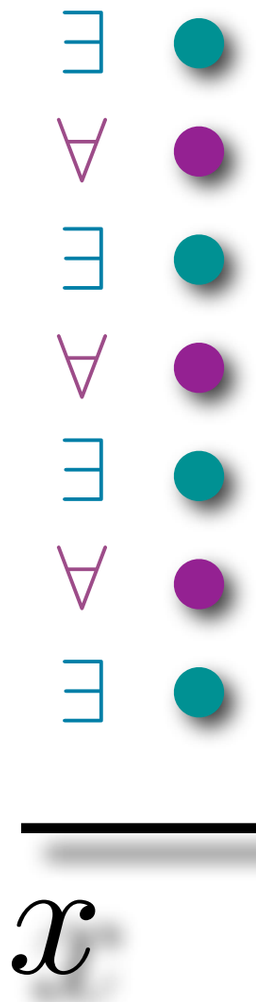
Alice has a w.s. $\iff$

$$\exists \; \bullet$$
$$\forall \; \bullet$$
$$\exists \; \bullet$$
$$\forall \; \bullet$$
$$\exists \; \bullet$$
$$\forall \; \bullet$$
$$\exists \; \bullet$$

$$\frac{}{x}$$

UNIL | Université de Lausanne

Alice has a w.s. $\iff$



$$\frac{}{x}$$

UNIL | Université de Lausanne

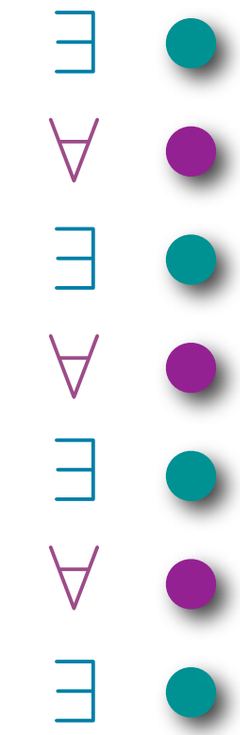
Alice has a w.s. $\iff$

$$\exists \; \bullet$$
$$\forall \; \bullet$$
$$\exists \; \bullet$$
$$\forall \; \bullet$$
$$\exists \; \bullet$$
$$\forall \; \bullet$$
$$\exists \; \bullet$$

$$\overline{\qquad\qquad\qquad\qquad}$$

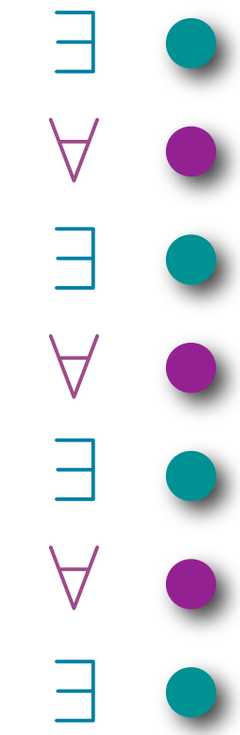
$$x \in A$$

UNIL | Université de Lausanne



$$\frac{}{x \in A}$$

UNIL | Université de Lausanne



$$\frac{}{x \in A}$$

UNIL | Université de Lausanne

Alice has no w.s. $\iff$

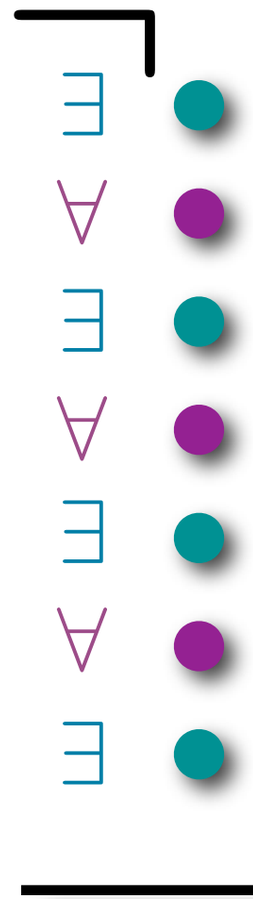
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$

$$\overline{\qquad\qquad}$$

$$x \in A$$

UNIL | Université de Lausanne

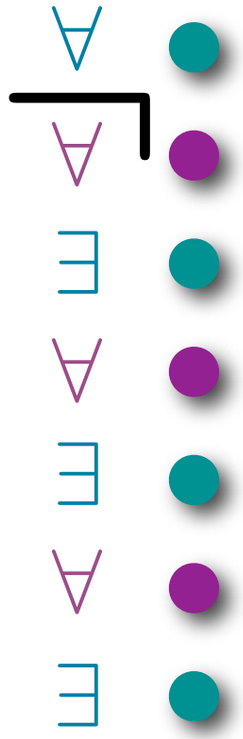
Alice has no w.s. $\iff$



$$x \in A$$

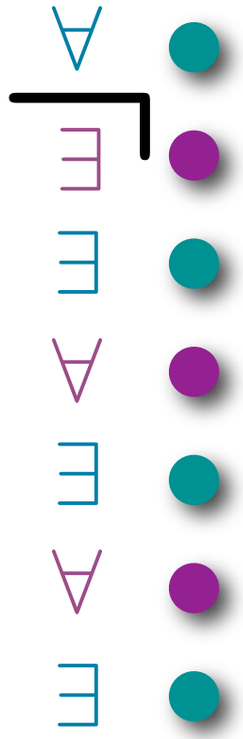
Unil
UNIL | Université de Lausanne

Alice has no w.s. $\iff$



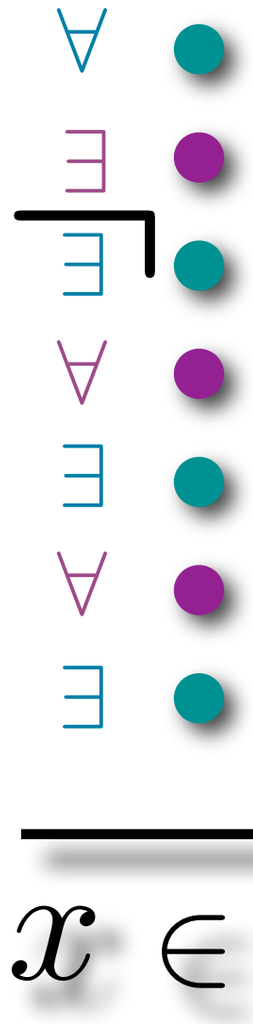
$$\frac{}{x \in A}$$

Alice has no w.s. $\iff$



$$\frac{}{x \in A}$$

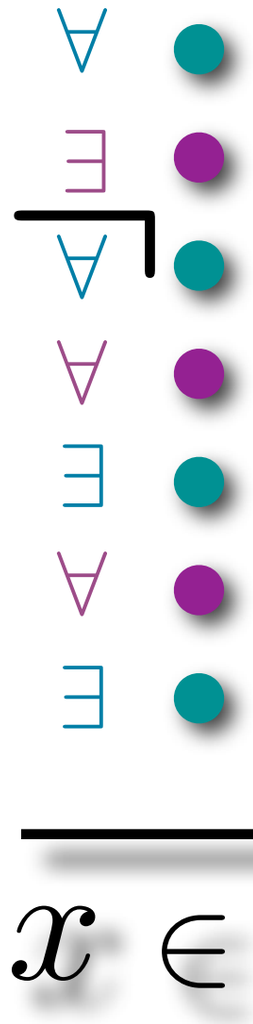
Alice has no w.s. $\iff$



$$\frac{}{x \in A}$$

UNIL | Université de Lausanne

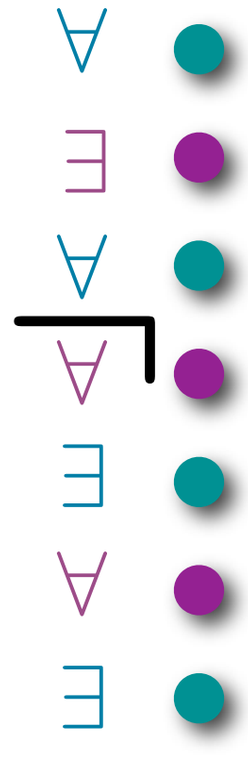
Alice has no w.s. $\Longleftrightarrow$



$$\frac{}{x \in A}$$

UNIL | Université de Lausanne

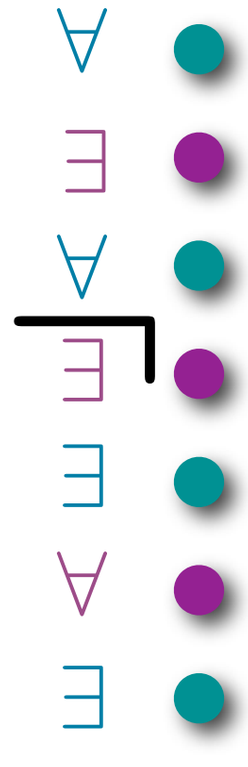
Alice has no w.s. $\Longleftrightarrow$



$$\frac{}{x \in A}$$

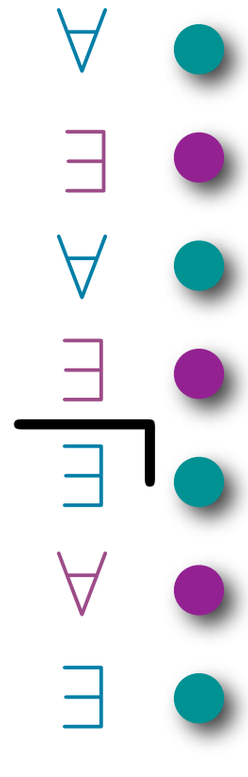
UNIL | Université de Lausanne

Alice has no w.s. $\iff$



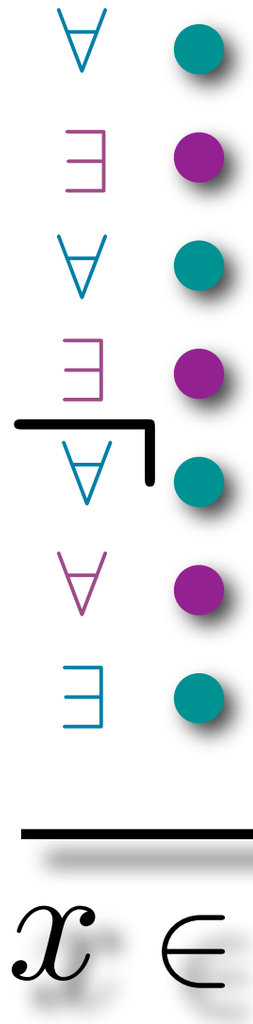
$$\frac{}{x \in A}$$

Alice has no w.s. $\Longleftrightarrow$



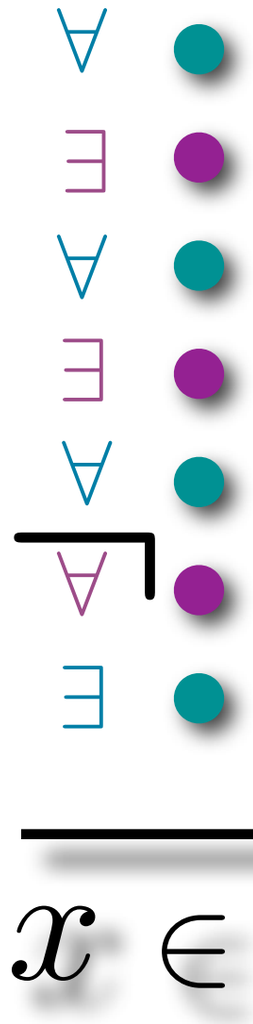
$$\frac{}{x \in A}$$

Alice has no w.s. $\iff$



$$\frac{}{x \in A}$$

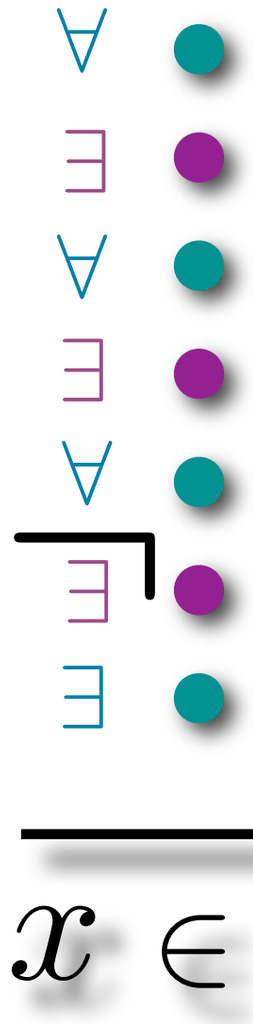
Alice has no w.s. $\Longleftrightarrow$



$$\frac{}{x \in A}$$

UNIL | Université de Lausanne

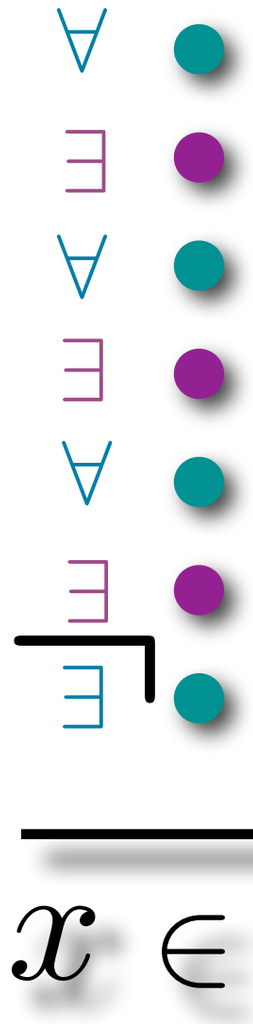
Alice has no w.s. $\Longleftrightarrow$



$$\frac{}{x \in A}$$

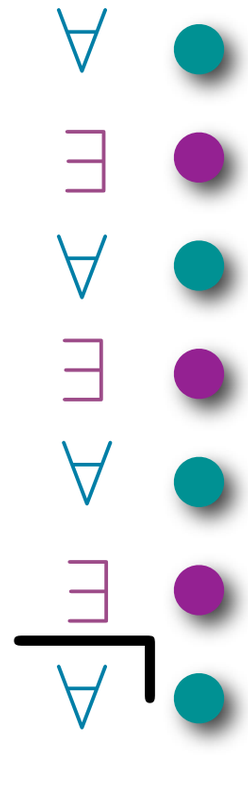
UNIL | Université de Lausanne

Alice has no w.s. $\iff$



$$\frac{}{x \in A}$$

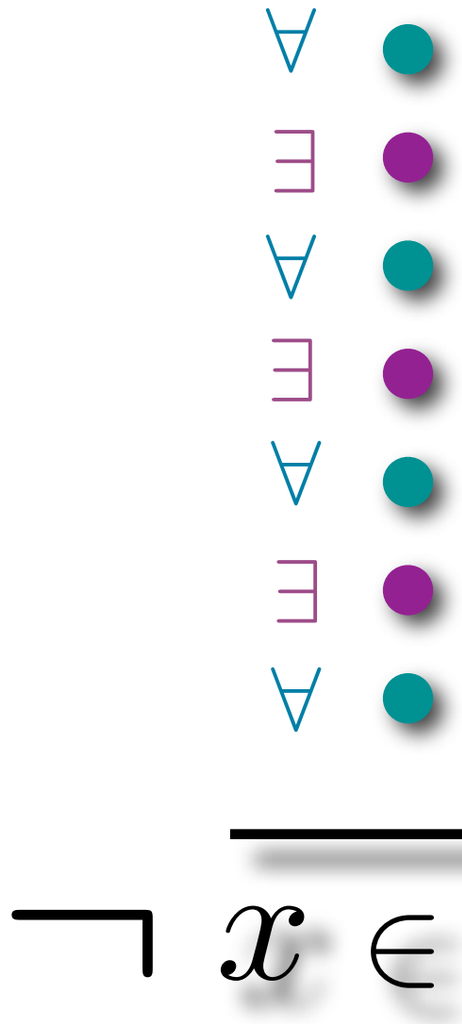
Alice has no w.s. $\iff$



$$\frac{}{x \in A}$$

UNIL | Université de Lausanne

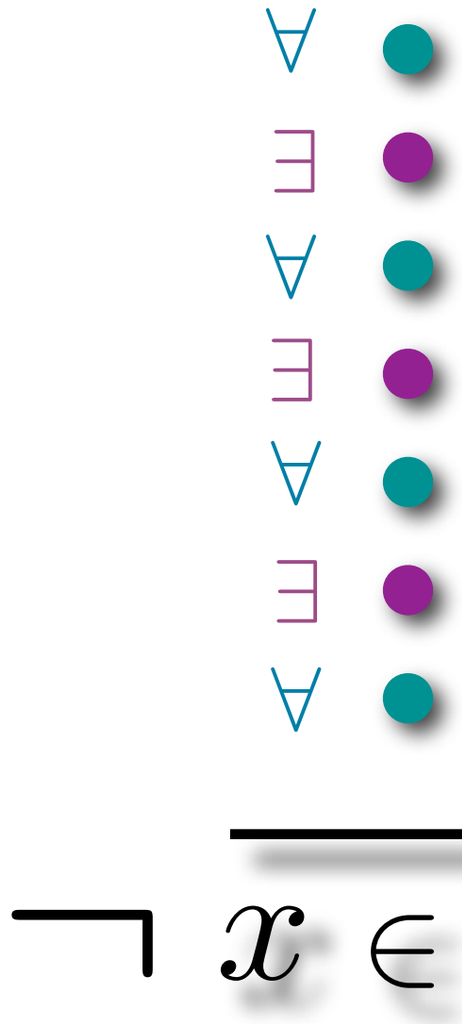
Alice has no w.s. $\Longleftrightarrow$



$$\neg\ x \in A$$

UNIL | Université de Lausanne

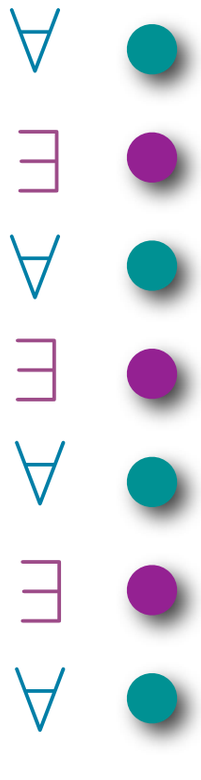
Alice has no w.s. $\Longleftrightarrow$



$$\neg\ x \in A$$

UNIL | Université de Lausanne

Alice has no w.s. $\iff$

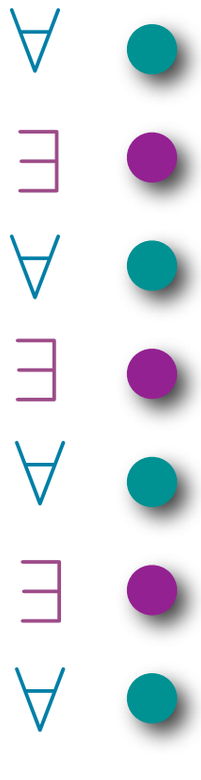


$$\overline{\qquad\qquad\qquad}$$

$$x \notin A$$

UNIL | Université de Lausanne

Alice has no w.s. $\iff$



$$\overline{x \notin A}$$

Unil
UNIL | Université de Lausanne

Alice has no w.s. $\iff$ Bob has a w.s.

$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$

$$\frac{}{x \notin A}$$

Alice has no w.s. $\iff$ Bob has a w.s.

$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$
$$\exists \bullet$$
$$\forall \bullet$$

$$\frac{}{x \notin A}$$

@ 1st order logic provides determinacy !

UNIL | Université de Lausanne



UNIL | Université de Lausanne

@ when the game is infinite

$$\mathcal{G}(A)$$

Alice

Bob



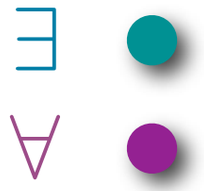
UNIL | Université de Lausanne

Alice has a w.s. $\Longleftrightarrow$

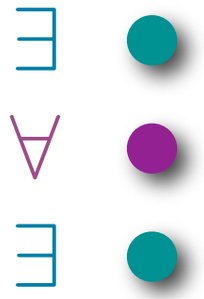
Alice has a w.s. $\iff$



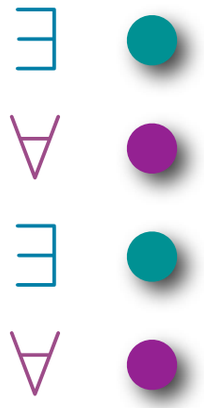
Alice has a w.s. $\iff$



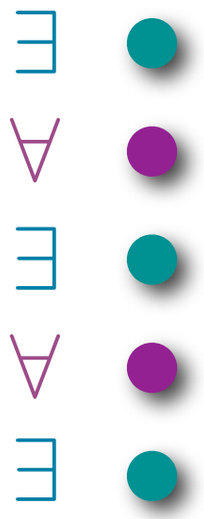
Alice has a w.s. $\iff$



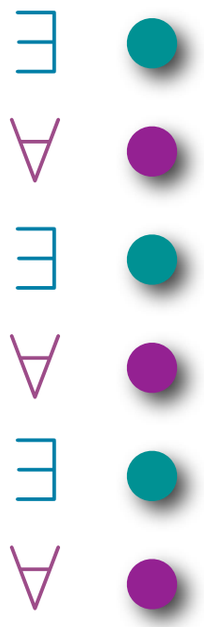
Alice has a w.s. $\iff$



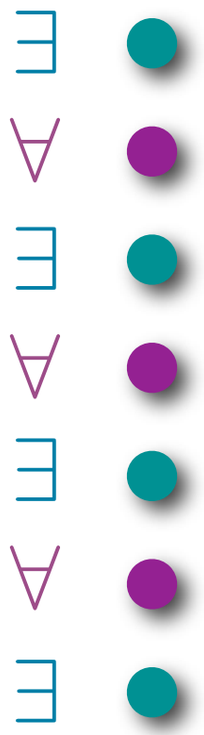
Alice has a w.s. $\iff$



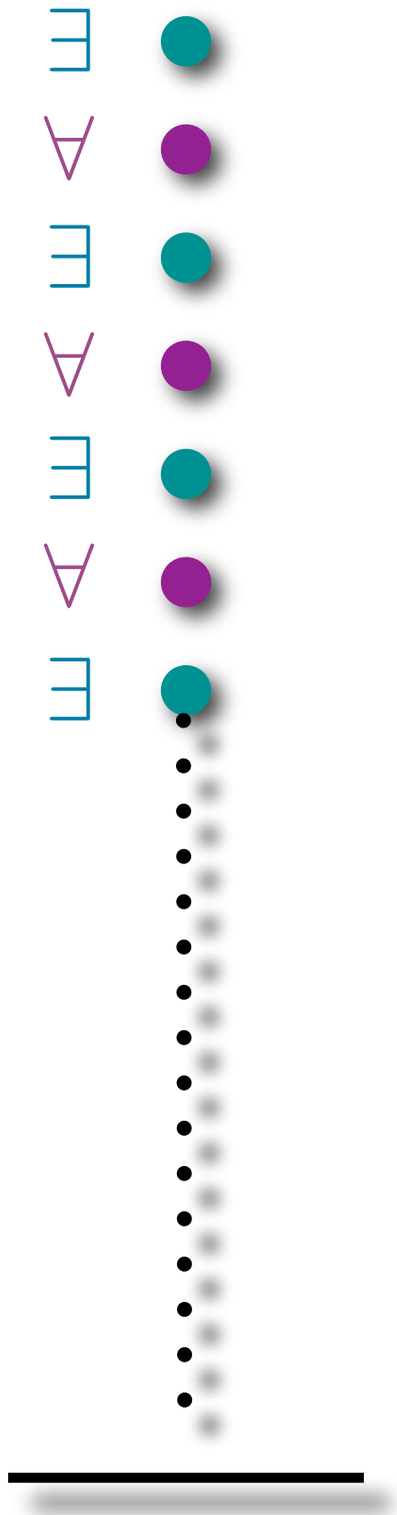
Alice has a w.s. $\iff$



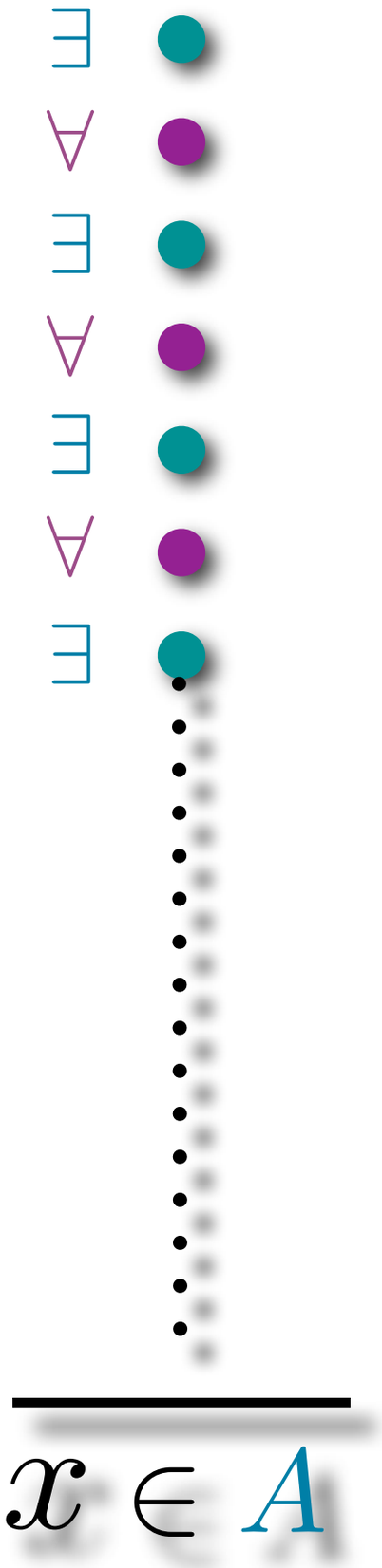
Alice has a w.s. $\iff$



Alice has a w.s. $\iff$

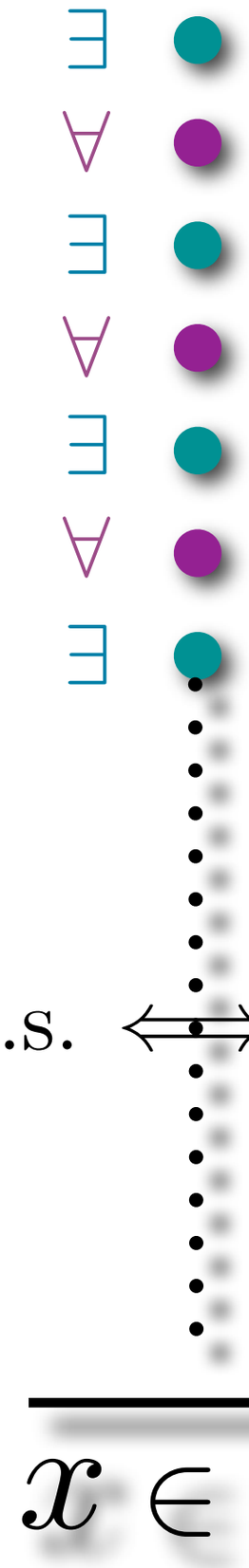


Alice has a w.s. $\iff$



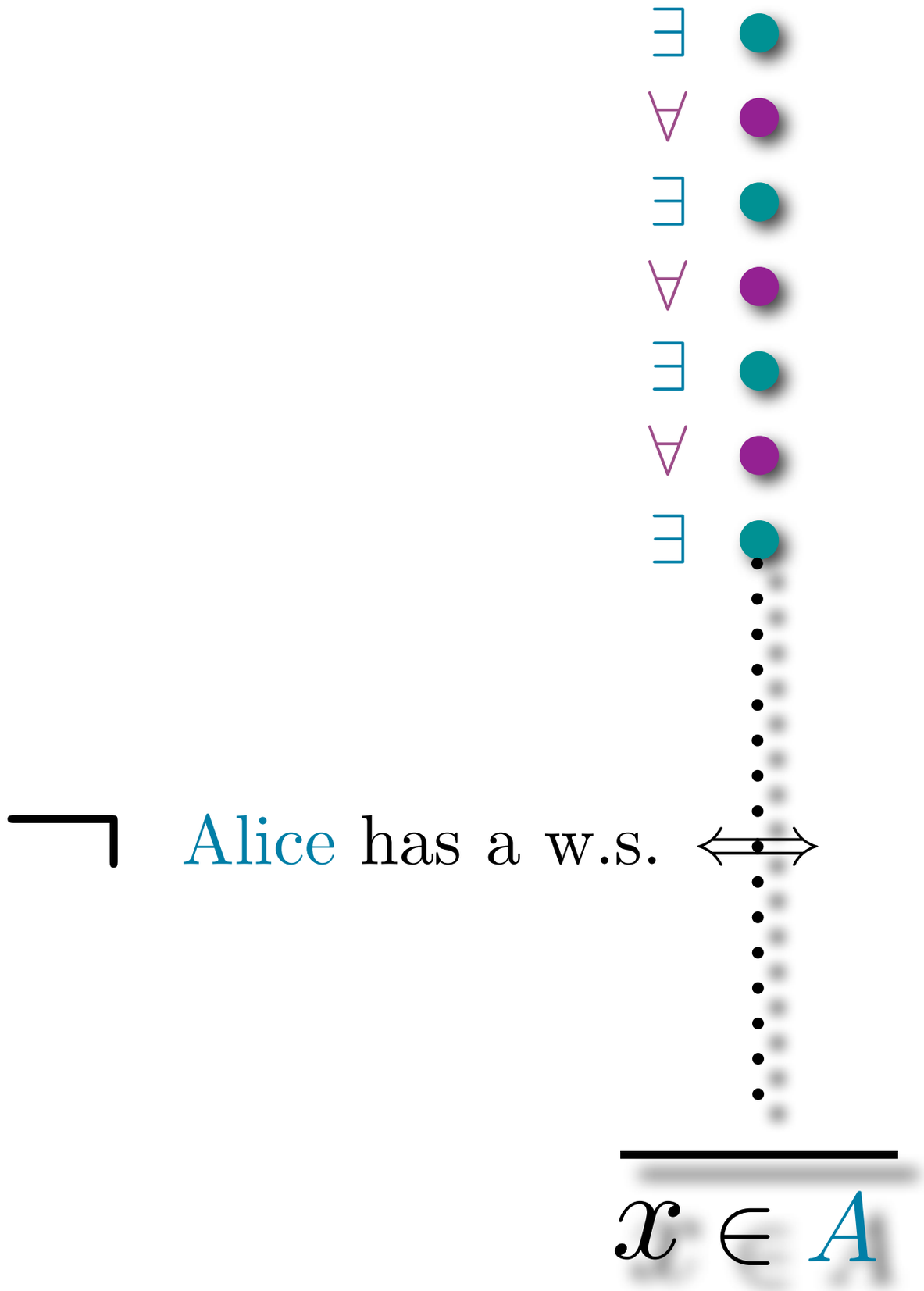
$$\overline{x \in A}$$

Unil
UNIL | Université de Lausanne

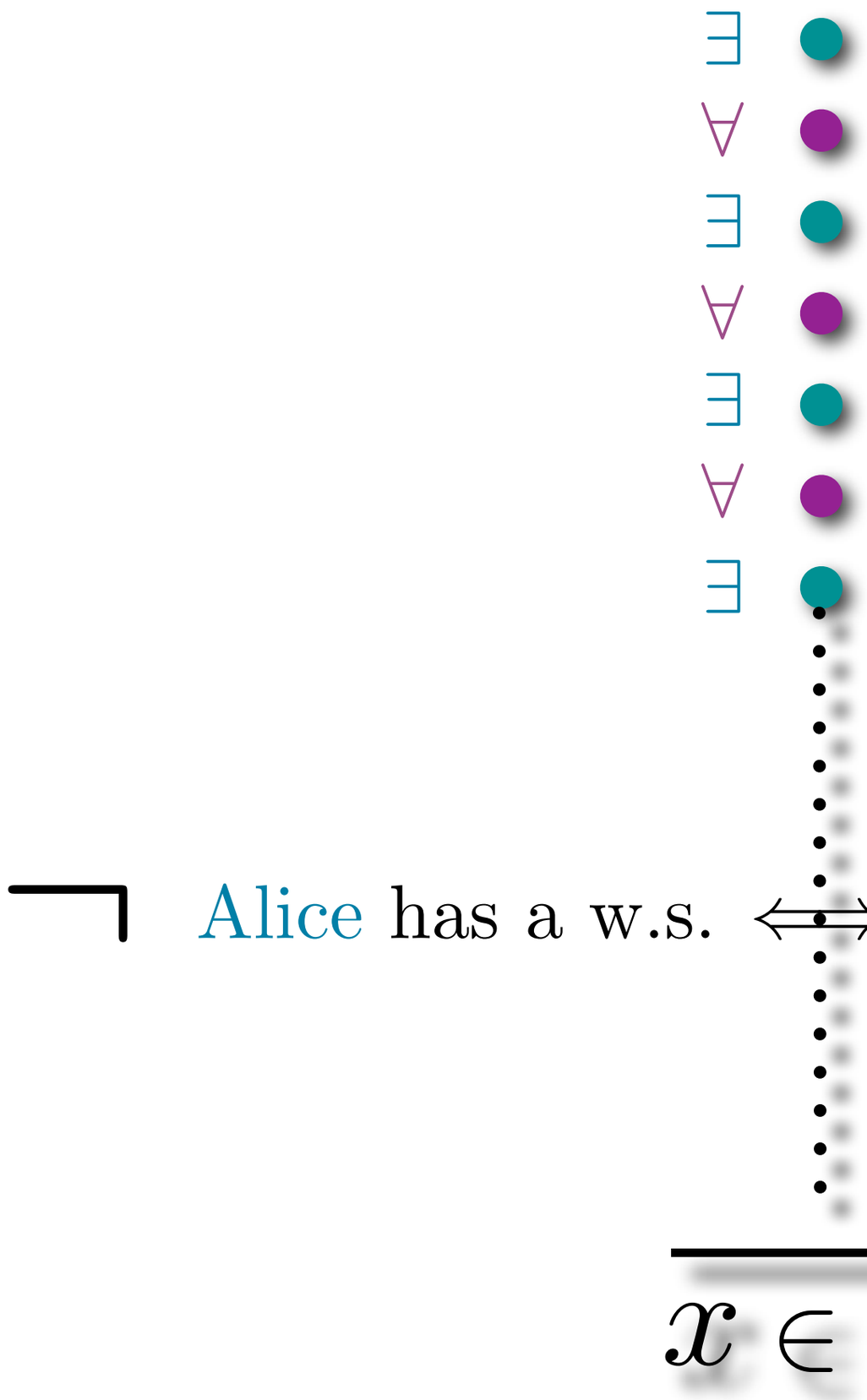


Alice has a w.s. $\Longleftrightarrow$

$$\overline{x \in A}$$



UNIL | Université de Lausanne



¬ Alice has a w.s. $\iff$ Bob has a w.s.

$$\overline{x \in A}$$

UNIL | Université de Lausanne

$\neg$ Alice has a w.s. $\iff$ Bob has a w.s.

UNIL | Université de Lausanne

$\neg$ Alice has a w.s. $\iff$ Bob has a w.s.

UNIL | Université de Lausanne

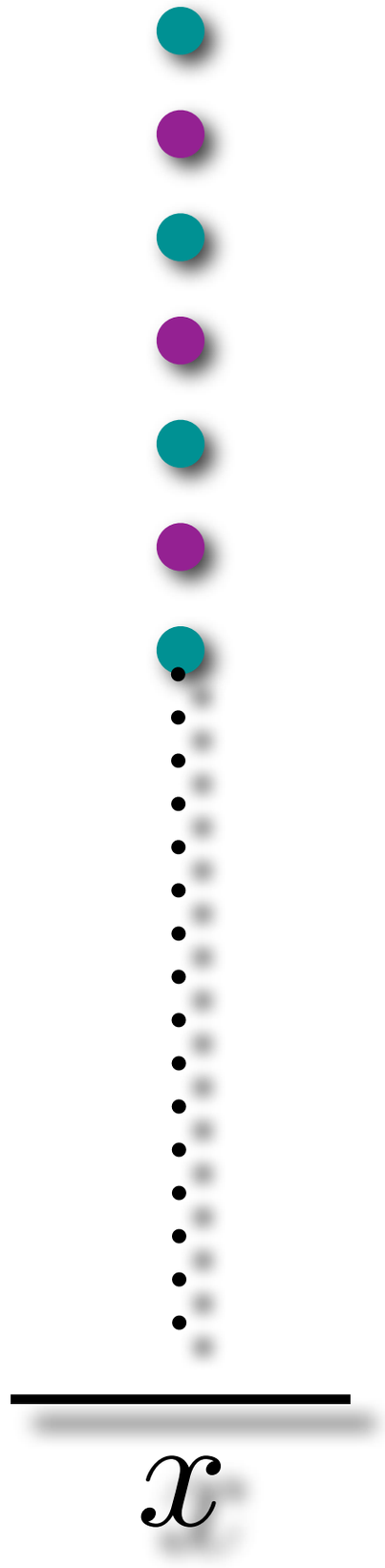
$\neg$ Alice has a w.s. $\iff$ Bob has a w.s.

@ *This is determinacy!* @

@ *It highly depends on* A @

UNIL | Université de Lausanne

$\mathcal{G}(A)$

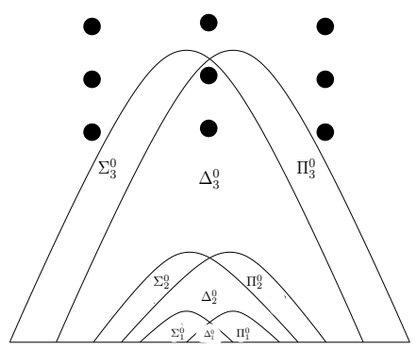


x

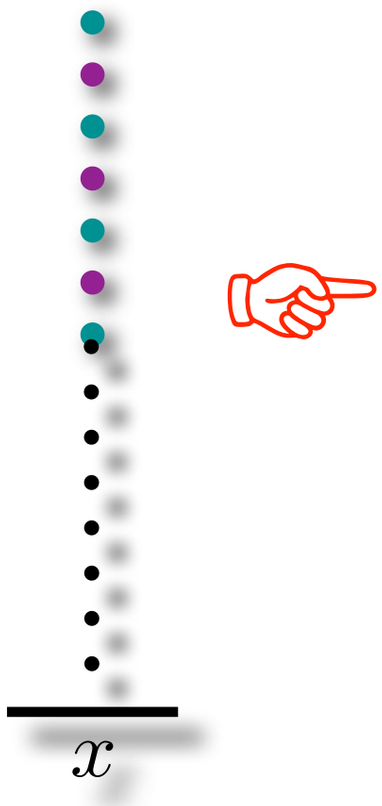


An example: evaluation game for 1st order logic

non determined game

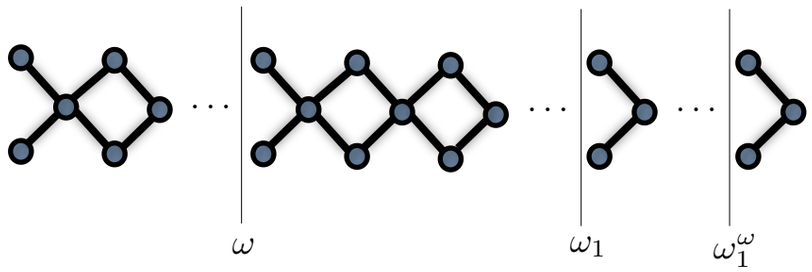


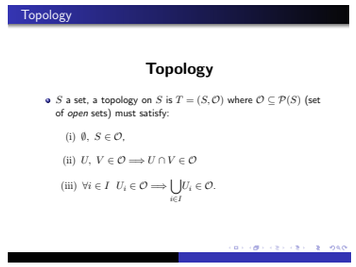
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

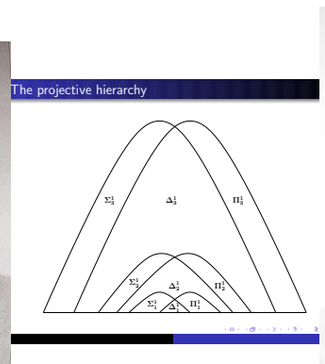
Games that construct mappings



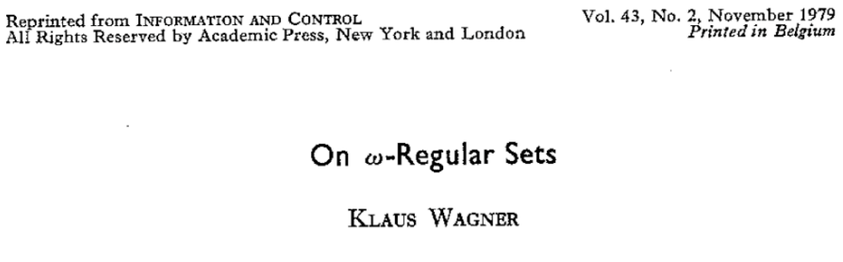










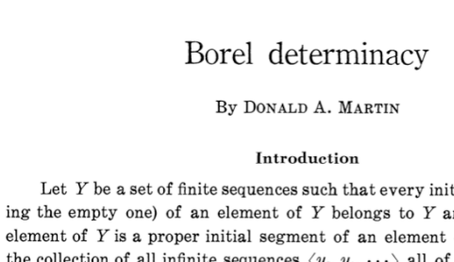


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

@ What about infinite duration ?

- A first approach: games that are essentially finite

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially ' = we know the winner after finitely many moves (even if the game has infinite duration)

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially ^{1} = we know the winner after finitely many moves (even if the game has infinite duration)

  - essentially ^{2} = we know the winner after finitely many moves, otherwise Bob wins

UNIL | Université de Lausanne

- *essentially* = we know the winner after finitely many moves (even if the game has infinite duration)

- *essentially* = we know the winner after finitely many moves (even if the game has infinite duration)

@ This means there exist $W, W \subseteq \mathbb{N}^*$ s.t.

- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W

@ *This means there exist* $W, W \subseteq \mathbb{N}^*$ *s.t.*

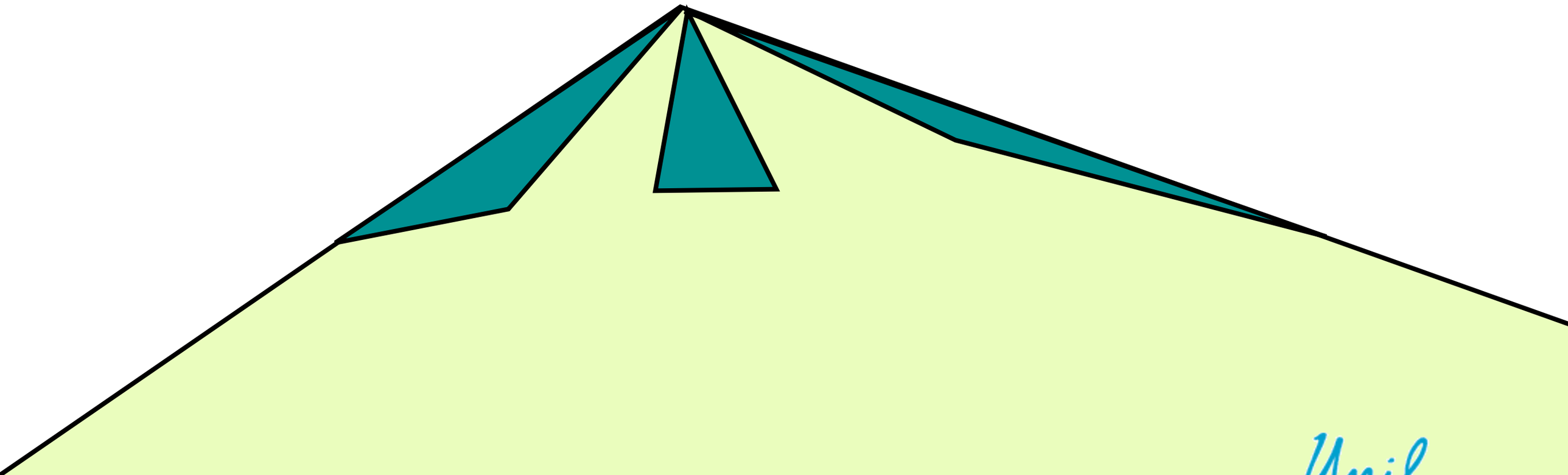
- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W

@ *This means there exist* $W, W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W

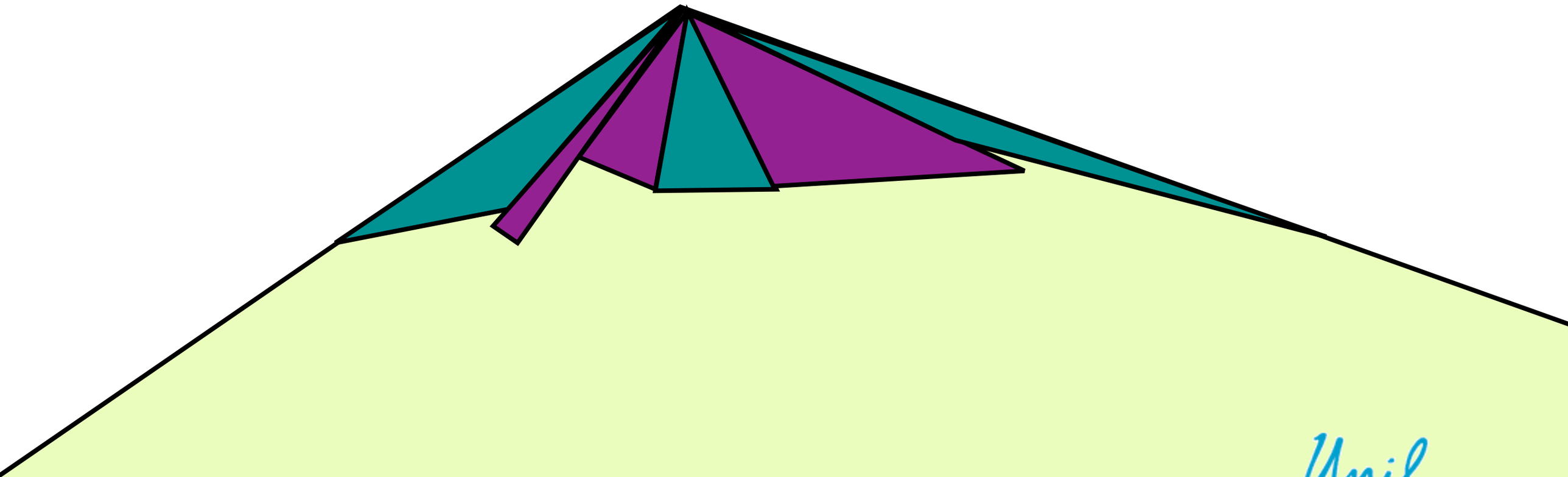
@ *This means there exist* $W, W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W



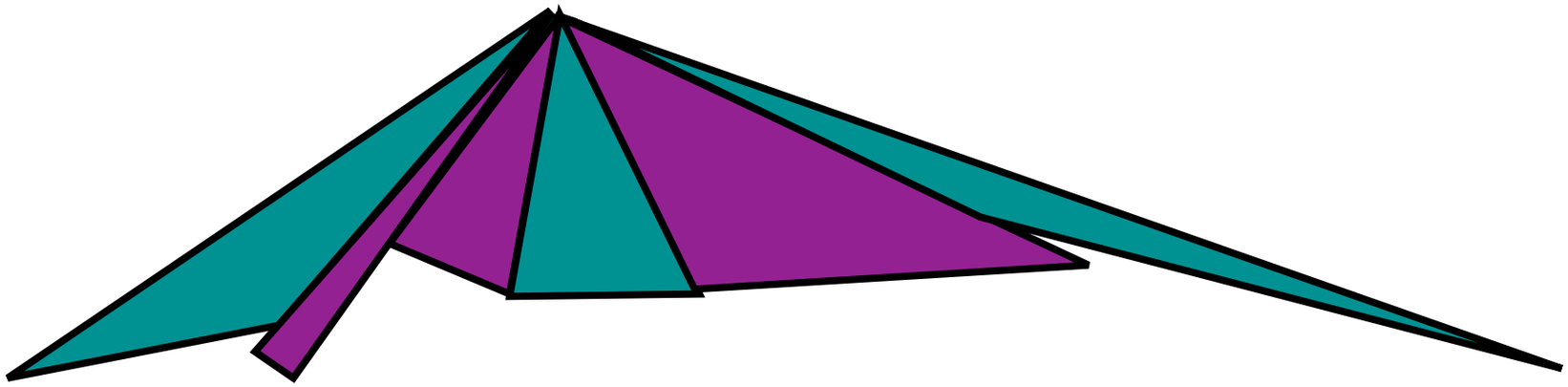
@ *This means there exist* $W, W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W



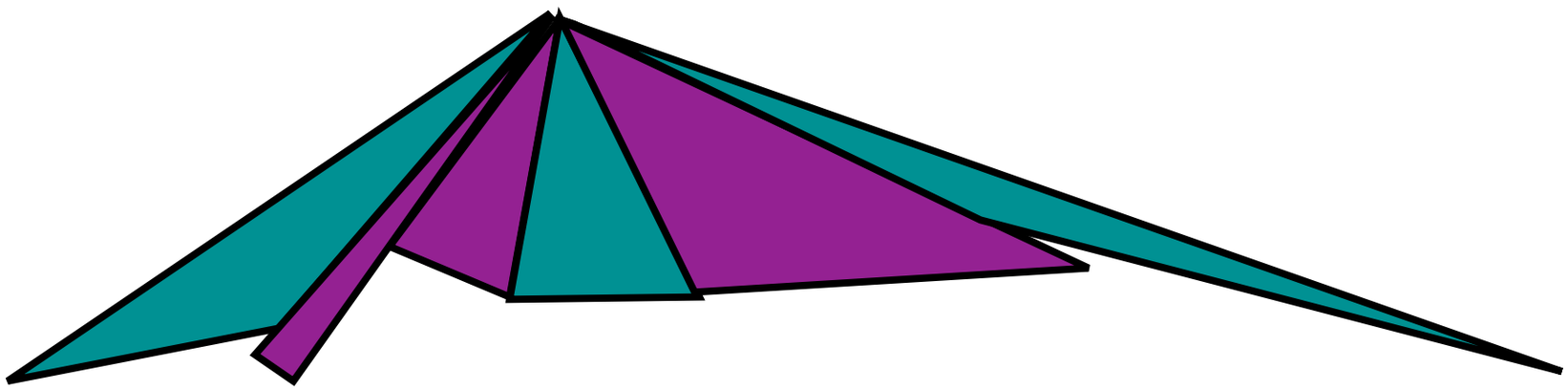
@ *This means there exist* $W, W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W



@ This means there exist $W, W \subseteq \mathbb{N}^*$ s.t.

- Alice wins $\iff$ x has a prefix inside W
- Bob wins $\iff$ x has a prefix inside W



@ This is nothing but a finite game ! Hence determined

- essentially [2] = we know the winner after finitely many moves, otherwise Bob wins

- essentially [2] = we know the winner after finitely many moves, otherwise Bob wins

@ This means there exist $W \subseteq \mathbb{N}^*$ s.t.

- Alice wins $\iff$ x has a prefix inside W

UNIL | Université de Lausanne

- *This means there exist* $W \subseteq \mathbb{N}^*$ *s.t.*
  - Alice wins $\iff$ x has a prefix inside W

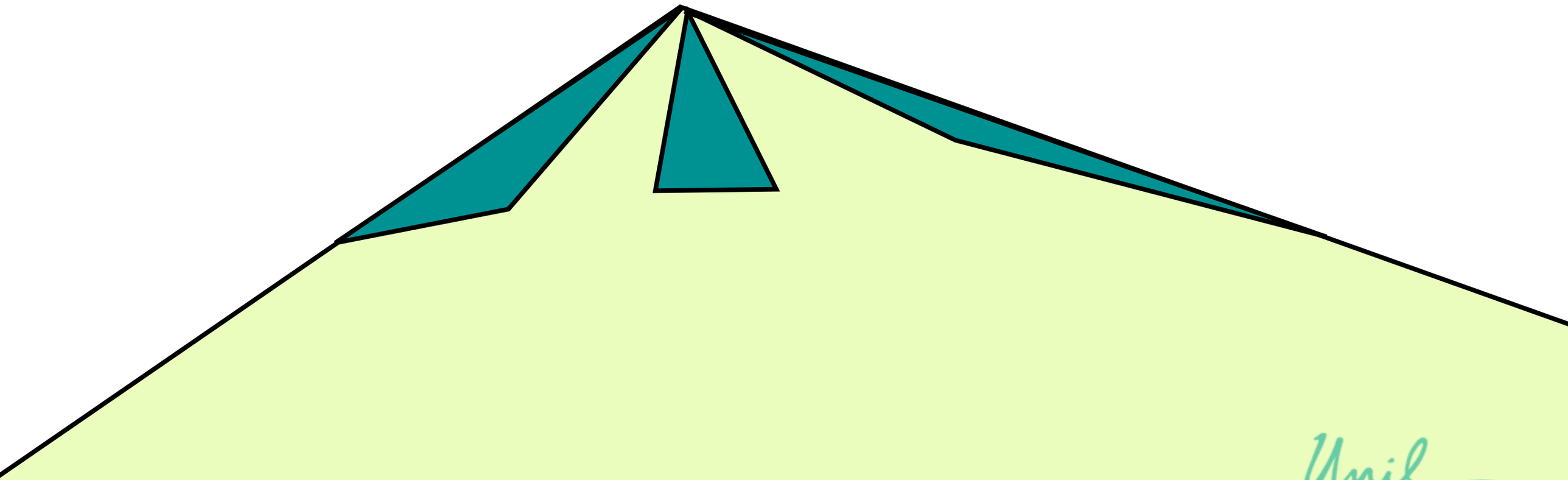
@ *This means there exist* $W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W

UNIL | Université de Lausanne

@ *This means there exist* $W \subseteq \mathbb{N}^*$ *s.t.*

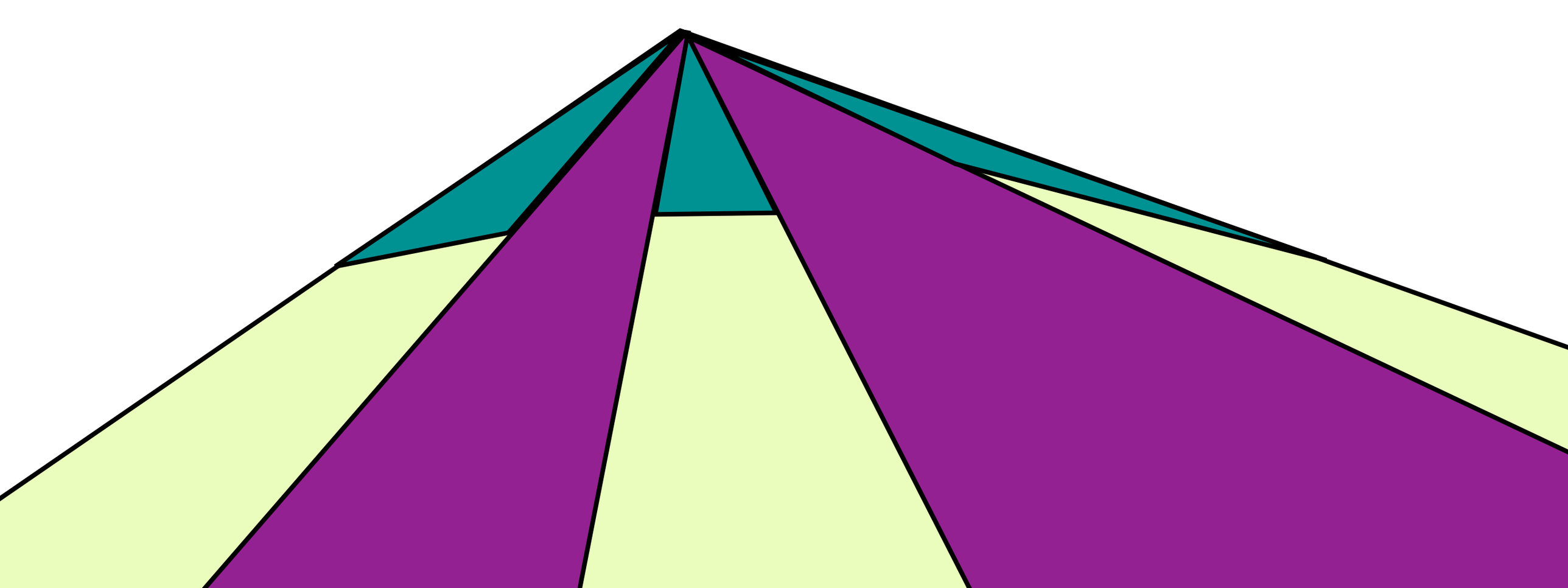
- Alice wins $\iff$ x has a prefix inside W



UNIL | Université de Lausanne

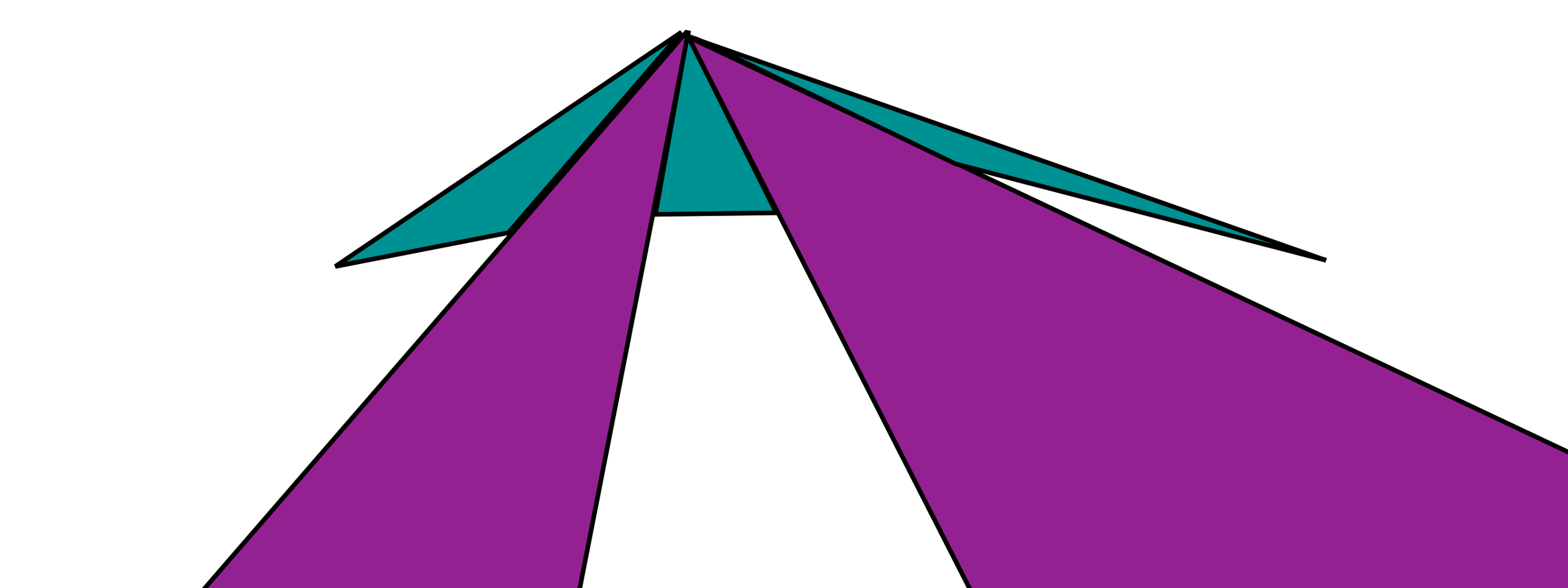
@ *This means there exist* $W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W



@ *This means there exist* $W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W

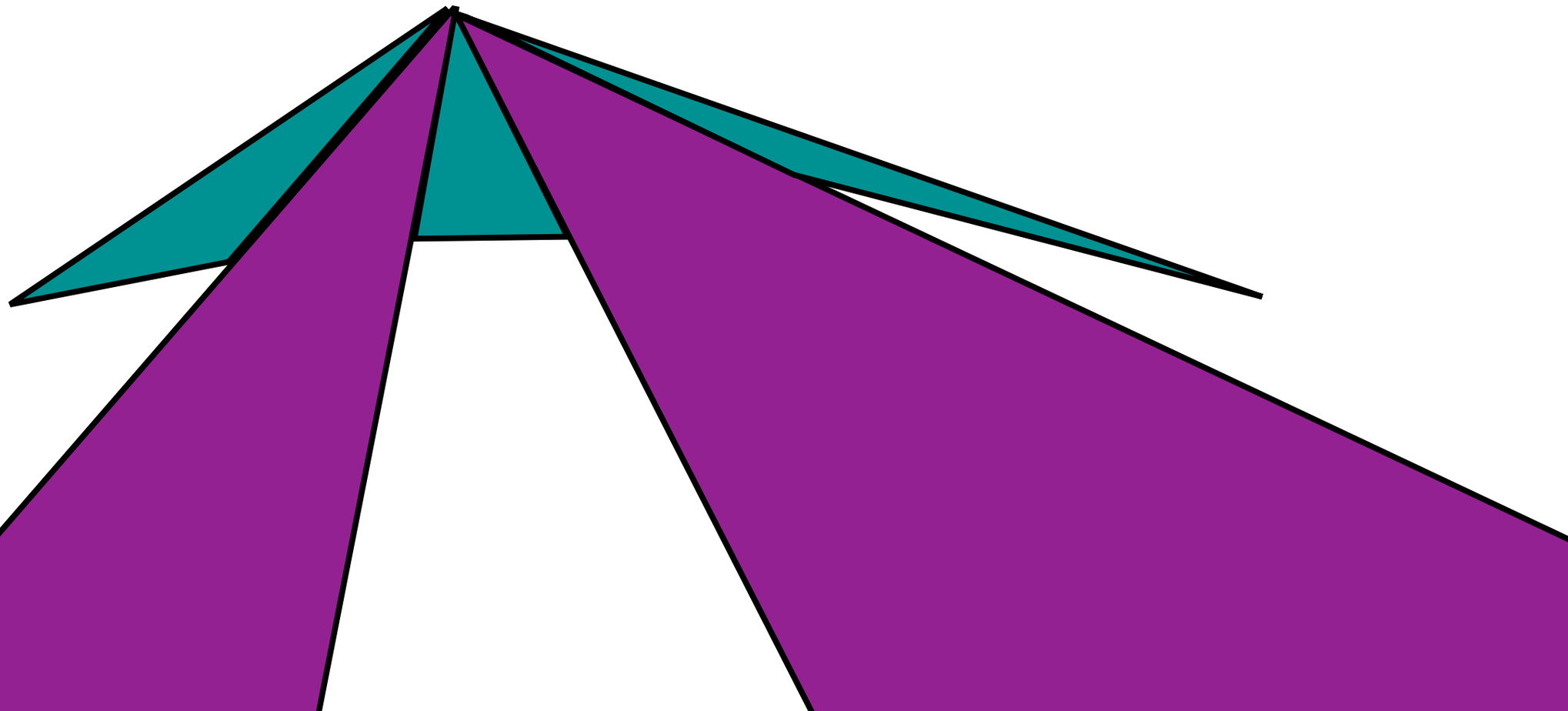


@ This means there exist $W \subseteq \mathbb{N}^*$ s.t.

- Alice wins $\iff$ x has a prefix inside W

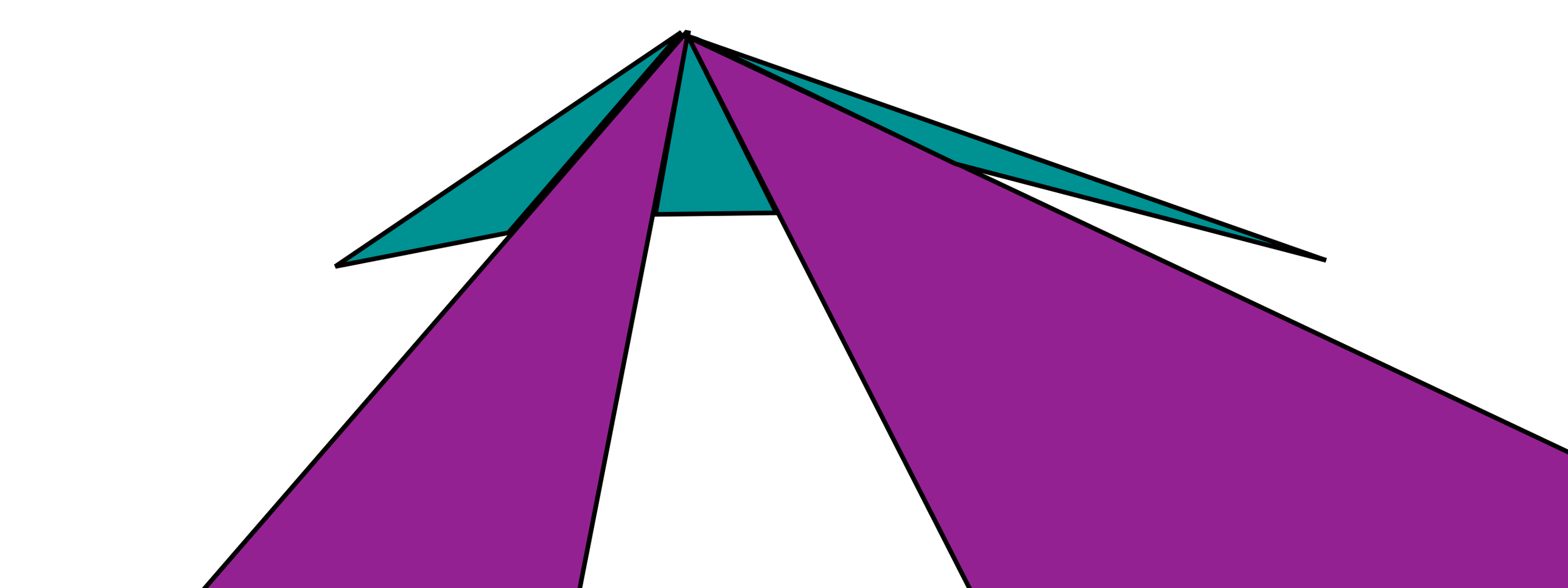
@ This is no more a finite game.

@ Is it determined ?



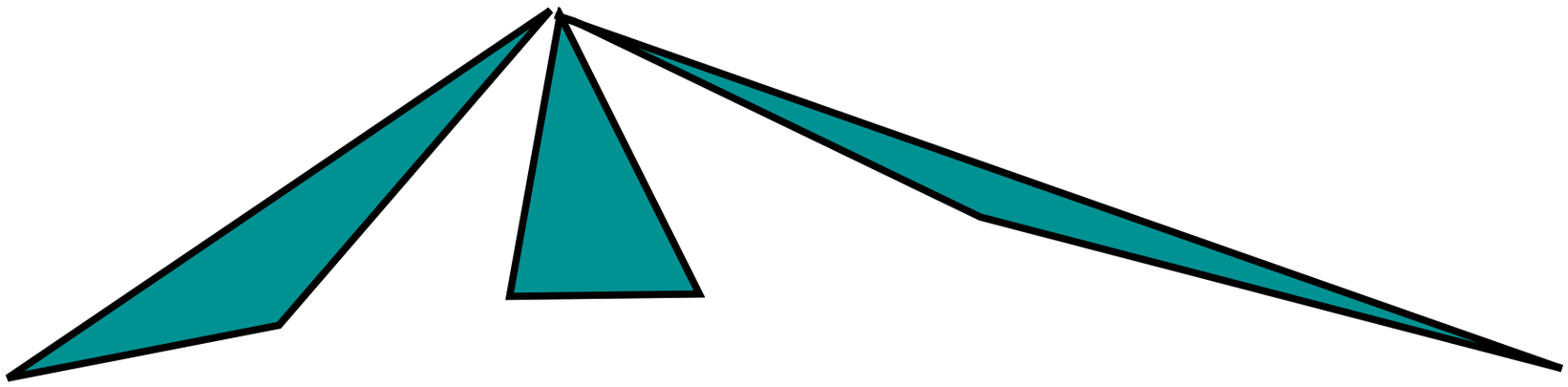
@ *This means there exist* $W \subseteq \mathbb{N}^*$ *s.t.*

- Alice wins $\iff$ x has a prefix inside W



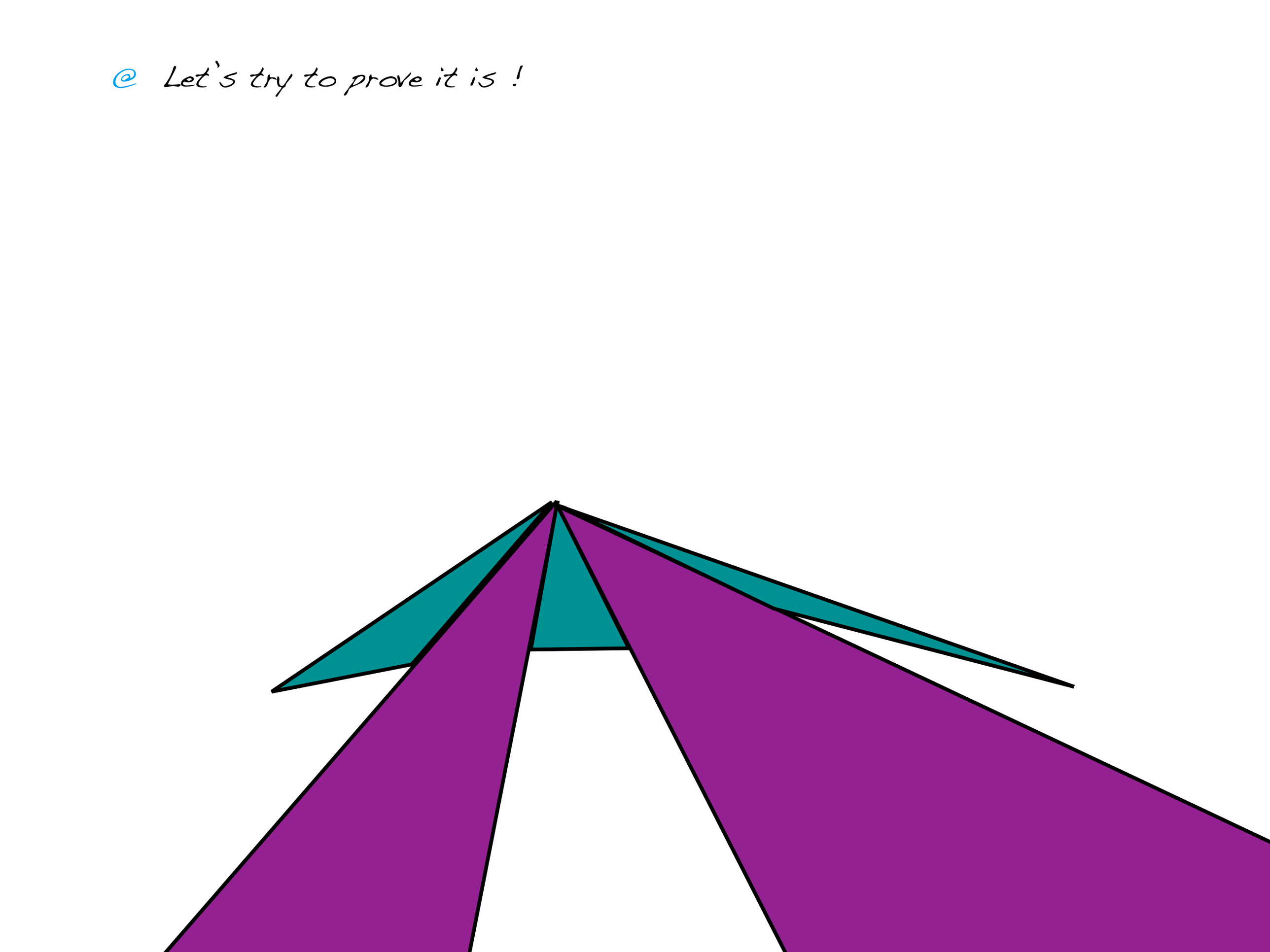
@ Let's try to prove it is !

@ Let's try to prove it is !



UNIL | Université de Lausanne

@ Let's try to prove it is !





UNIL | Université de Lausanne

- We assume Alice has no w.s.

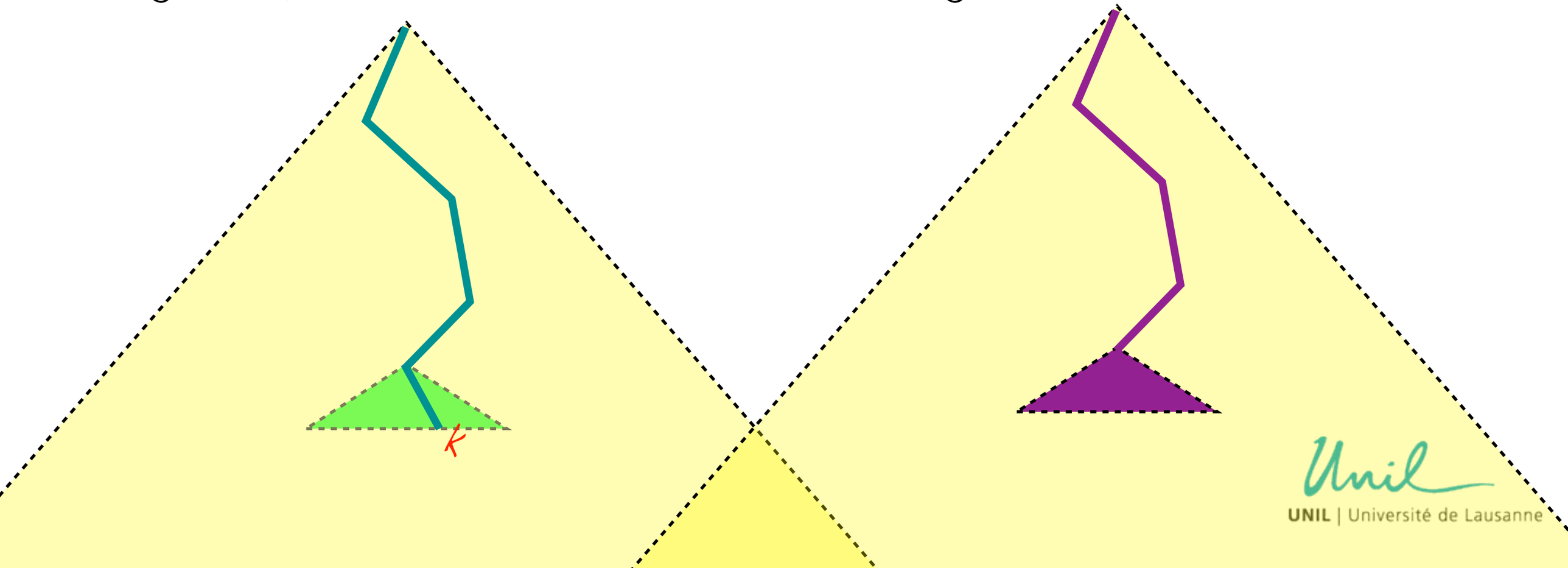
- But what is a w.s. for Alice ?

- Let T be the set of all positions

- a strategy for Alice is a n.e.p. tree S s.t.

(1) for any position $u \in S$ for Alice, there exists a unique integer k s.t. $uk \in S$

(2) given any position $u \in S$ for Bob, for all integers k $uk \in S$



UNIL | Université de Lausanne



UNIL | Université de Lausanne

- We assume Alice has no w.s.

  - a position is "not losing" for Bob if Alice has no w.s. from it.

- We describe T := the set of all positions that are "not losing" for Bob

  (1) the initial position := $\varepsilon \in T$

  (2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$uk \in T$

UNIL | Université de Lausanne

- We assume Alice has no w.s.

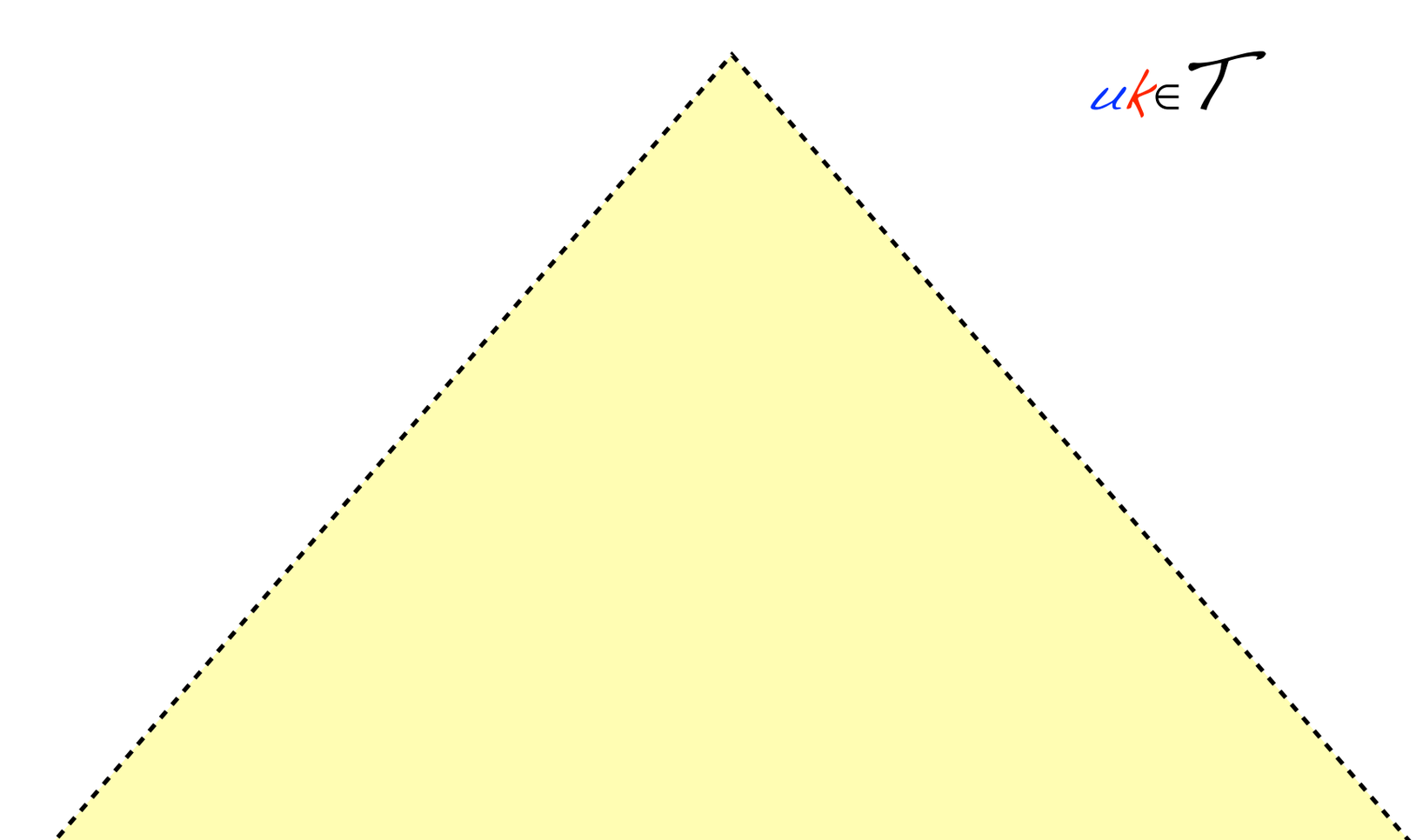
- a position is "not losing" for Bob if Alice has no w.s. from it.

- We describe T := the set of all positions that are "not losing" for Bob

(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$uk \in T$



UNIL | Université de Lausanne

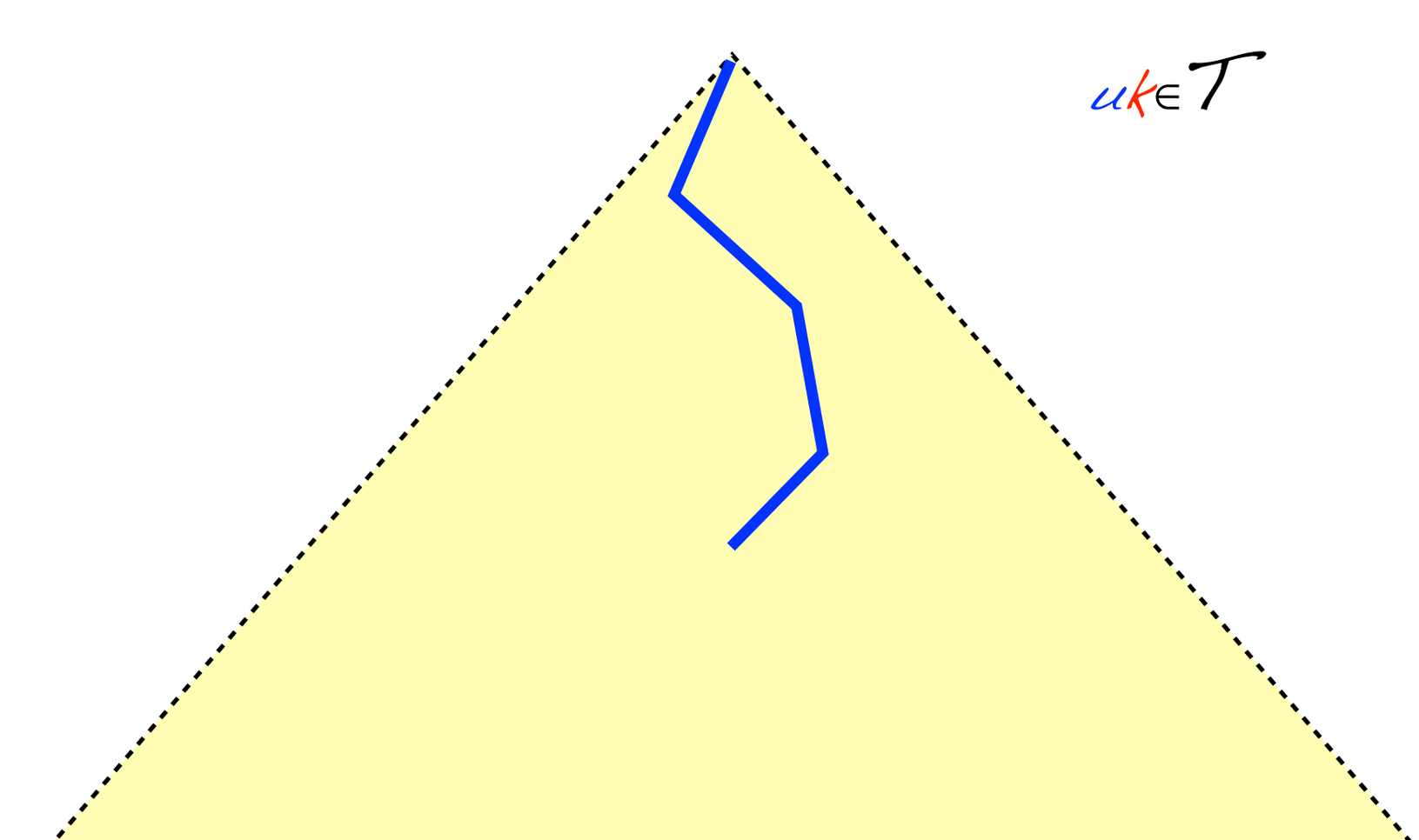
- We assume Alice has no w.s.

- a position is "not losing" for Bob if Alice has no w.s. from it.

- We describe T := the set of all positions that are "not losing" for Bob

(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.



UNIL | Université de Lausanne

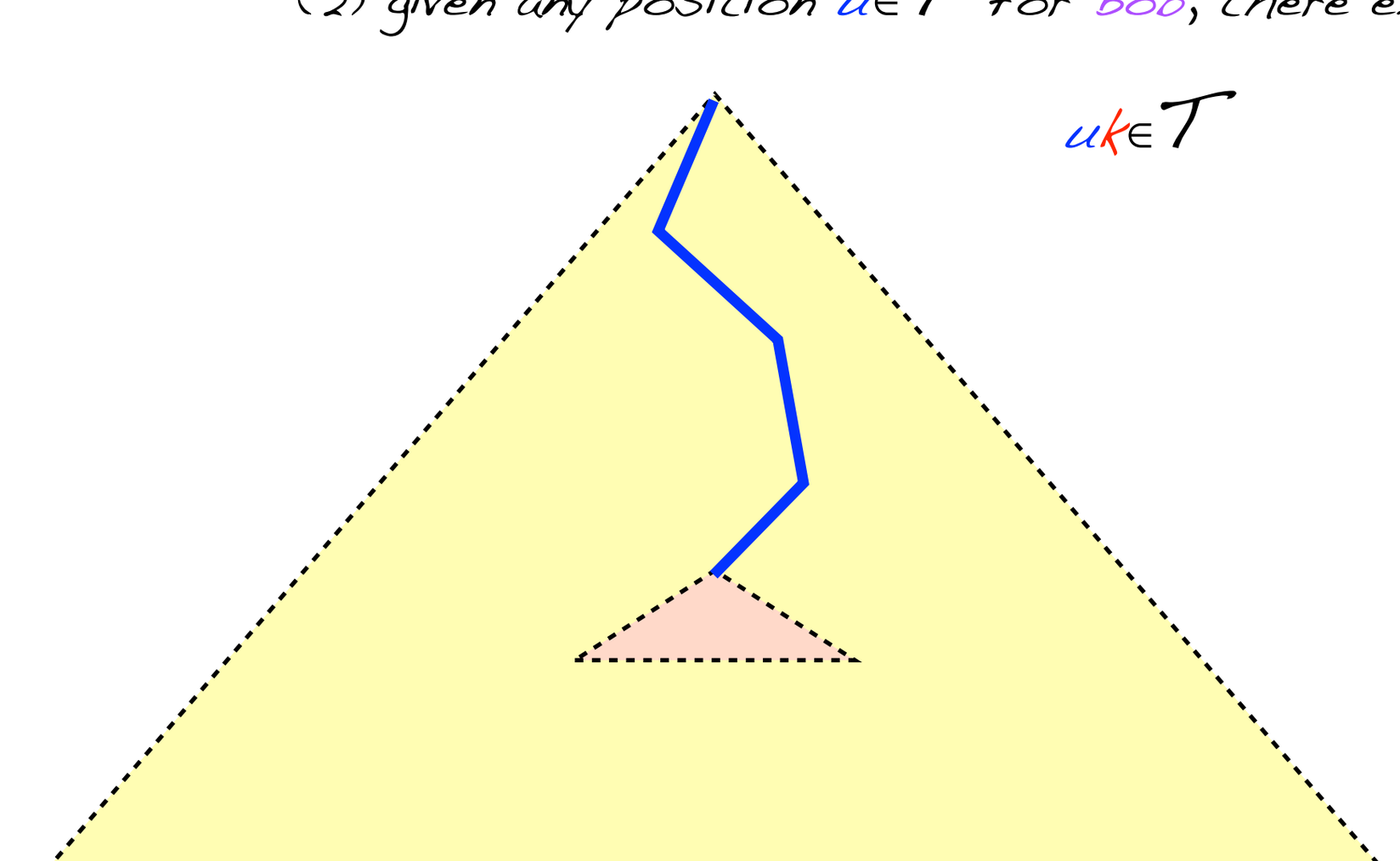
- We assume Alice has no w.s.

  - a position is "not losing" for Bob if Alice has no w.s. from it.

- We describe T := the set of all positions that are "not losing" for Bob

  (1) the initial position := $\varepsilon \in T$

  (2) given any position $u \in T$ for Bob, there exists an integer k s.t.



UNIL | Université de Lausanne

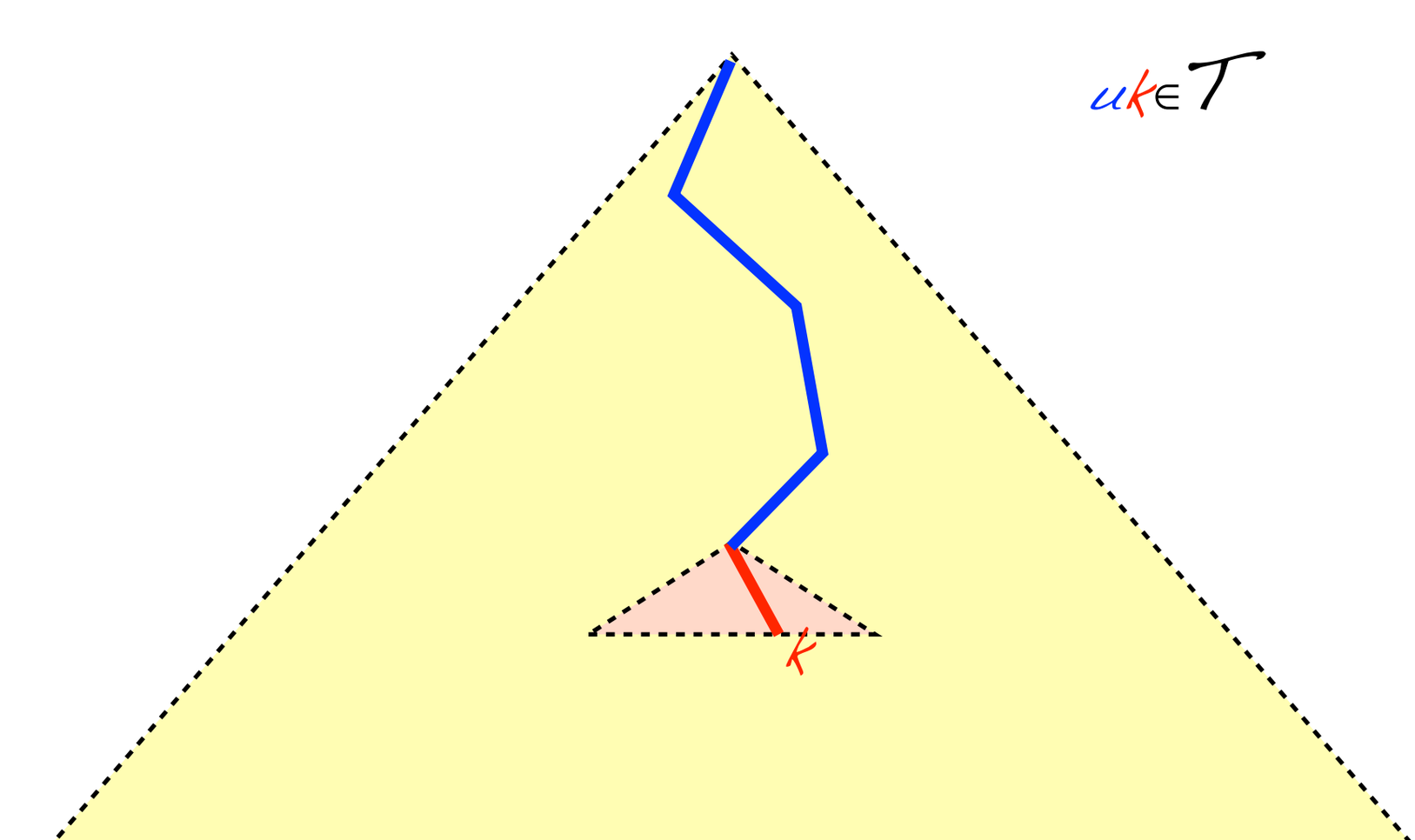
- We assume Alice has no w.s.

  - a position is "not losing" for Bob if Alice has no w.s. from it.

- We describe T := the set of all positions that are "not losing" for Bob

  (1) the initial position := $\varepsilon \in T$

  (2) given any position $u \in T$ for Bob, there exists an integer k s.t.



UNIL | Université de Lausanne

(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$$uk \in T$$

UNIL | Université de Lausanne

(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$$uk \in T$$

(3) given any position $u \in T$ for Alice, for all integers k

$$uk \in T$$

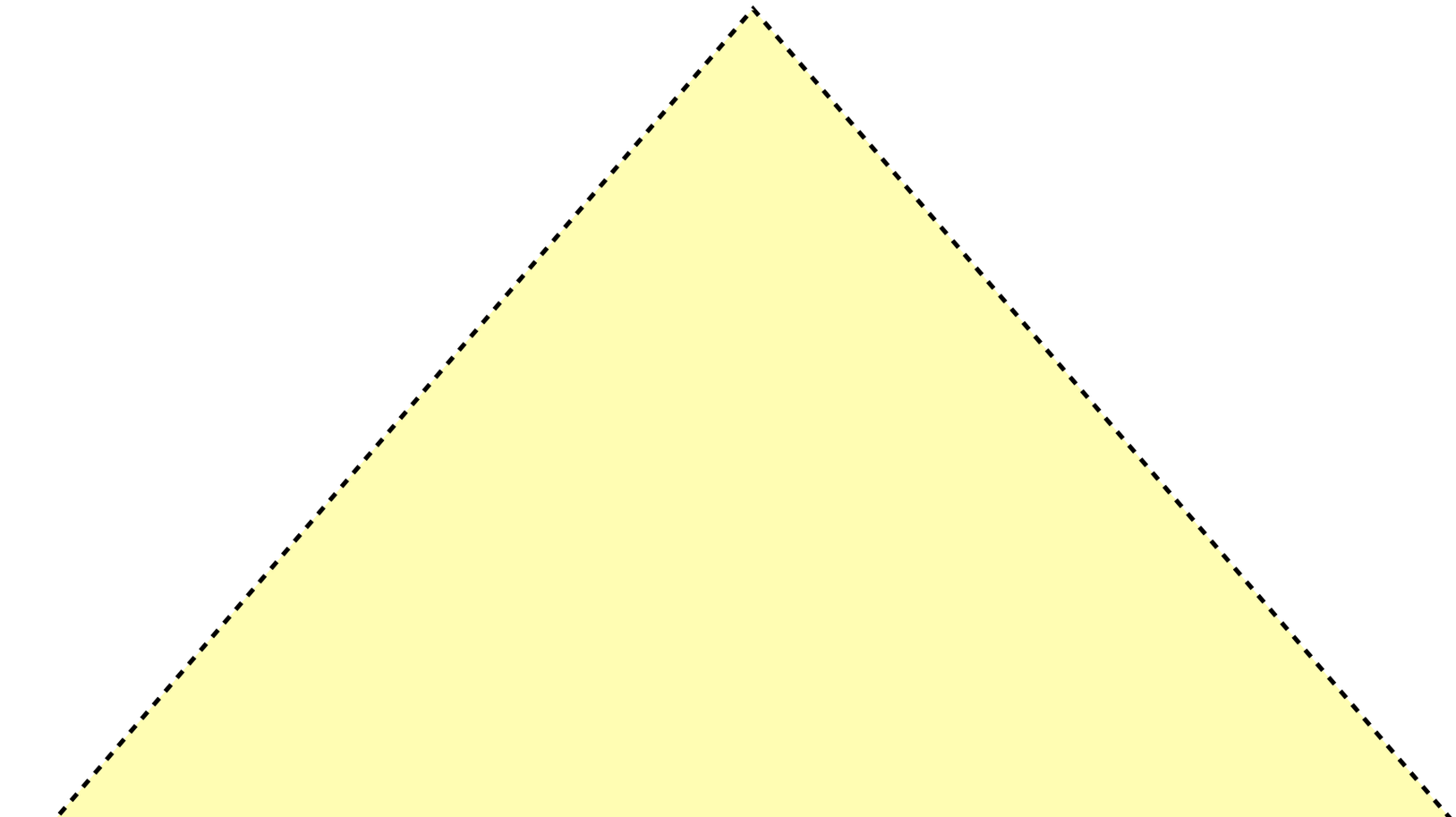
(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$$uk \in T$$

(3) given any position $u \in T$ for Alice, for all integers k

$$uk \in T$$



UNIL | Université de Lausanne

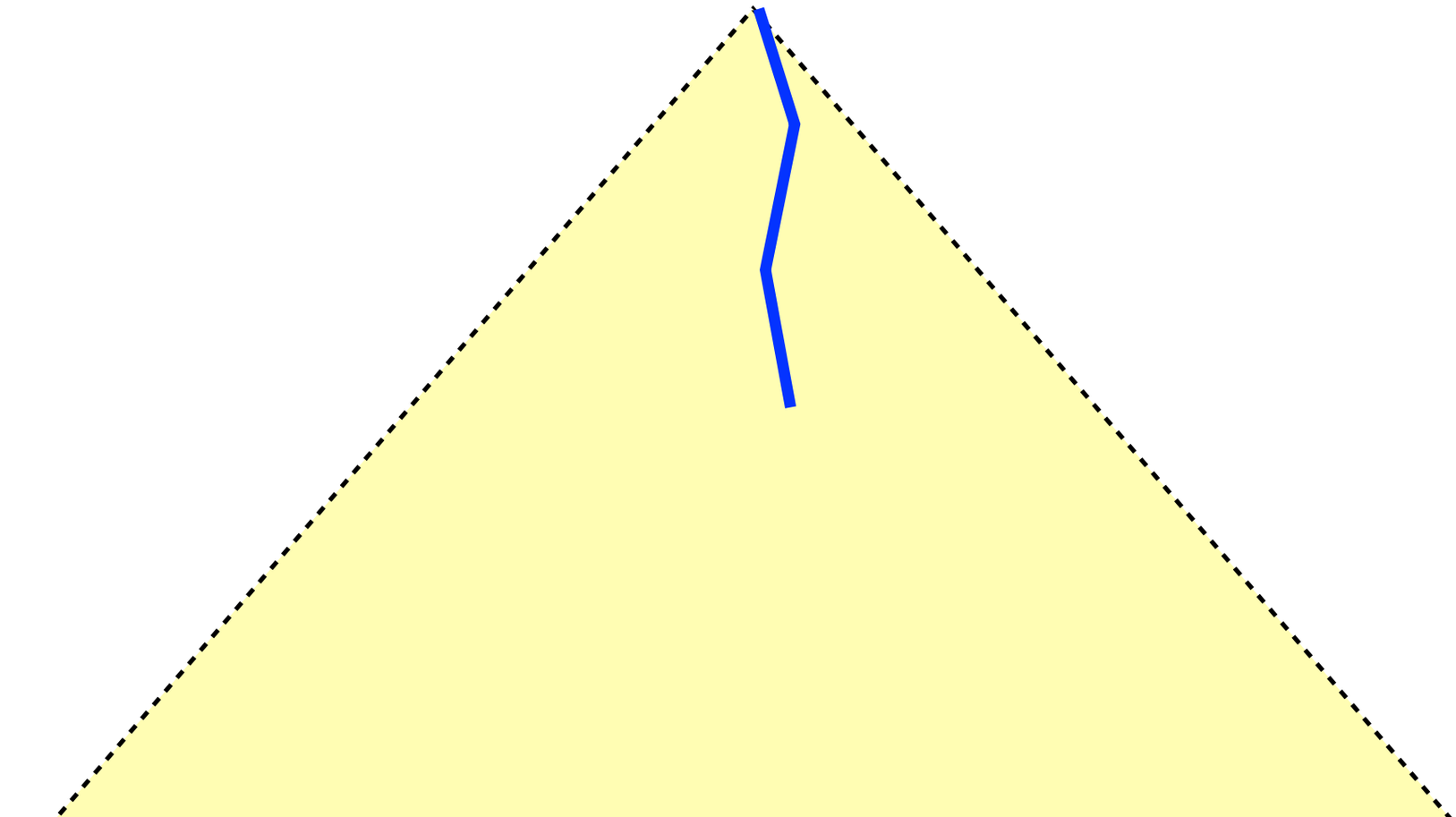
(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$$uk \in T$$

(3) given any position $u \in T$ for Alice, for all integers k

$$uk \in T$$



UNIL | Université de Lausanne

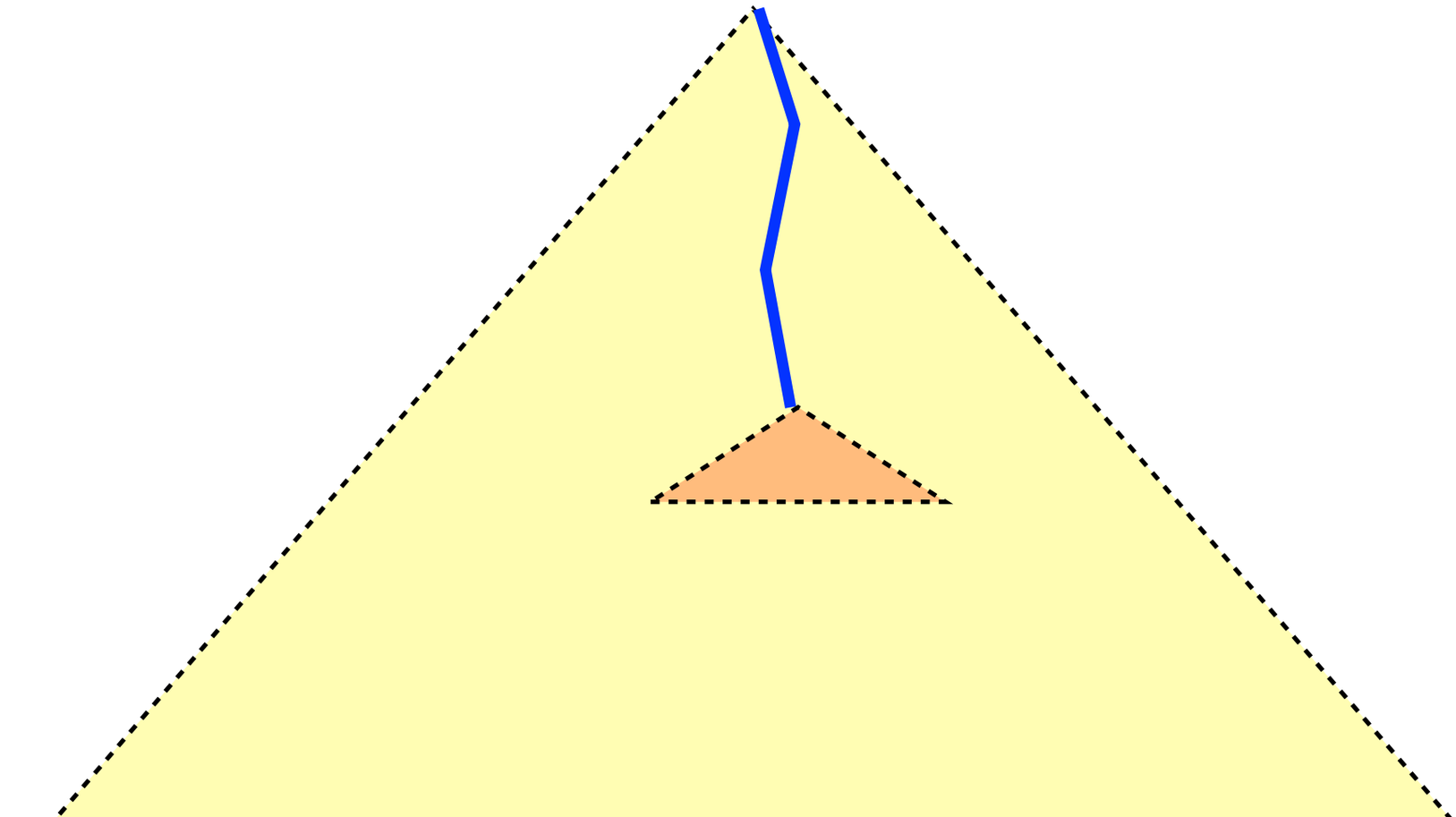
(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$$uk \in T$$

(3) given any position $u \in T$ for Alice, for all integers k

$$uk \in T$$



UNIL | Université de Lausanne

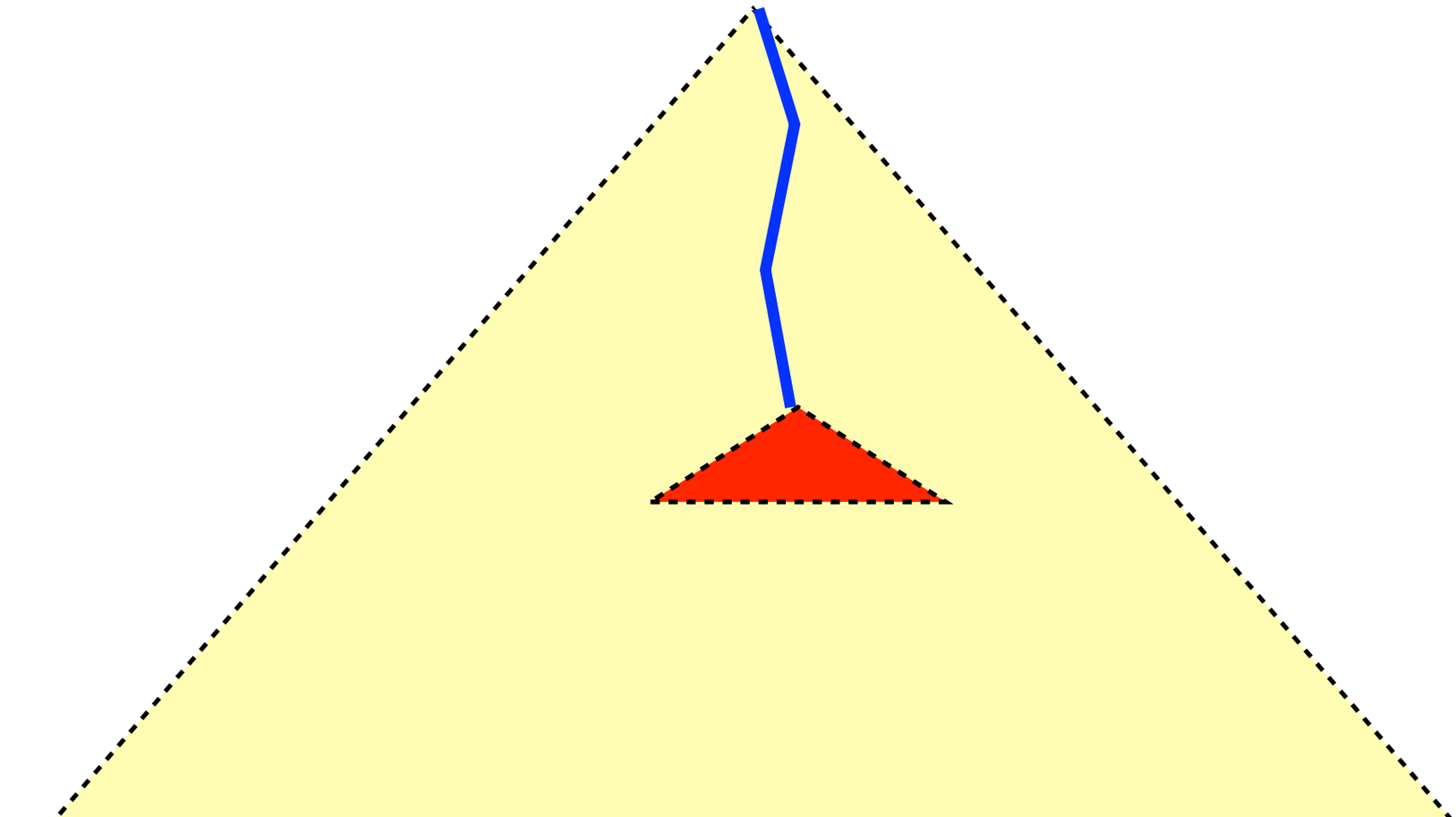
(1) the initial position := $\varepsilon \in T$

(2) given any position $u \in T$ for Bob, there exists an integer k s.t.

$$uk \in T$$

(3) given any position $u \in T$ for Alice, for all integers k

$$uk \in T$$

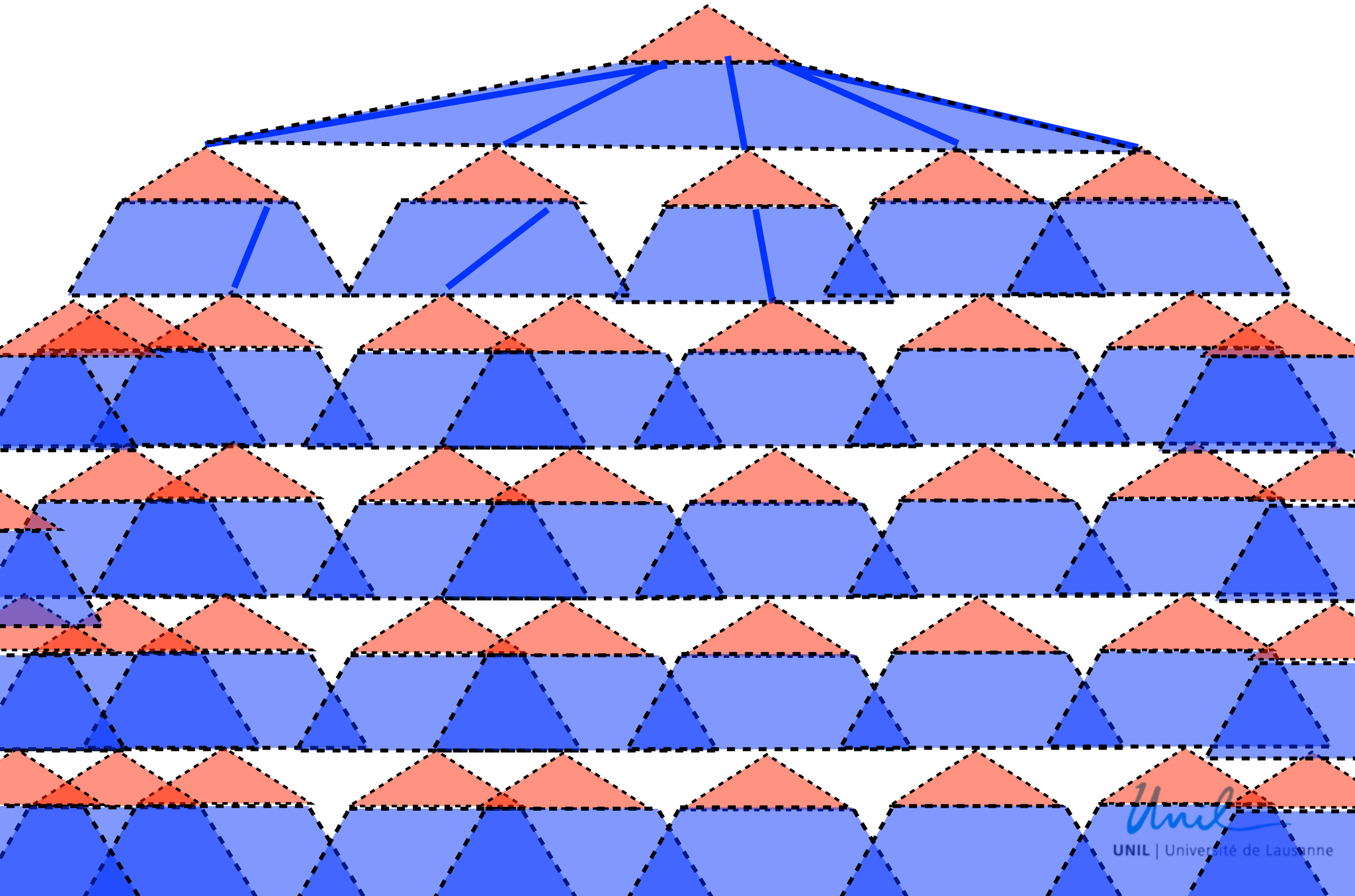


UNIL | Université de Lausanne

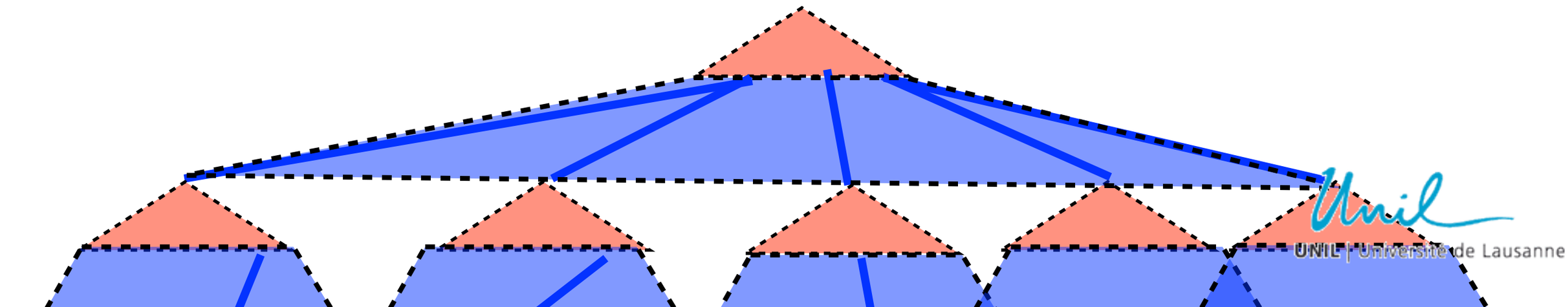


UNIL | Université de Lausanne

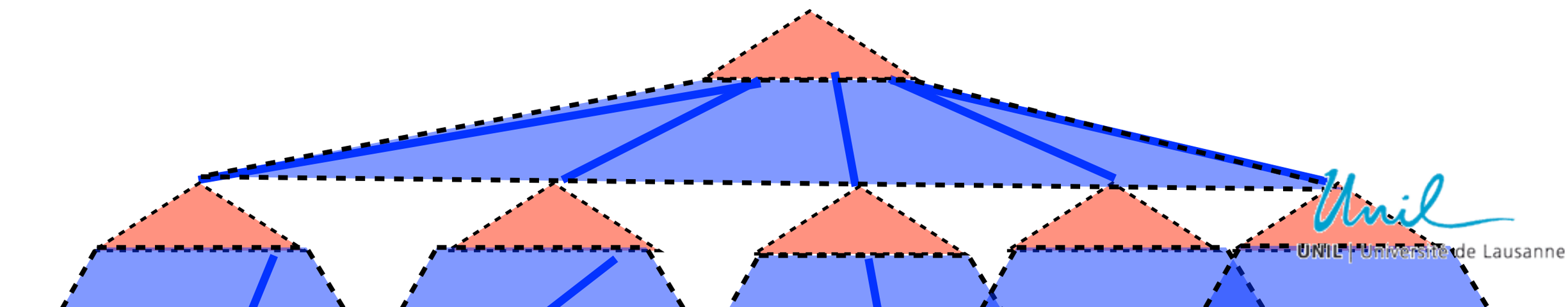
T looks like:



UNIL | Université de Lausanne


UNIL | Université de Lausanne

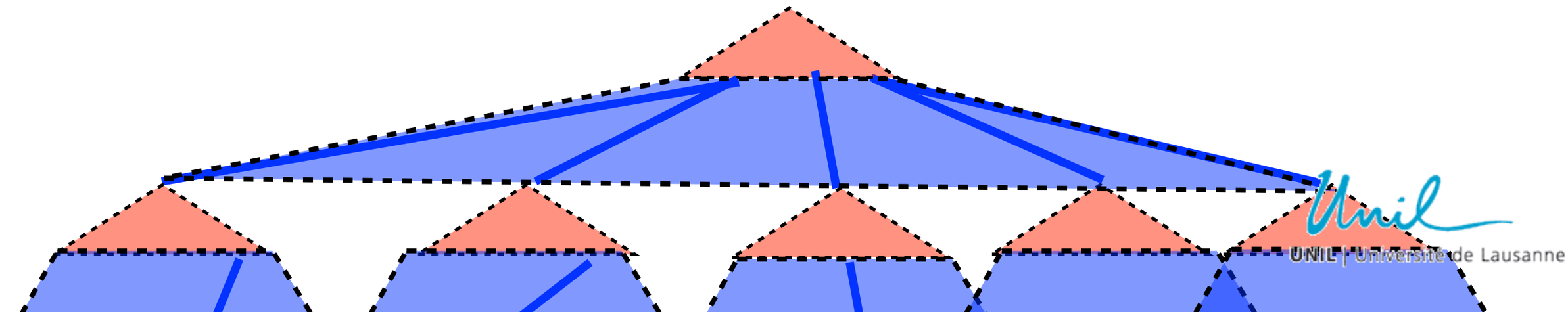
- T is a non-empty pruned tree



UNIL | Université de Lausanne

- T is a non-empty pruned tree

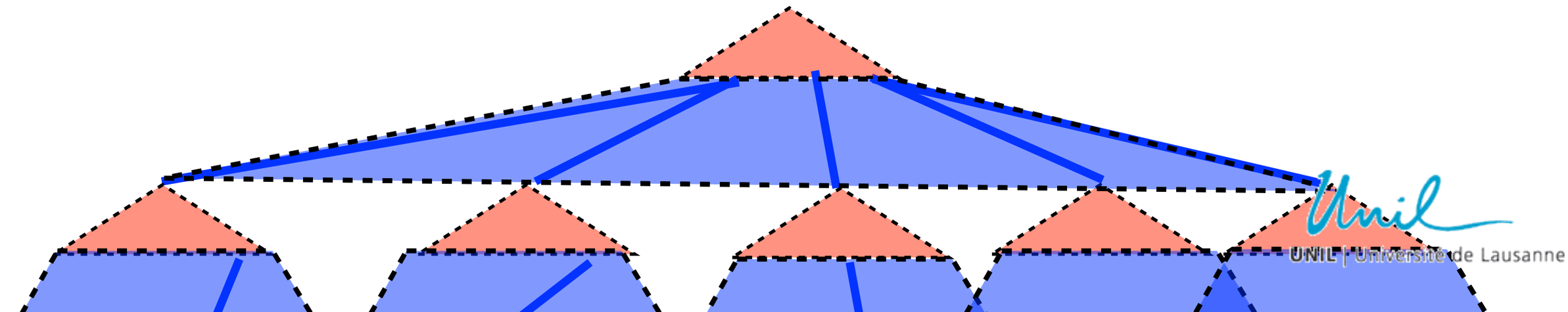
- T is a quasi-strategy for Bob



- T is a non-empty pruned tree

- T is a quasi-strategy for Bob

- with enough choice, we may

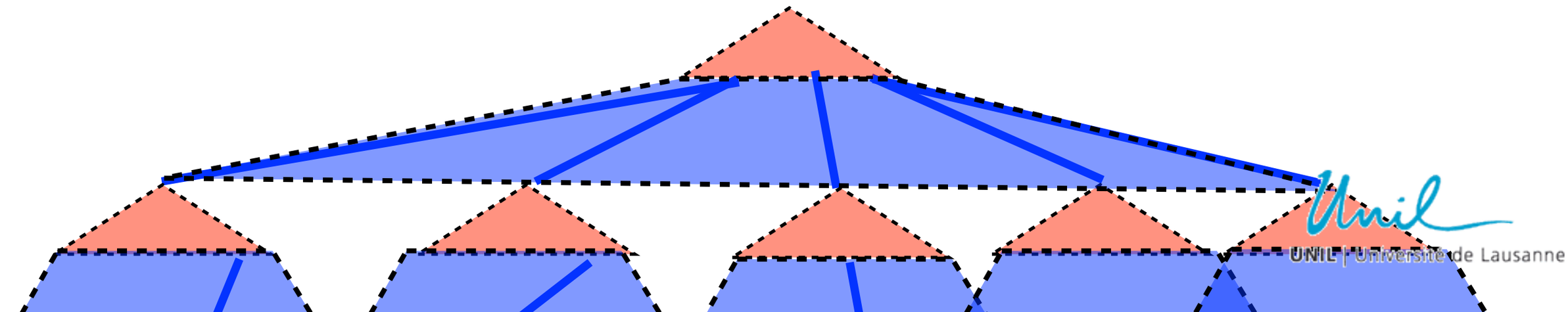


- T is a non-empty pruned tree

- T is a quasi-strategy for Bob

- with enough choice, we may

turn T into a strategy for Bob S



UNIL | Université de Lausanne

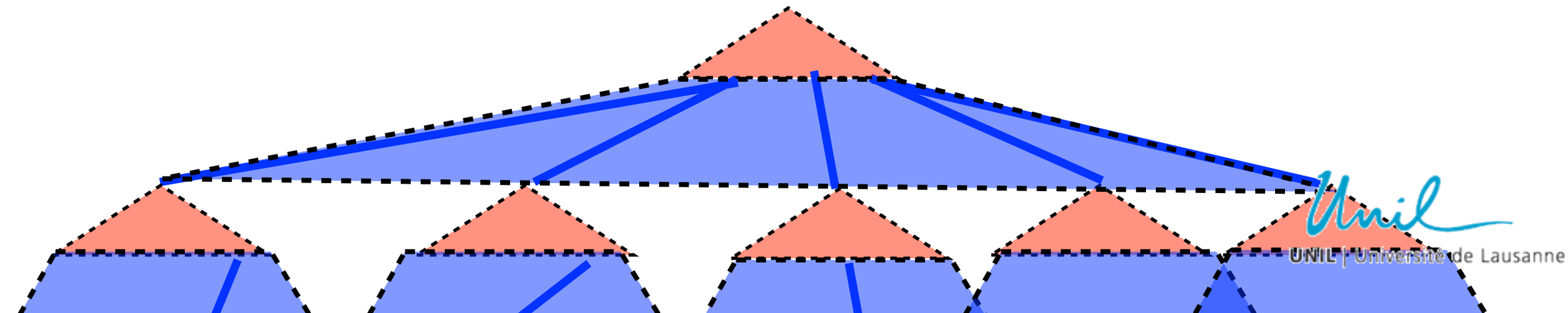
- T is a non-empty pruned tree

- T is a quasi-strategy for Bob

- with enough choice, we may

turn T into a strategy for Bob S

$$S \cap W = \emptyset$$



UNIL | Université de Lausanne

- T is a non-empty pruned tree

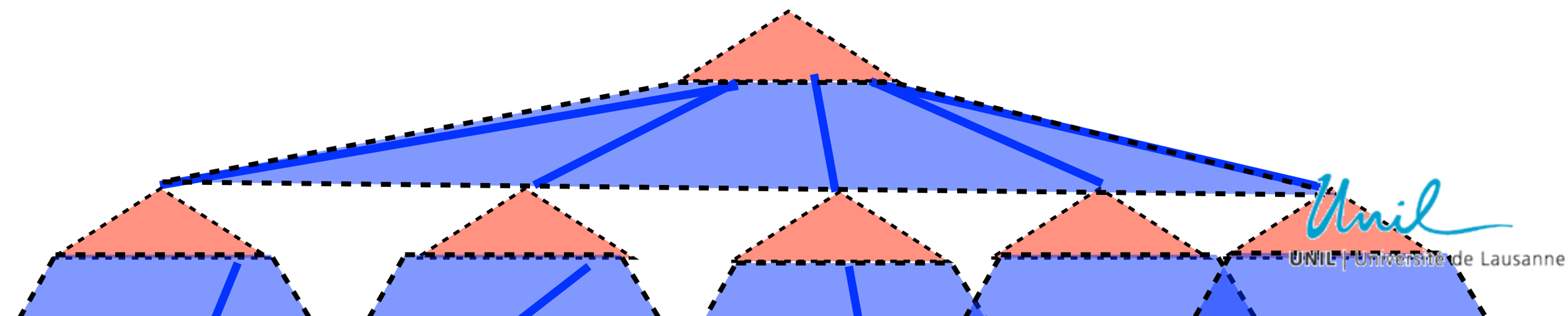
- T is a quasi-strategy for Bob

- with enough choice, we may

turn T into a strategy for Bob S

$$S \cap W = \emptyset$$

hence, S is a w.s. for Bob



UNIL | Université de Lausanne

- T is a non-empty pruned tree

- T is a quasi-strategy for Bob

- with enough choice, we may

turn T into a strategy for Bob S

$$S \cap W = \emptyset$$

hence, S is a w.s. for Bob

Therefore this game is determined!...

UNIL | Université de Lausanne

UNIL | Université de Lausanne

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially 1 we know the winner after finitely many moves (even if the game has infinite duration)

  - essentially 2 we know the winner after finitely many moves, otherwise Bob wins

UNIL | Université de Lausanne

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially 1 Bob wins inside $W\mathbb{N}^\omega$

    Alice wins inside $W\mathbb{N}^\omega$

  - essentially 2 we know the winner after finitely many moves, otherwise Bob wins

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially 1 — Bob wins inside $W\mathbb{N}^\omega$
    Alice wins inside $W\mathbb{N}^\omega$

  - essentially 2 — Bob wins inside $[T]$
    Alice wins inside $W\mathbb{N}^\omega$

Unil
UNIL | Université de Lausanne

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially 1 — Bob wins inside $W\mathbb{N}^\omega$ / Alice wins inside $W\mathbb{N}^\omega$

  - essentially 2 — Bob wins inside $[T]$ / Alice wins inside $W\mathbb{N}^\omega$ — the payoff is CLOSED

UNIL | Université de Lausanne

@ What about infinite duration ?

- A first approach: games that are essentially finite

  - essentially 1 — Bob wins inside $W\mathbb{N}^\omega$ / Alice wins inside $W\mathbb{N}^\omega$

  - essentially 2 — Bob wins inside $[T]$ (the payoff is CLOSED) / Alice wins inside $W\mathbb{N}^\omega$ (the payoff is OPEN)

@ What about infinite duration ?

- A first approach: games that are essentially finite

- essentially 1 — Bob wins inside $W\mathbb{N}^\omega$ — the payoff (payoff) are CLOPEN
  Alice wins inside $W\mathbb{N}^\omega$

- essentially 2 — Bob wins inside $[T]$ — the payoff is CLOSED
  Alice wins inside $W\mathbb{N}^\omega$ — the payoff is OPEN



UNIL | Université de Lausanne

The product topology of the discrete topology over $\mathbb{N}$

UNIL | Université de Lausanne

The product topology of the discrete topology over $\mathbb{N}$ turns $\mathbb{N}^{\omega}$ into the Baire Space

UNIL | Université de Lausanne

The product topology of the discrete topology over $\mathbb{N}$

turns $\mathbb{N}^{\omega}$ into the Baire Space

@ homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$

UNIL | Université de Lausanne

The product topology of the discrete topology over $\mathbb{N}$ turns $\mathbb{N}^{\omega}$ into the Baire Space

UNIL | Université de Lausanne

The product topology of the discrete topology over $\mathbb{N}$ turns $\mathbb{N}^\omega$ into the Baire Space

- A set is open if it is of the form $W\mathbb{N}^\omega$ for some $W \subseteq \mathbb{N}^*$

Unil
UNIL | Université de Lausanne

The product topology of the discrete topology over $\mathbb{N}$

turns $\mathbb{N}^\omega$ *into the Baire Space*

- A set is open if it is of the form $W\mathbb{N}^\omega$ for some $W \subseteq \mathbb{N}^*$

(we may get in but once we're in we're trapped !)

The product topology of the discrete topology over $\mathbb{N}$

turns $\mathbb{N}^\omega$ *into the Baire Space*

- A set is open if it is of the form $W\mathbb{N}^\omega$ for some $W \subseteq \mathbb{N}^*$

(*we may get in but once we're in we're trapped !*)

- A set is closed if it is of the form $[T]$ for some tree $T \subseteq \mathbb{N}^*$
  if it is the complement of an open set

The product topology of the discrete topology over $\mathbb{N}$

turns $\mathbb{N}^\omega$ *into the Baire Space*

- A set is open if it is of the form $W\mathbb{N}^\omega$ for some $W \subseteq \mathbb{N}^*$

(*we may get in but once we're in we're trapped !*)

- A set is closed if it is of the form $[T]$ for some tree $T \subseteq \mathbb{N}^*$
  if it is the complement of an open set

(*we may leave it, but if we do, we don't come back !*)

- A first approach: games that are essentially finite

  - essentially ^{1} = we know the winner after finitely many moves (even if the game has infinite duration)

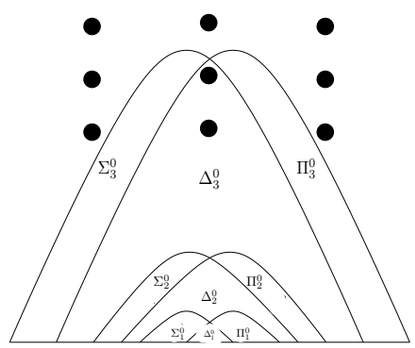
  - essentially ^{2} = we know the winner after finitely many moves, otherwise Bob wins

UNIL | Université de Lausanne

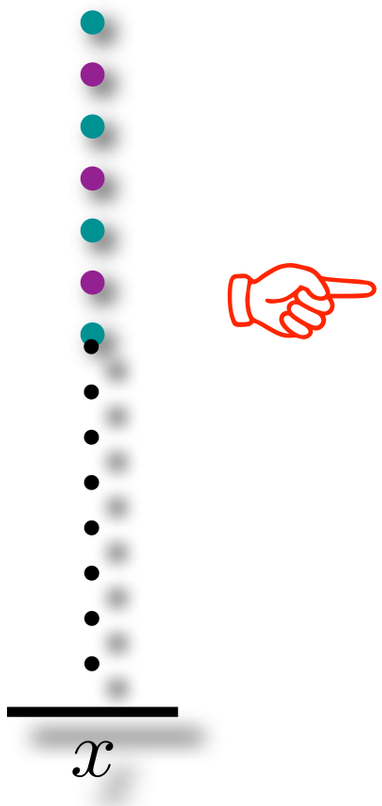


An example: evaluation game for 1st order logic

non determined game

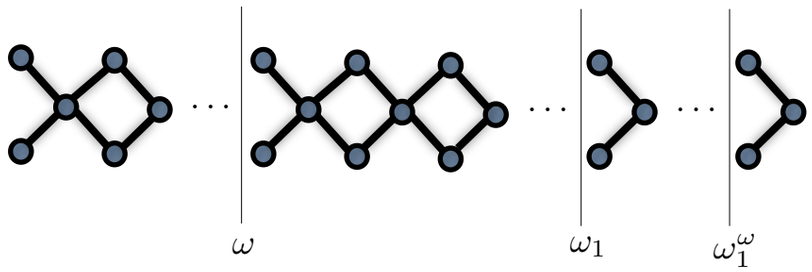


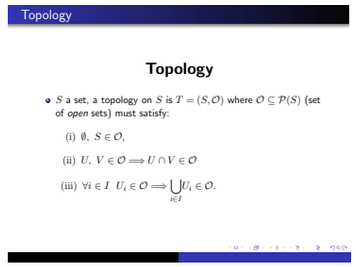
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

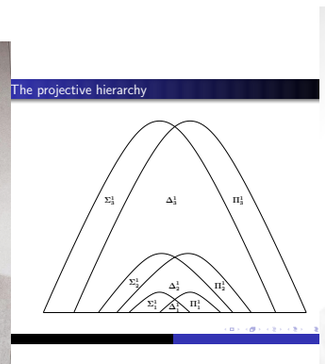
Games that construct mappings



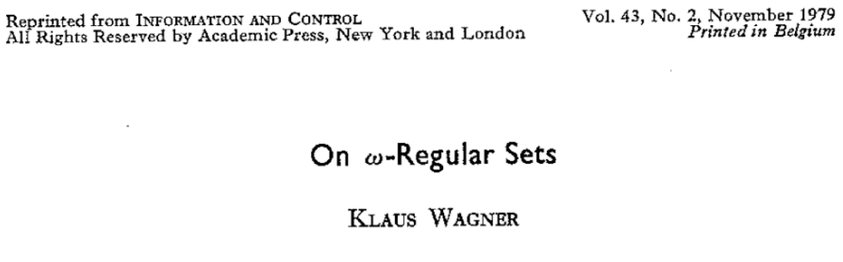










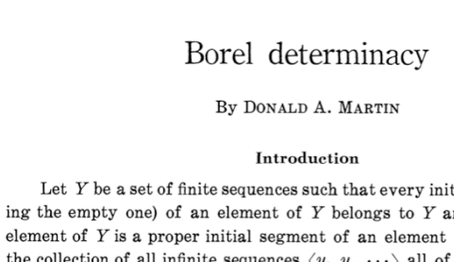


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

There exists a non determined game

There exists a non determined game

there exists $A \subset \mathbb{N}^\omega$ s.t.

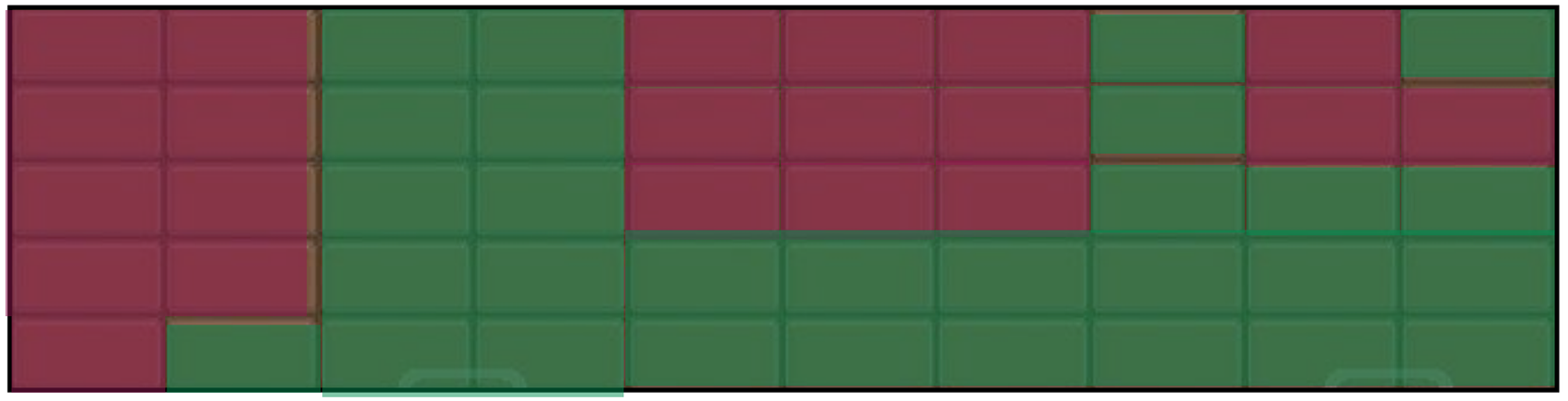
neither Alice nor Bob has a w.s. in $\mathcal{G}(A)$

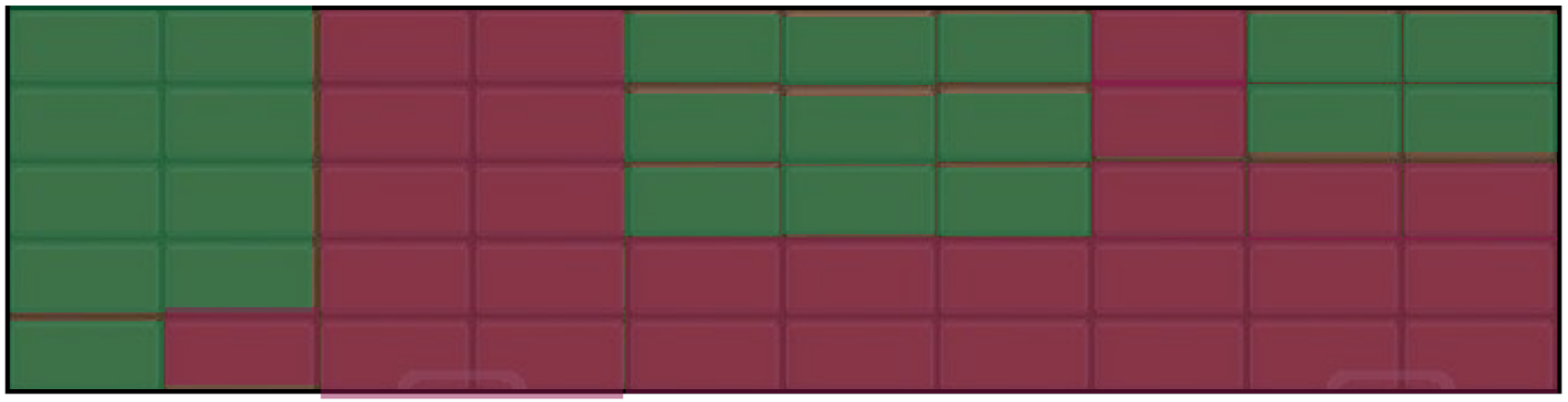
@ *different approaches*

(1)
- *enumerate all strategies for Alice ($2^{\aleph_0}$ many)*
- *enumerate all strategies for Bob ($2^{\aleph_0}$ many)*
- *enumerate all $x \in \mathbb{N}^\omega$ ($2^{\aleph_0}$ many)*
- *by transfinite induction and a diagonal argument, construct payoff A s.t. neither Bob nor Alice has a w.s.*

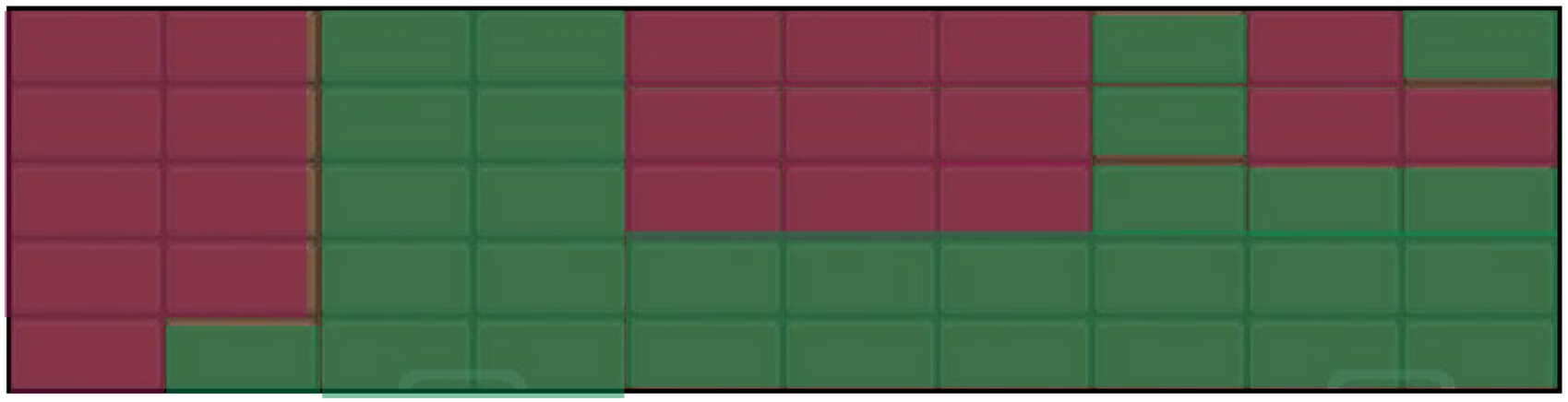
(2) - *similar to the swiss chocolate-game ?*

UNIL | Université de Lausanne





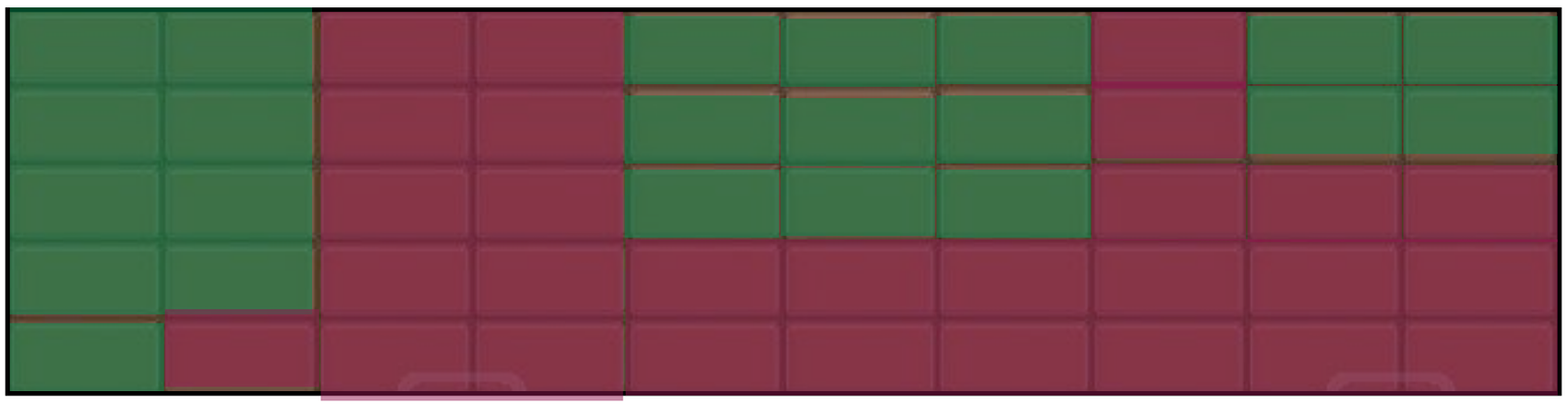
UNIL | Université de Lausanne



@ Except for the first piece of chocolate:

Green ↑ = Red ↓ and Green ↓ = Red ↑ therefore

Bob wins ↓ ⇒ Alice wins ↑



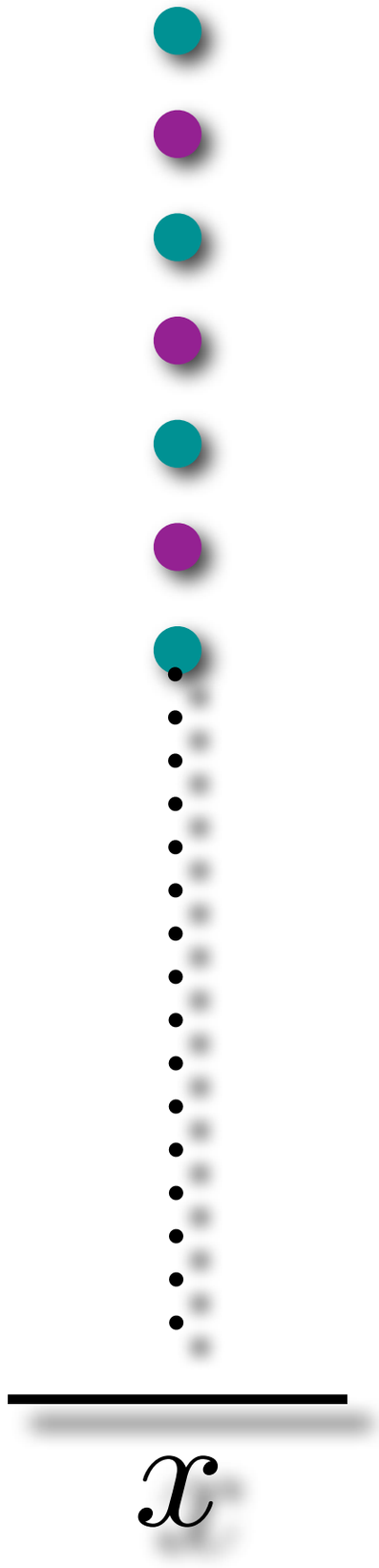
UNIL | Université de Lausanne

UNIL | Université de Lausanne

$\mathcal{G}(A)$

Alice

Bob



x

UNIL | Université de Lausanne

UNIL | Université de Lausanne

$\mathcal{G}(A)$



$\mathcal{G}(A)$



Alice has a w.s. $\Longrightarrow$ Bob has a w.s.

UNIL | Université de Lausanne

$\mathcal{G}(A)$

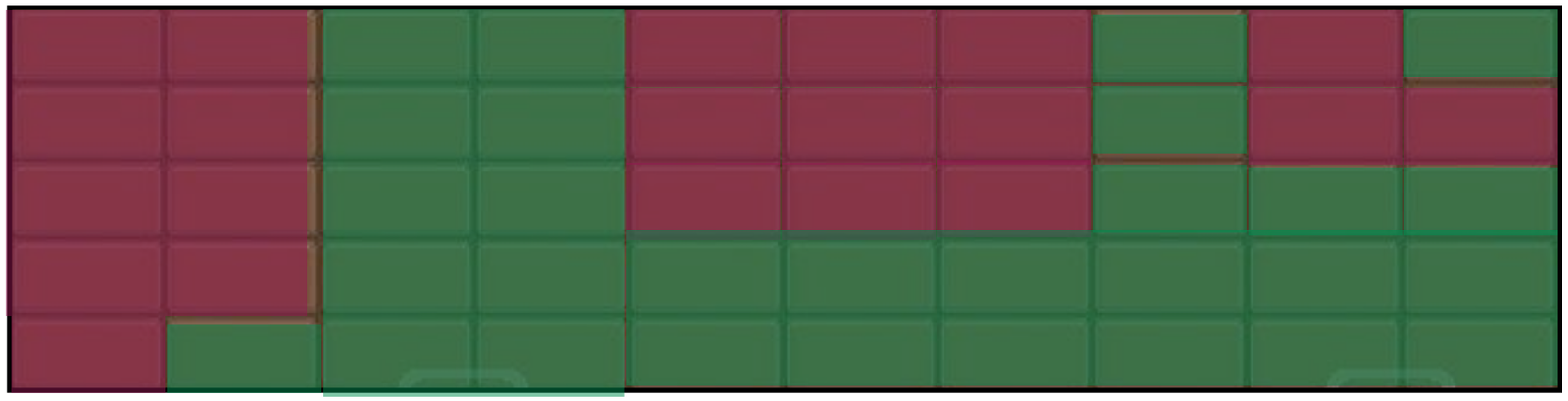


$\mathcal{G}(A)$

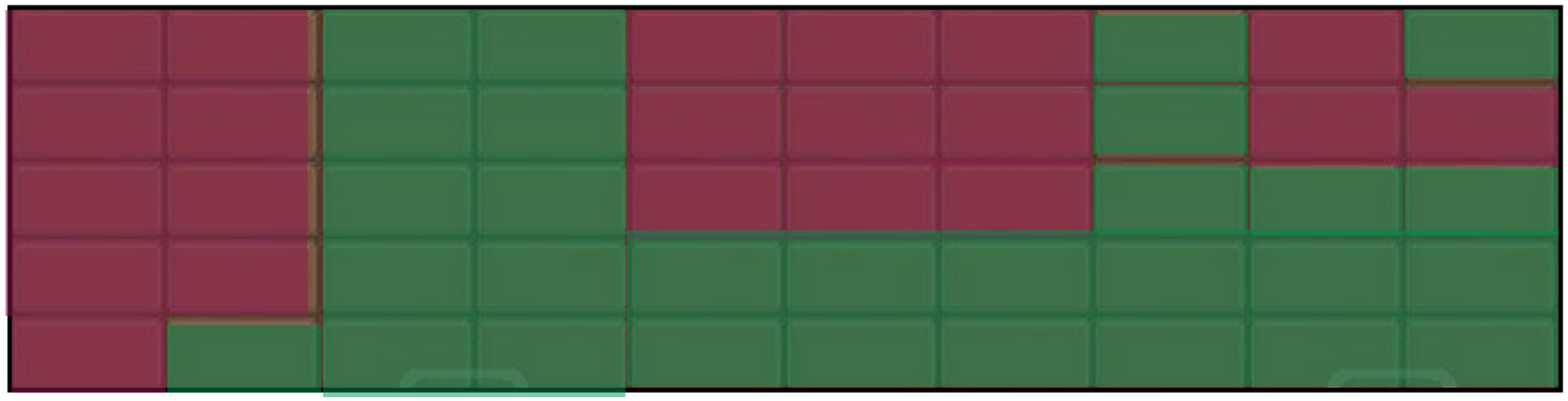


Bob has a w.s. $\Longrightarrow$ Alice has a w.s.

UNIL | Université de Lausanne

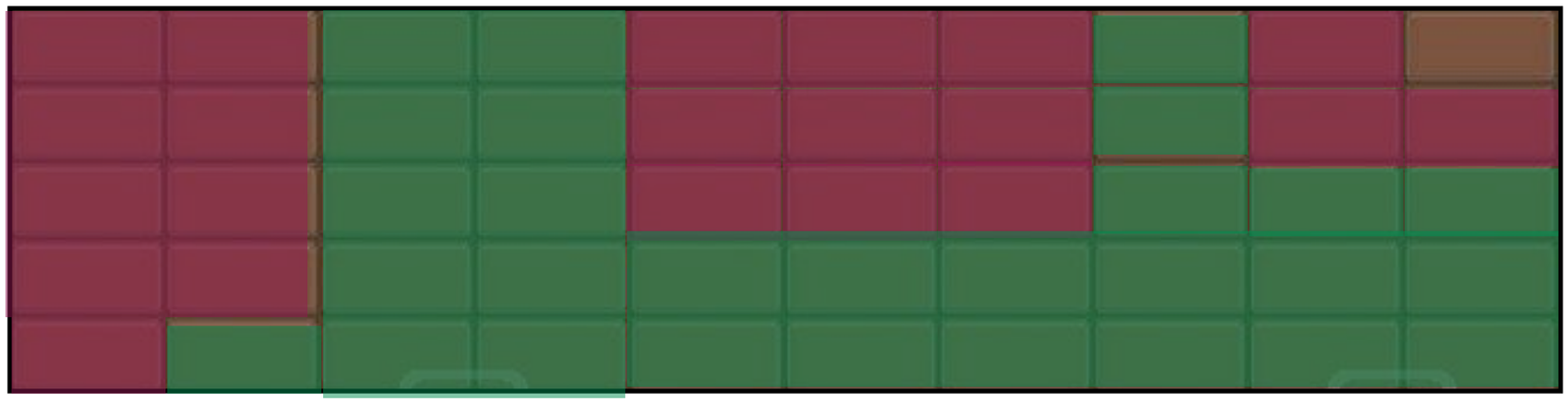


UNIL | Université de Lausanne



Alice's =

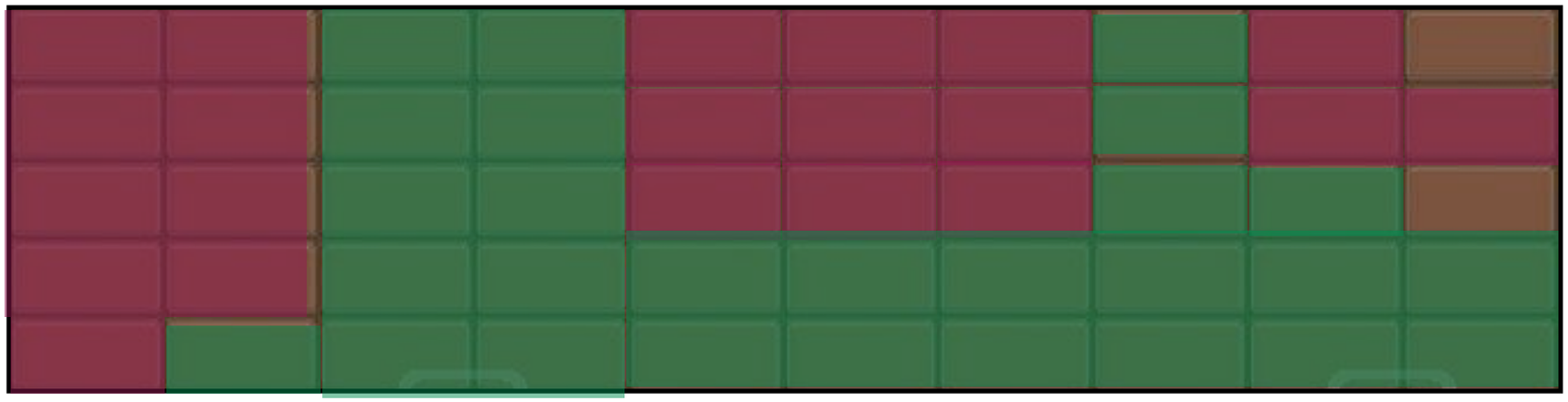
Bob's =



Alice's =

Bob's =

UNIL | Université de Lausanne

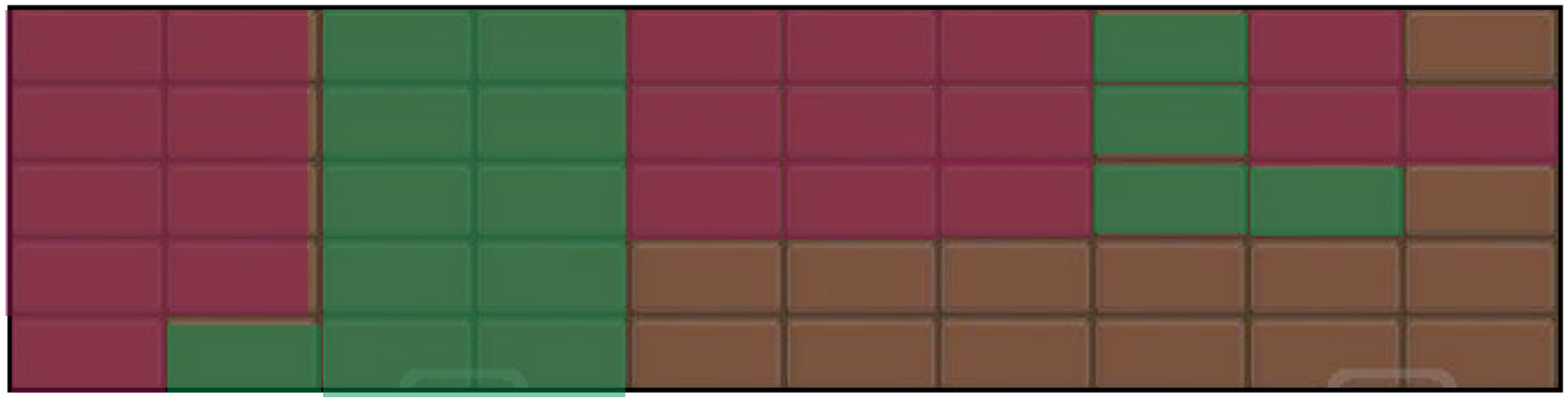


Alice's =

Bob's =



UNIL | Université de Lausanne



Alice's =

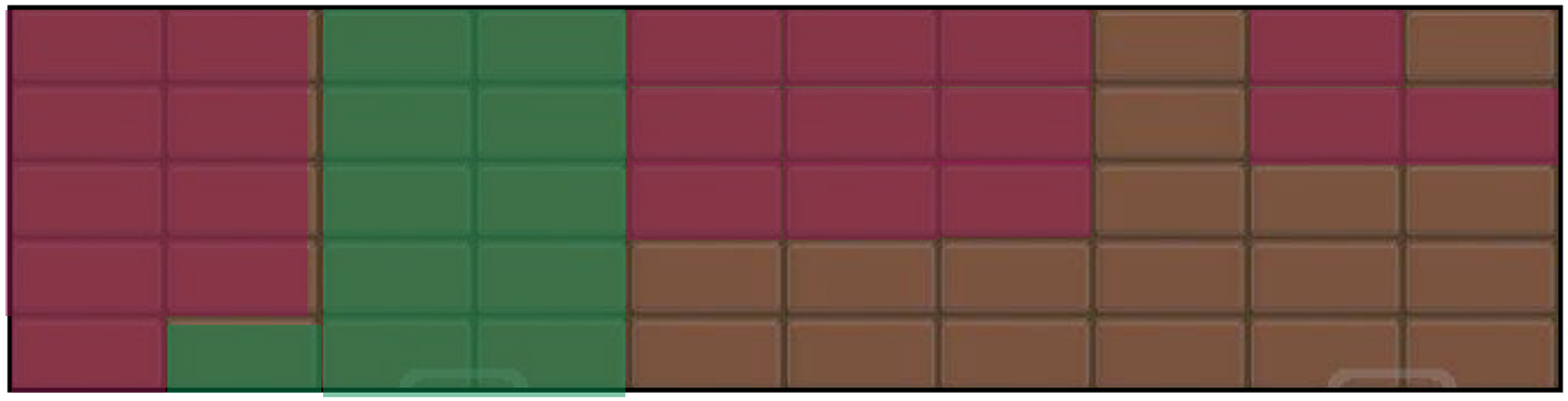




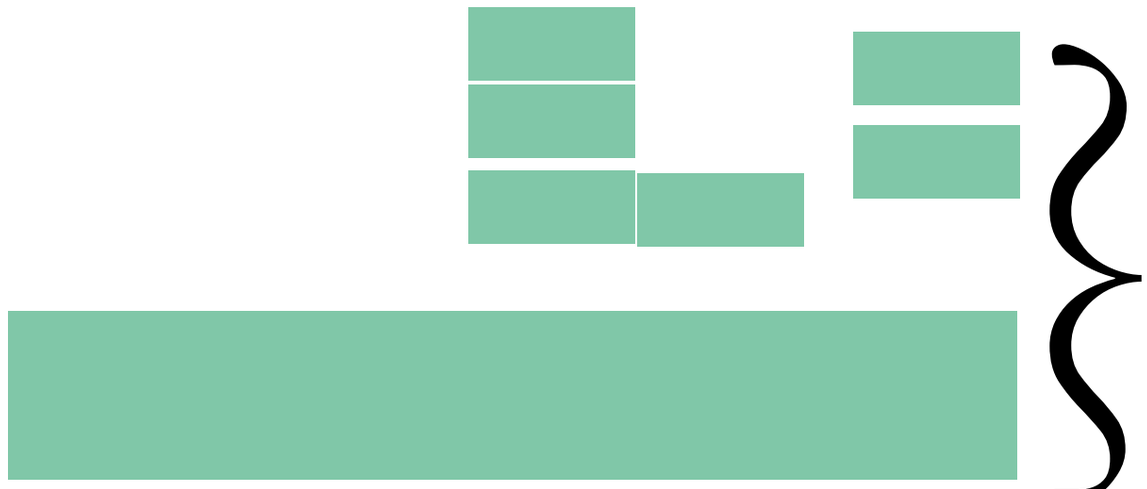
Bob's =



UNIL | Université de Lausanne



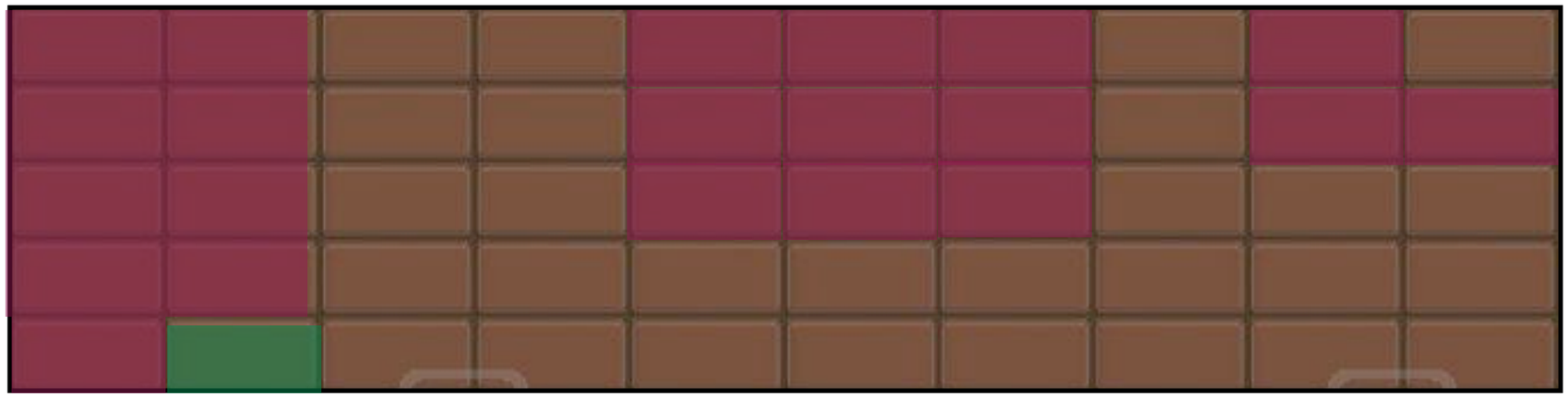
Alice's =



Bob's =



UNIL | Université de Lausanne



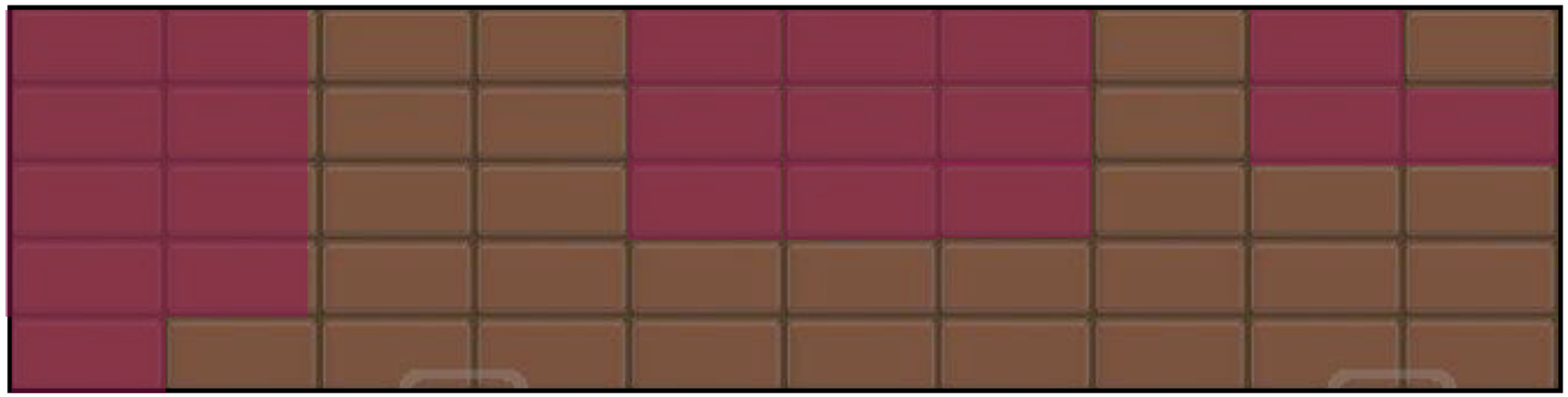
Alice's =



Bob's =



UNIL | Université de Lausanne



Alice's = {

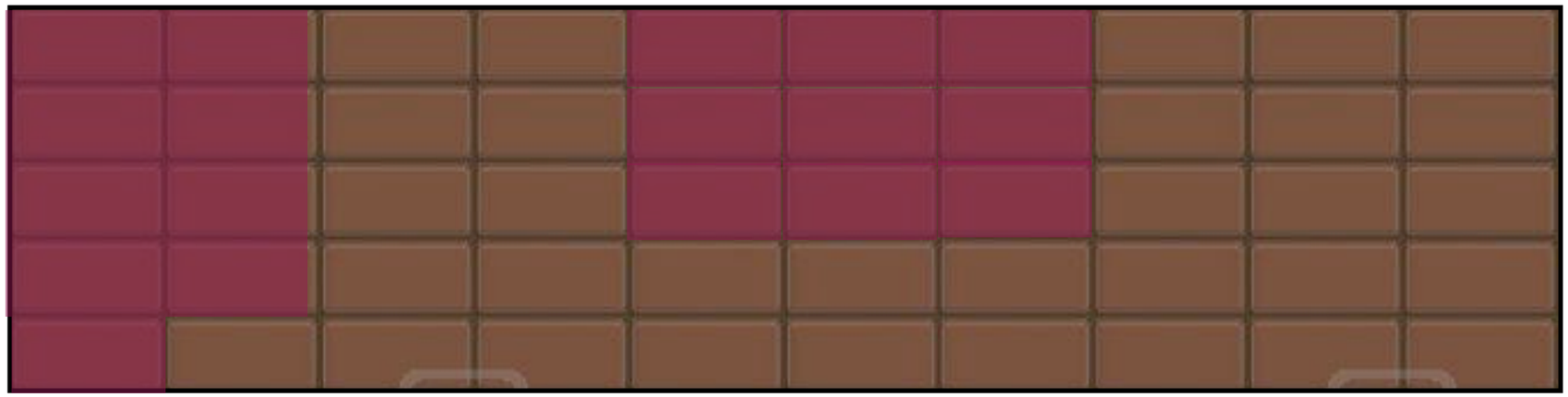


}

Bob's = {

}

Unil
UNIL | Université de Lausanne



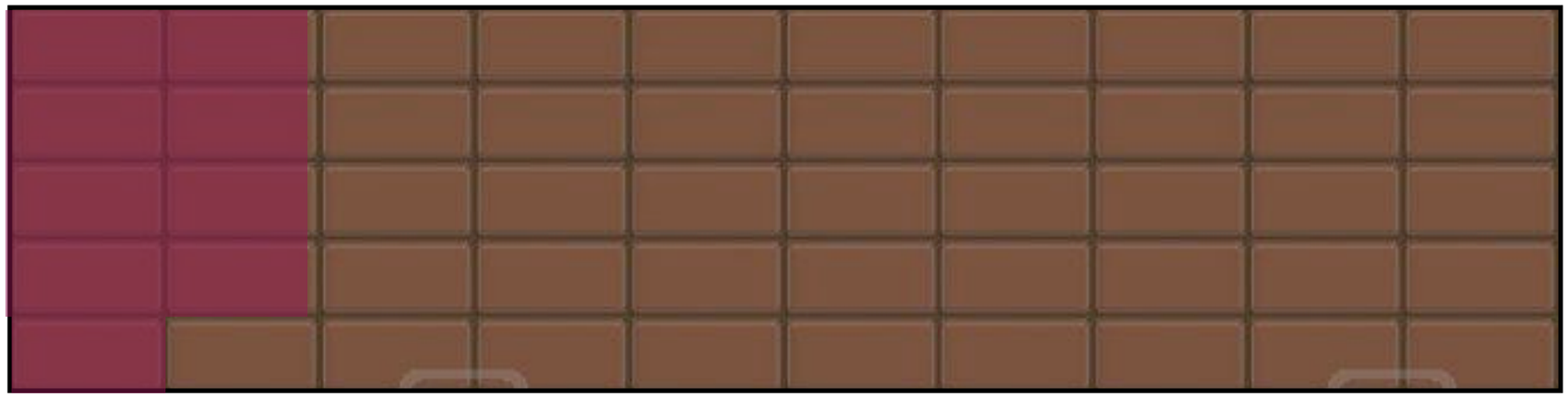
Alice's =



Bob's =



UNIL | Université de Lausanne



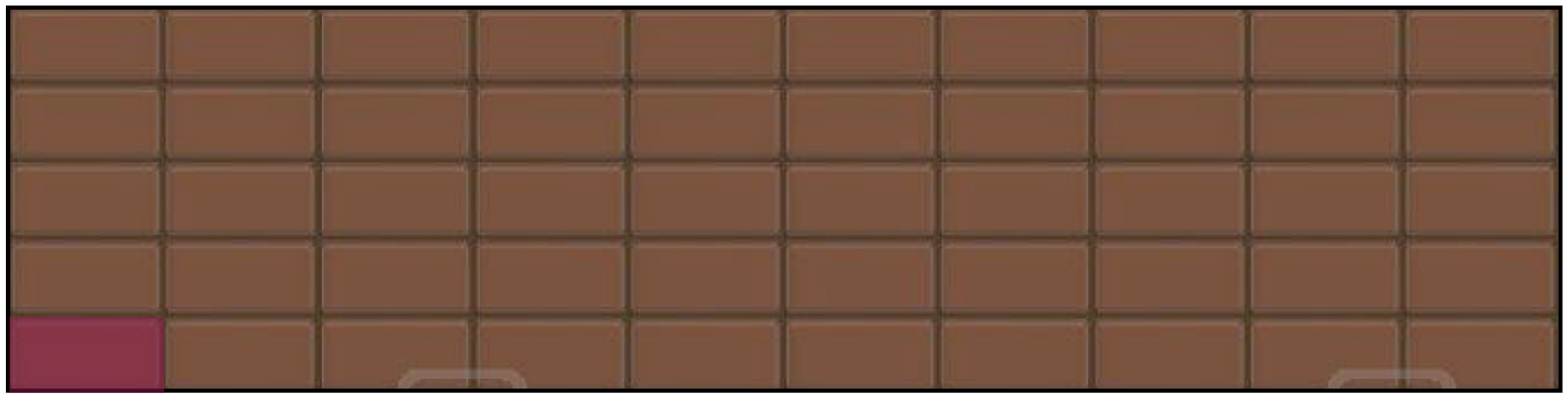
Alice's =



Bob's =



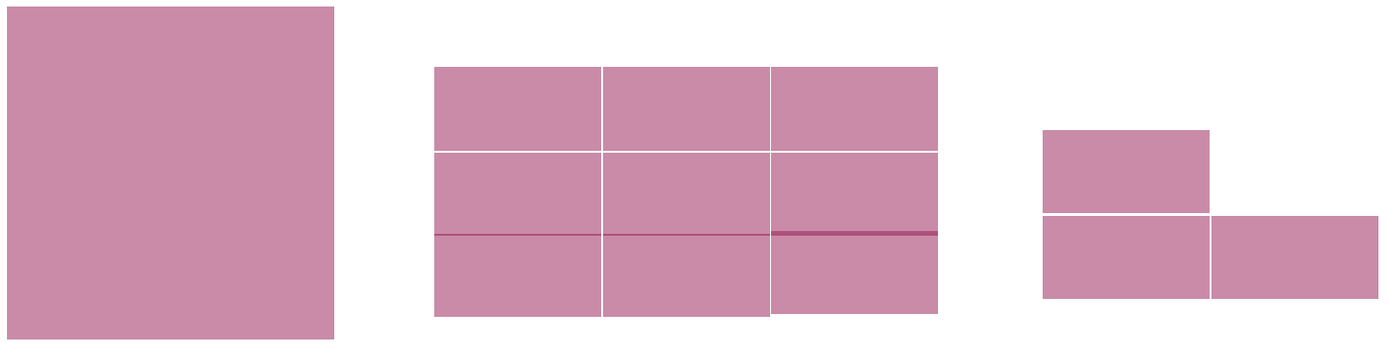
UNIL | Université de Lausanne



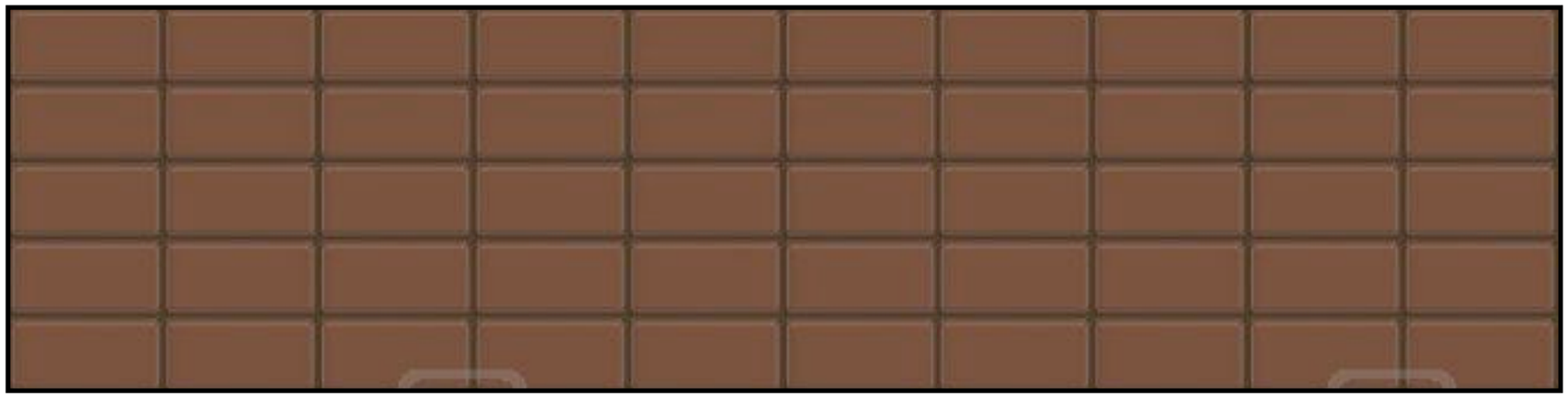
Alice's =



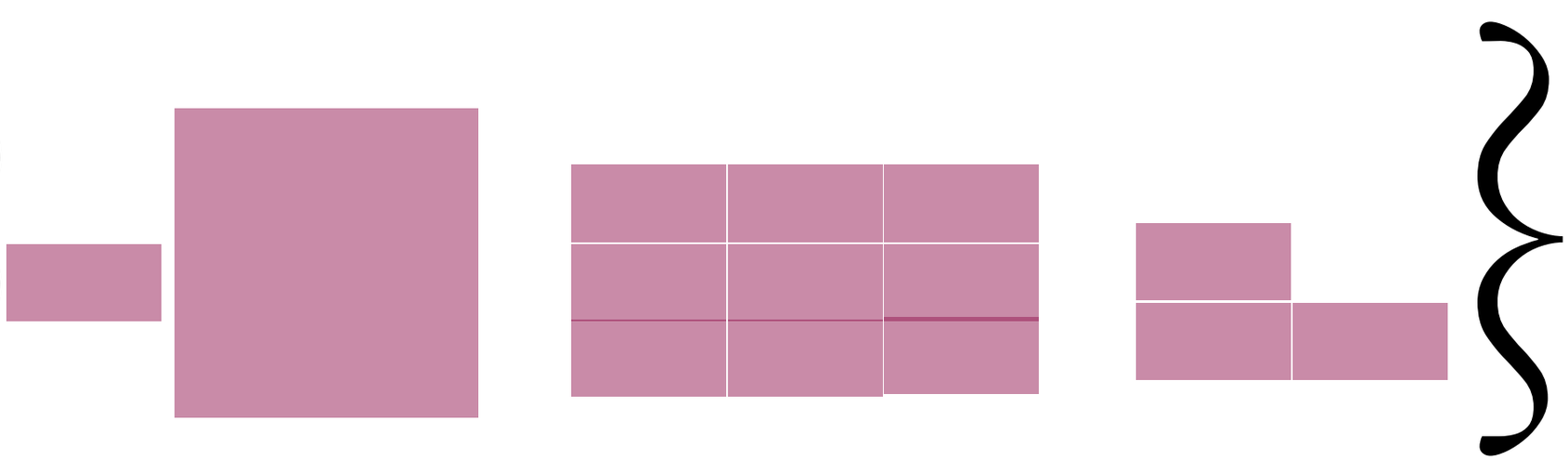
Bob's =



UNIL | Université de Lausanne



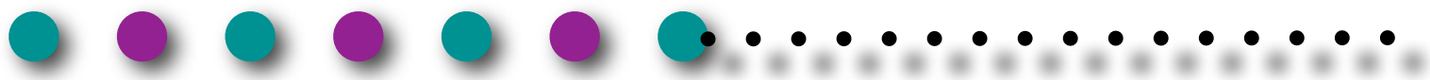
Alice's = 

Bob's = 

UNIL | Université de Lausanne

UNIL | Université de Lausanne

Alice



Bob

UNIL | Université de Lausanne

Alice

3 4 6 9 10 13 15 . . . . . strictly ↗

Bob

UNIL | Université de Lausanne

Alice

Bob

Alice

Bob

Alice's = {

Bob's = {

}

UNIL | Université de Lausanne

Alice

3

Bob

Alice's = { 0 1 2 3 }

Bob's = { }

Unil
UNIL | Université de Lausanne

Alice

3 4

Bob

Alice's = { 0 1 2 3 }

Bob's = { 4 }

Unil
UNIL | Université de Lausanne

Alice

3 4 6

Bob

Alice's = { 0 1 2 3 5 6 }

Bob's = { 4 }

UNIL | Université de Lausanne

Alice

3 4 6 9

Bob

Alice's = { 0 1 2 3 5 6 }

Bob's = { 4 7 8 9 }

UNIL | Université de Lausanne

Alice

3 4 6 9 10

Bob

Alice's = { 0 1 2 3 5 6 10 }

Bob's = { 4 7 8 9 11 12 }

Alice

3 4 6 9 10 13

Bob

Alice's = { 0 1 2 3 5 6 10 }

Bob's = { 4 7 8 9 11 12 13 }

Unil
UNIL | Université de Lausanne

Alice

3 4 6 9 10 13 15 ..... strictly ↗

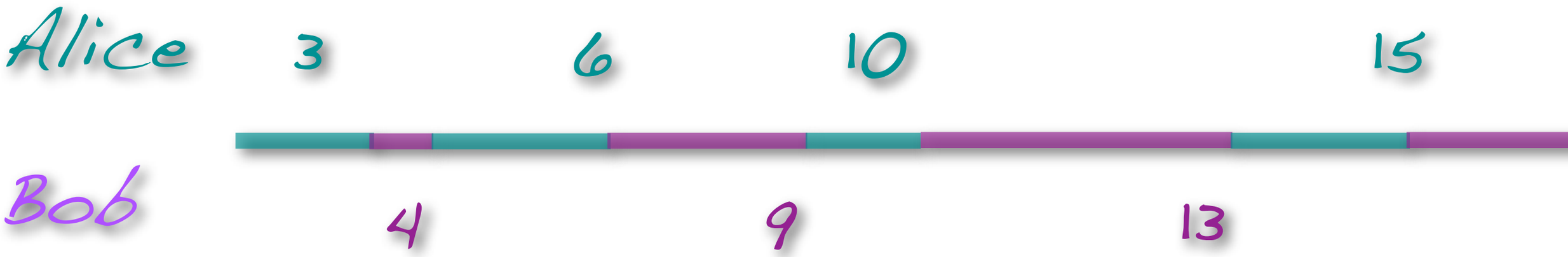
Bob

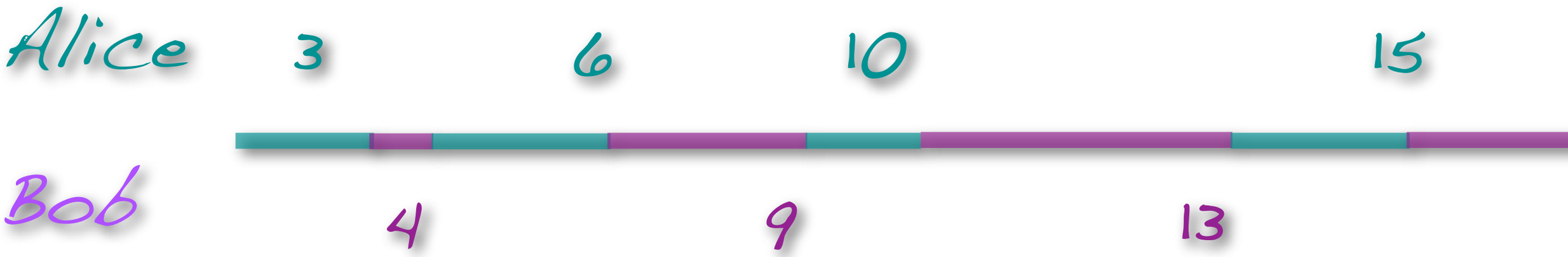
Alice's = { 0 1 2 3 5 6 10 14 15 }

Bob's = { 4 7 8 9 11 12 13 }

UNIL | Université de Lausanne

UNIL | Université de Lausanne

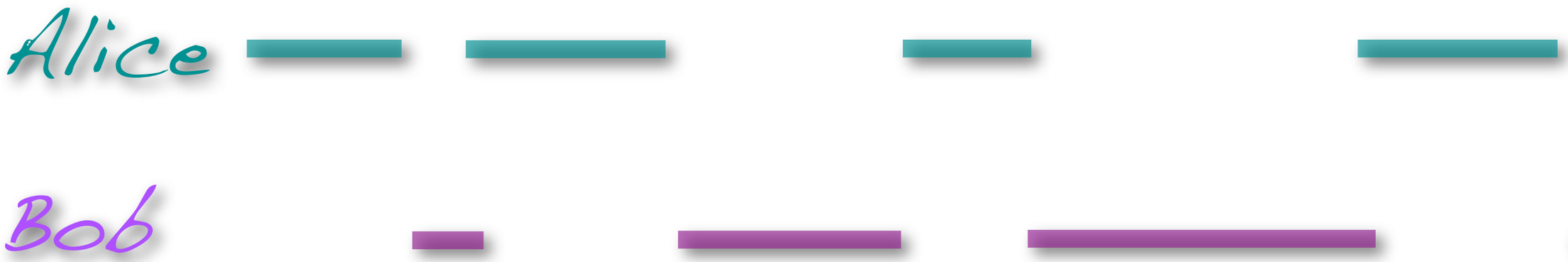

Alice
3
6
10
15
Bob
4
9
13

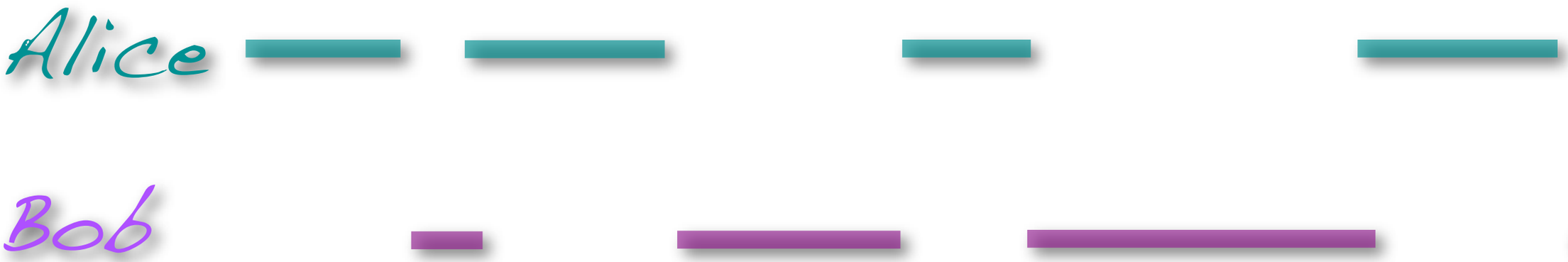

Alice
3
6
10
15
Bob
4
9
13

Alice



Bob


Alice
Bob



$\Leftrightarrow$

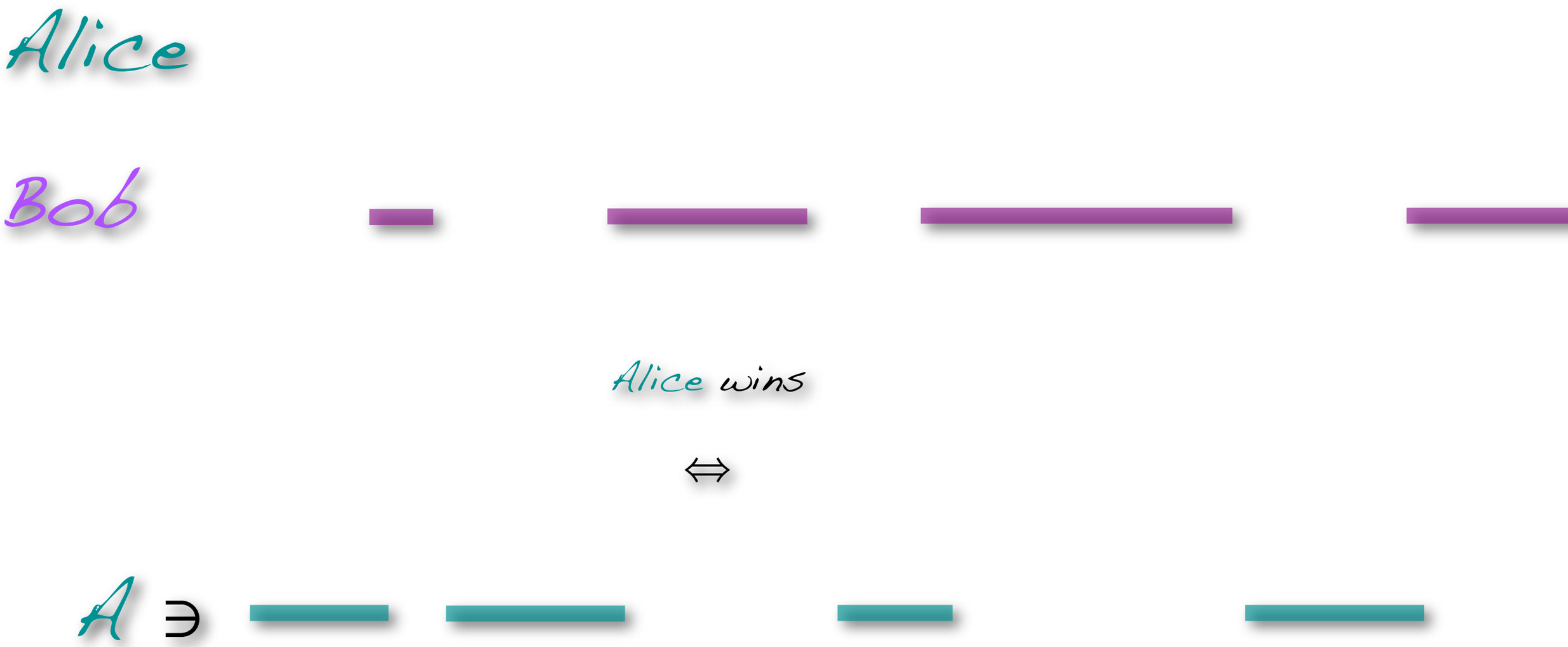
UNIL | Université de Lausanne



$\Leftrightarrow$

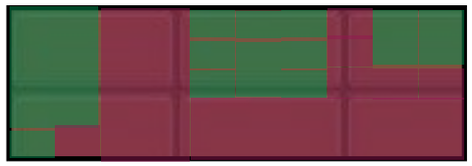


UNIL | Université de Lausanne

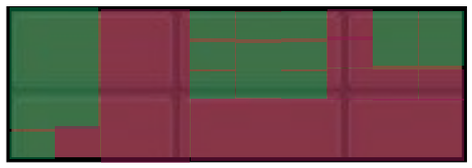

Alice
Bob
Alice wins
⇔
A ∋

(o) $\mathcal{A} \in \mathcal{P}(N^\omega)$

(0) $\mathcal{A} \in \mathcal{P}(N^\omega)$

mimicking   requires:

(0) $\mathcal{A} \in \mathcal{P}(N^\omega)$

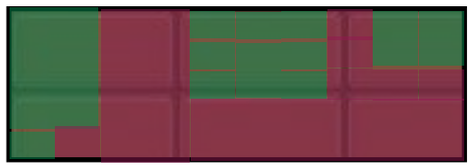
mimicking   requires:

(1) $\mathcal{A} \ni$ 

$\Leftrightarrow$

$\mathcal{A} \not\ni$ 

(0) $\mathcal{A} \in \mathcal{P}(N^\omega)$

mimicking   requires:

(1) $\mathcal{A}$ ∋ 

⇔

$\mathcal{A}$ ∌ 

(2) $\mathcal{A}$ ∋ 

⇒

$\mathcal{A}$ ∋ 

⇒

$\mathcal{A}$ ∋ 

UNIL | Université de Lausanne

If BoB has a w.s...

UNIL | Université de Lausanne

UNIL | Université de Lausanne

Assume BoB applies a w.s....

Alice

Bob

Assume BoB applies a w.s....

Alice

Bob

Alice plays against Bob....

Alice

Bob

Assume BoB applies a w.s....

Alice 3



Bob 4

Alice plays against Bob....

Alice

Bob

UNIL | Université de Lausanne

Assume BoB applies a w.s....



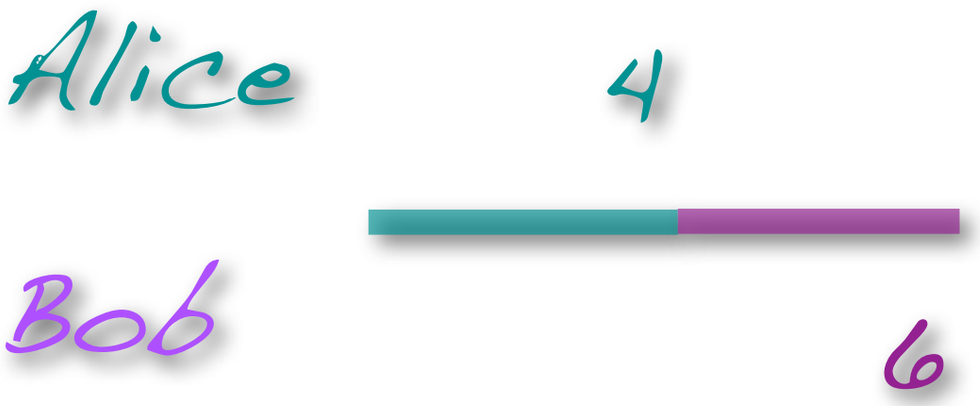
Alice plays against Bob....



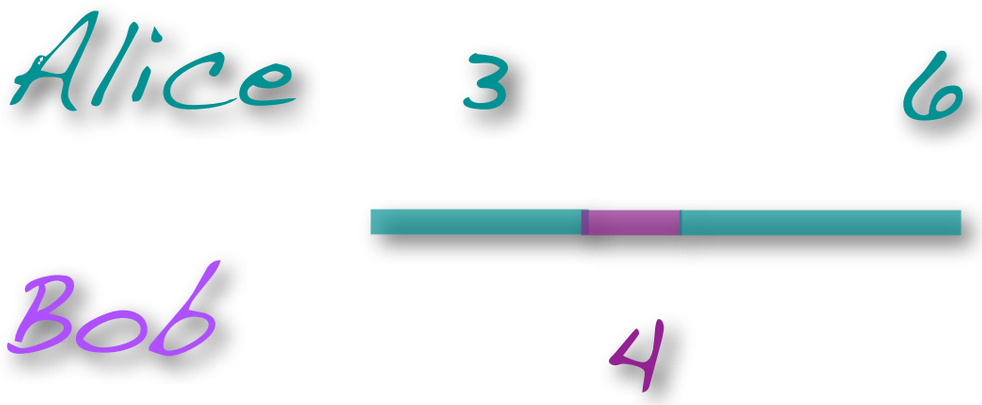
Assume BoB applies a w.s....



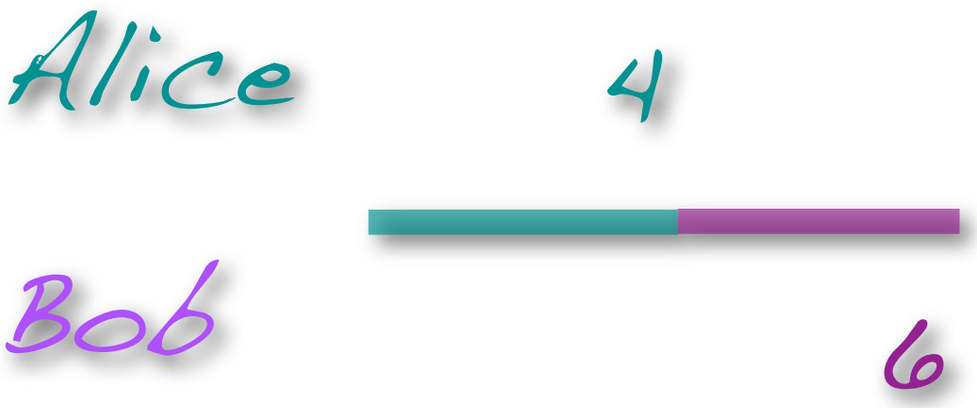
Alice plays against Bob....



Assume BoB applies a w.s....

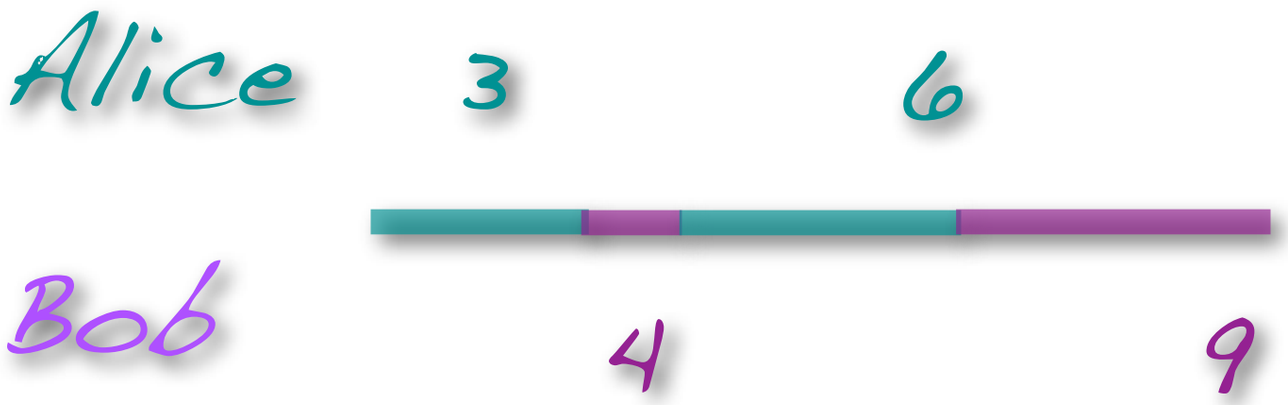


Alice plays against Bob....

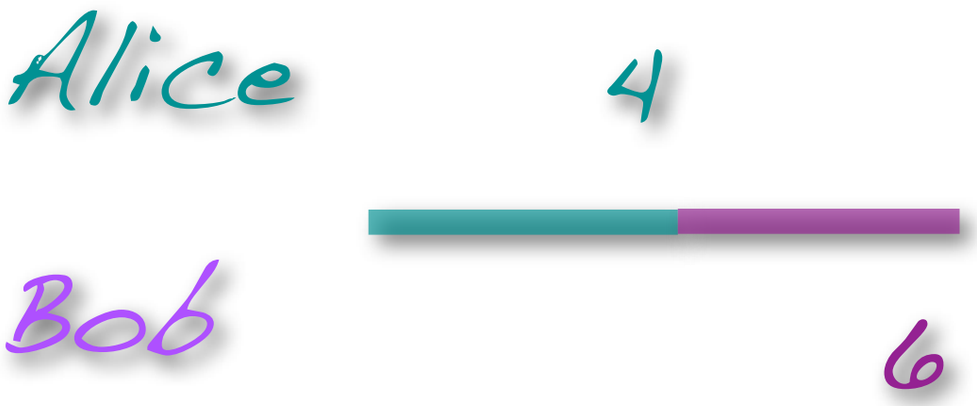


UNIL | Université de Lausanne

Assume BoB applies a w.s....

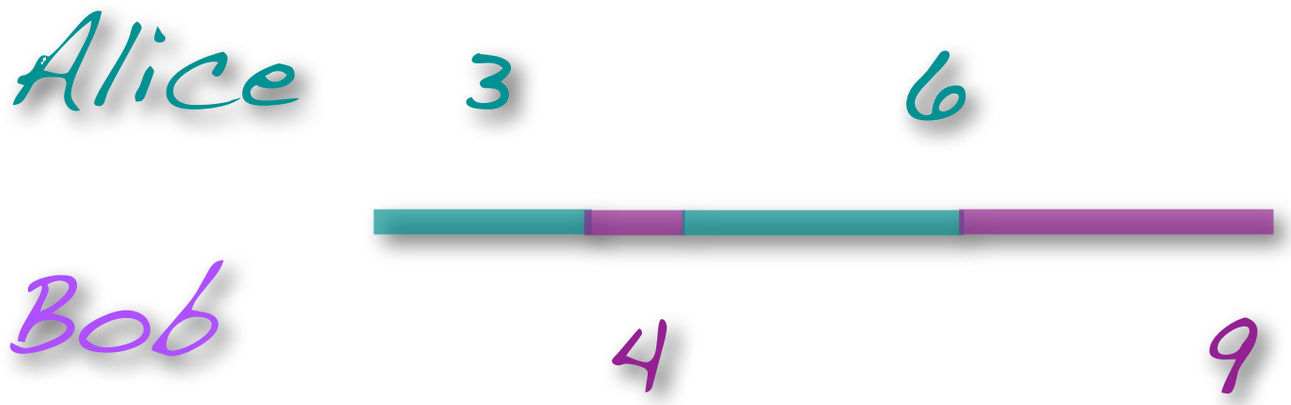


Alice plays against Bob....

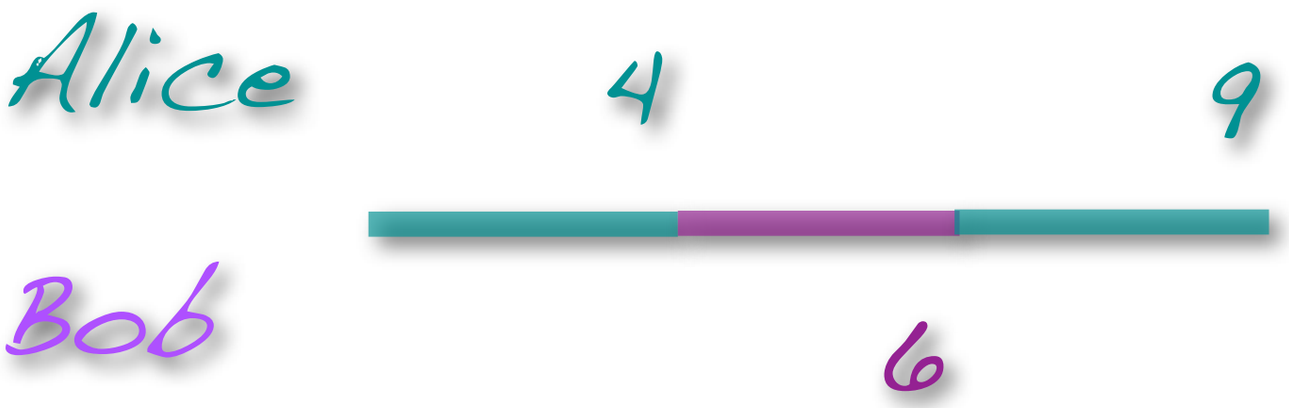


UNIL | Université de Lausanne

Assume BoB applies a w.s....

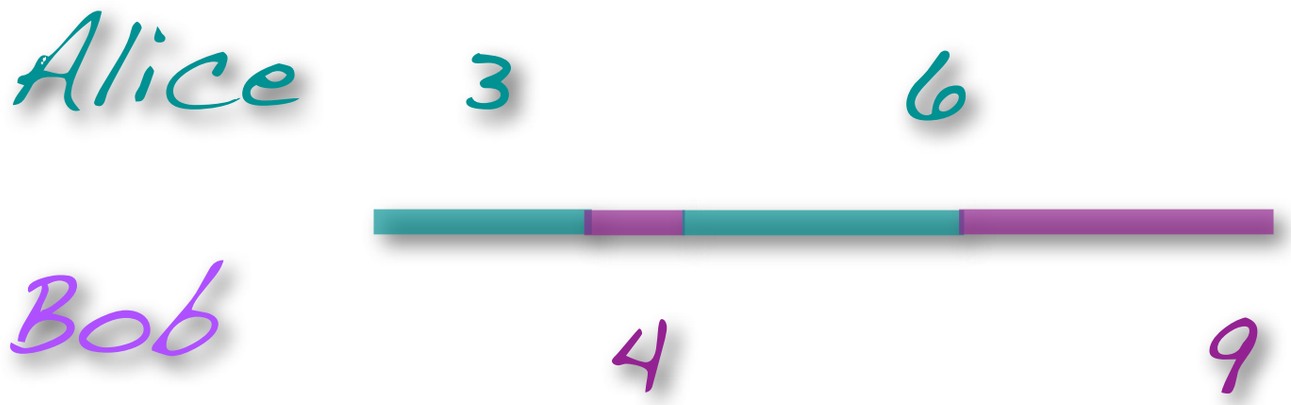


Alice plays against Bob....

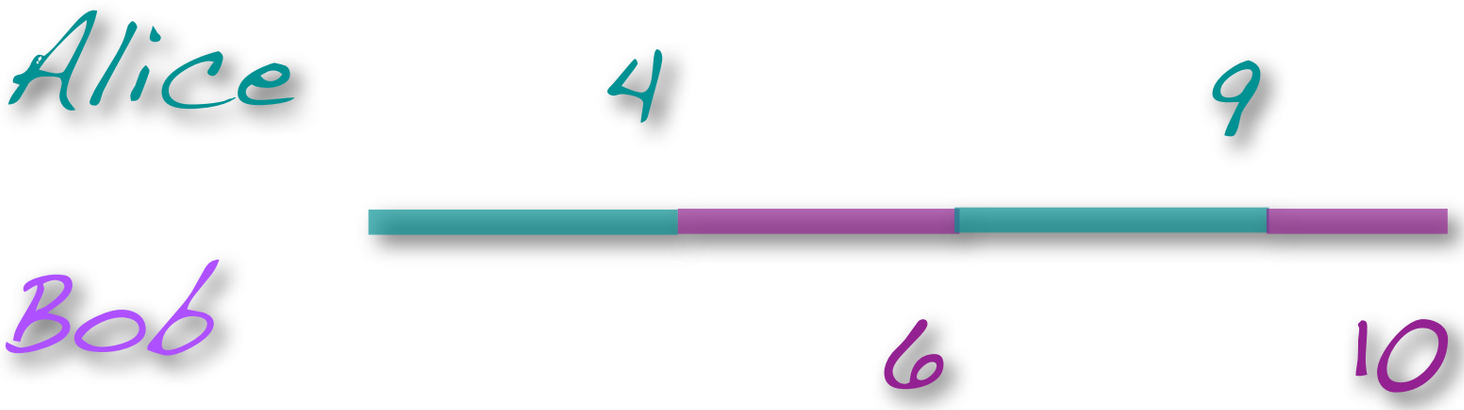


UNIL | Université de Lausanne

Assume BoB applies a w.s....

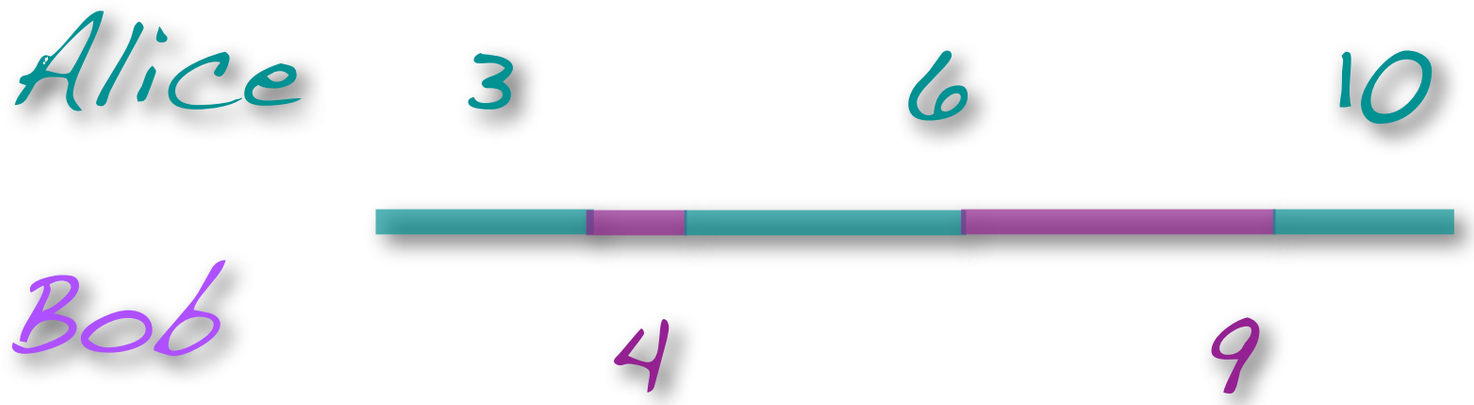


Alice plays against Bob....

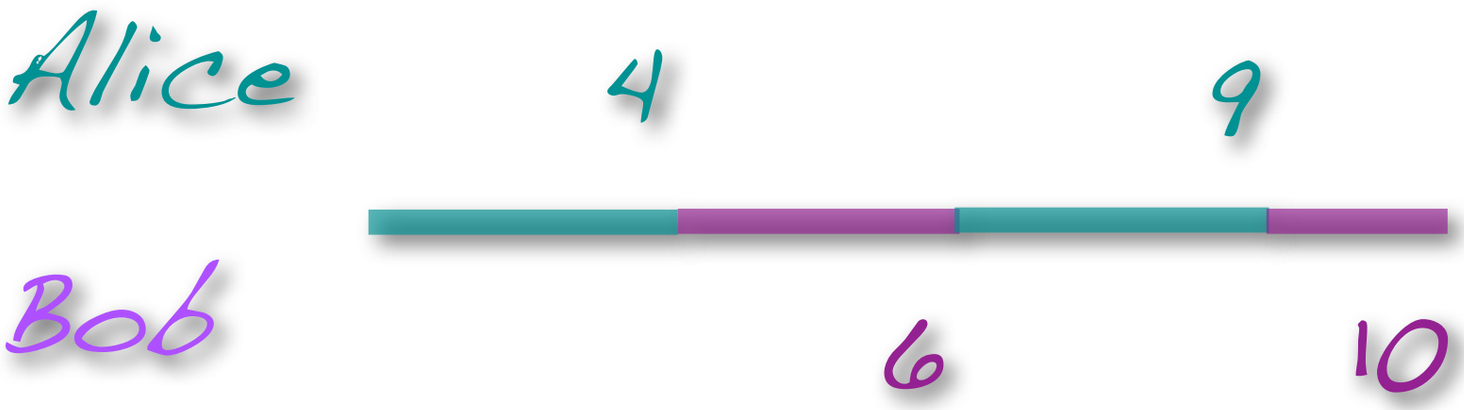


UNIL | Université de Lausanne

Assume BoB applies a w.s....

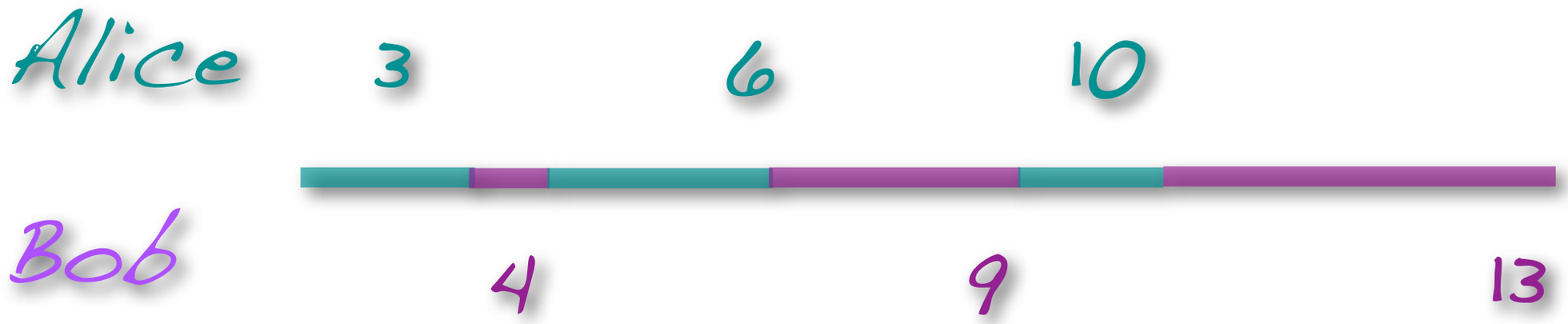


Alice plays against Bob....

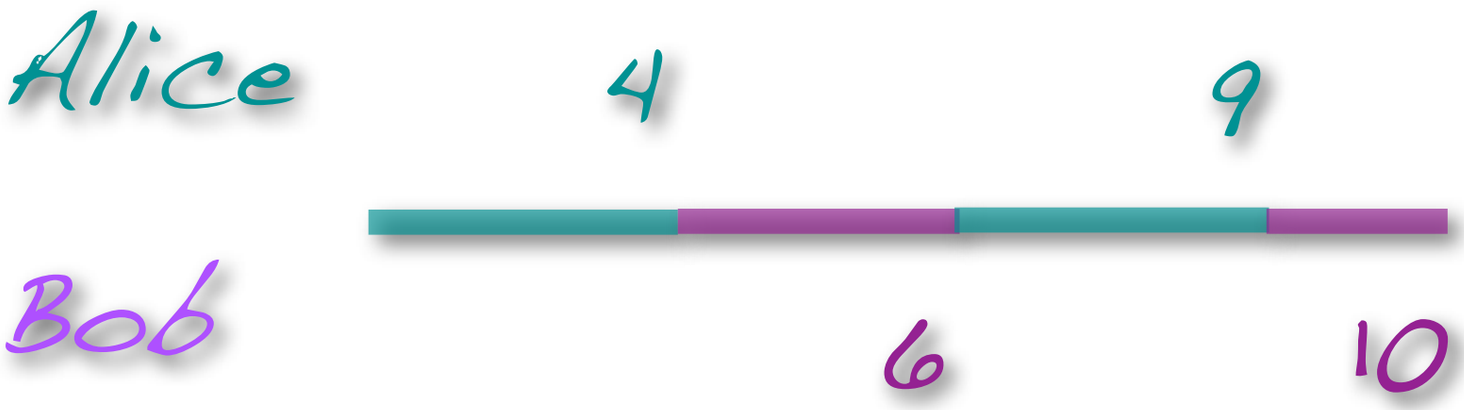


UNIL | Université de Lausanne

Assume BoB applies a w.s....

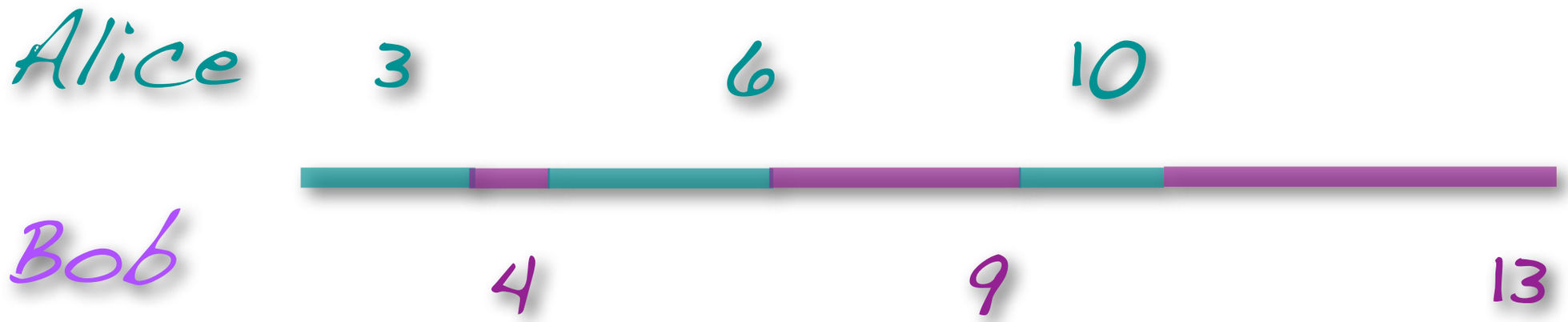


Alice plays against Bob....

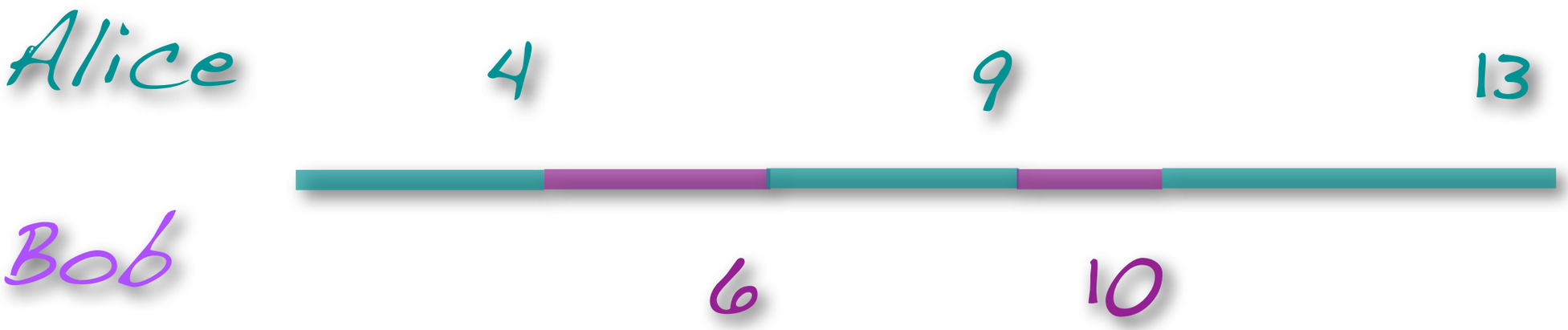


UNIL | Université de Lausanne

Assume BoB applies a w.s....

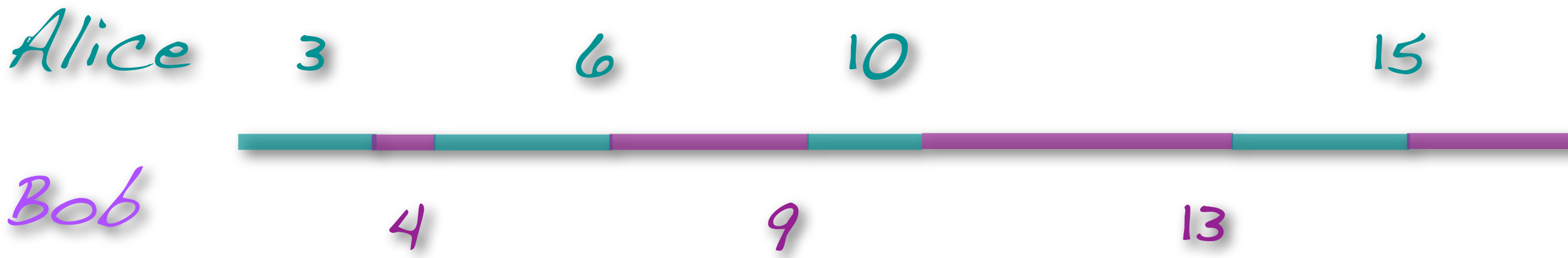


Alice plays against Bob....

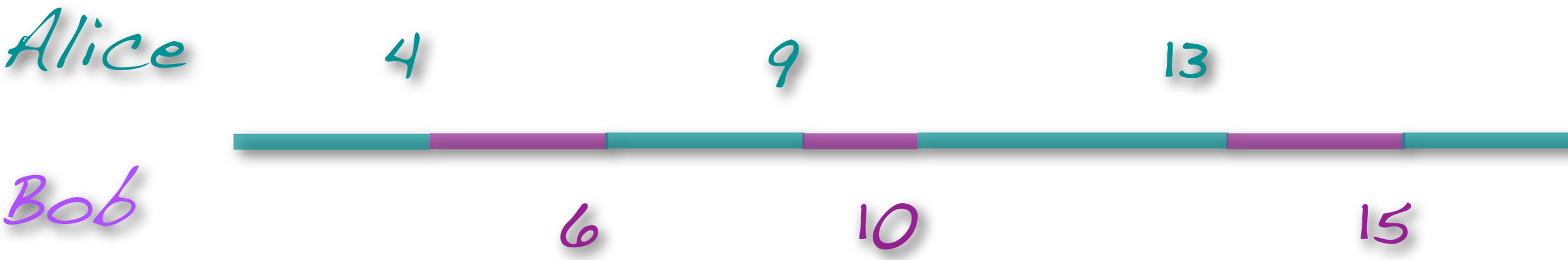


UNIL | Université de Lausanne

Assume BoB applies a w.s....

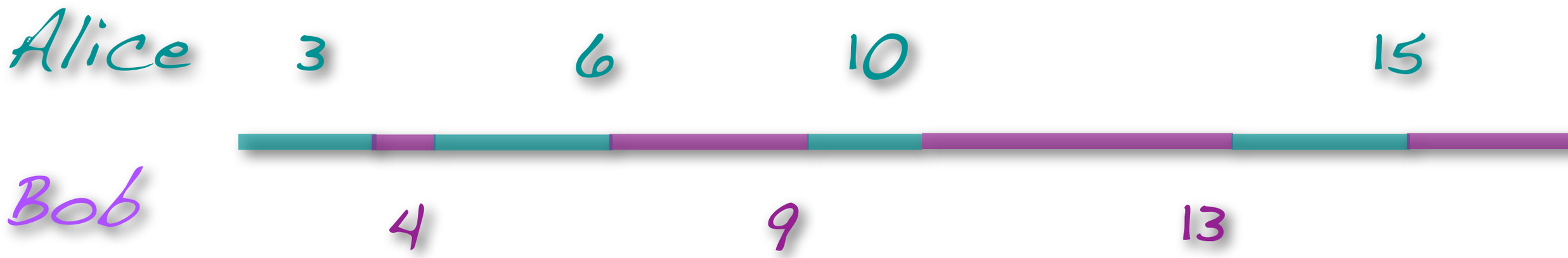


Alice plays against Bob....

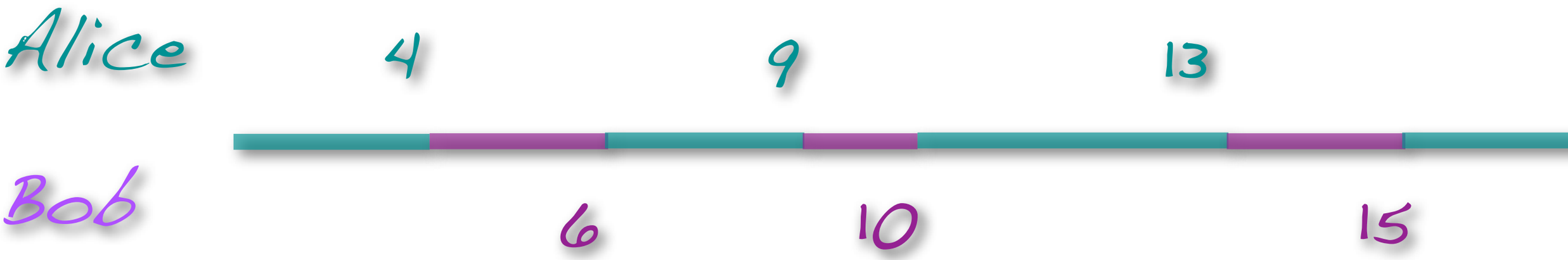


UNIL | Université de Lausanne

Assume BoB applies a w.s.... Hence BoB wins:

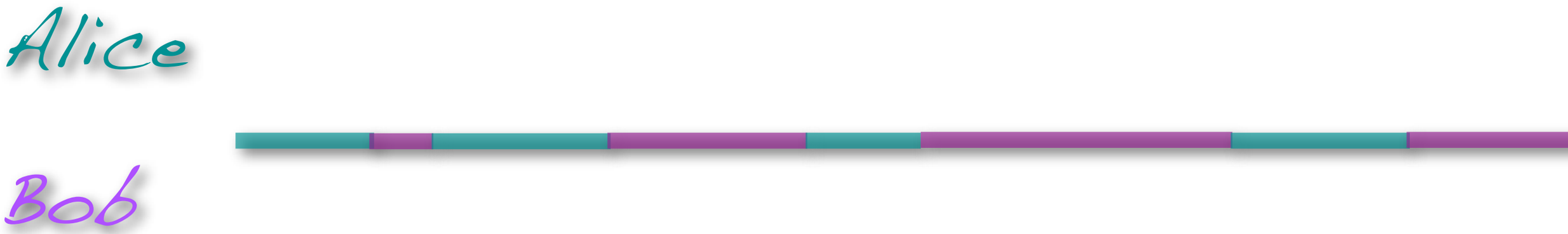


Alice plays against Bob....

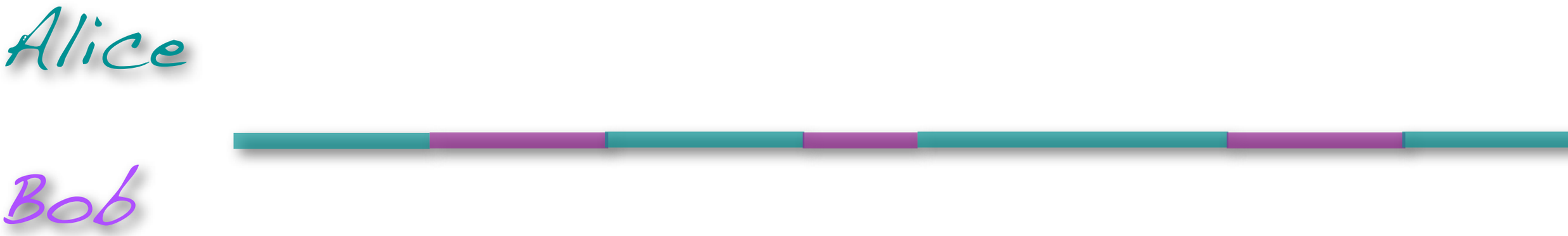


UNIL | Université de Lausanne

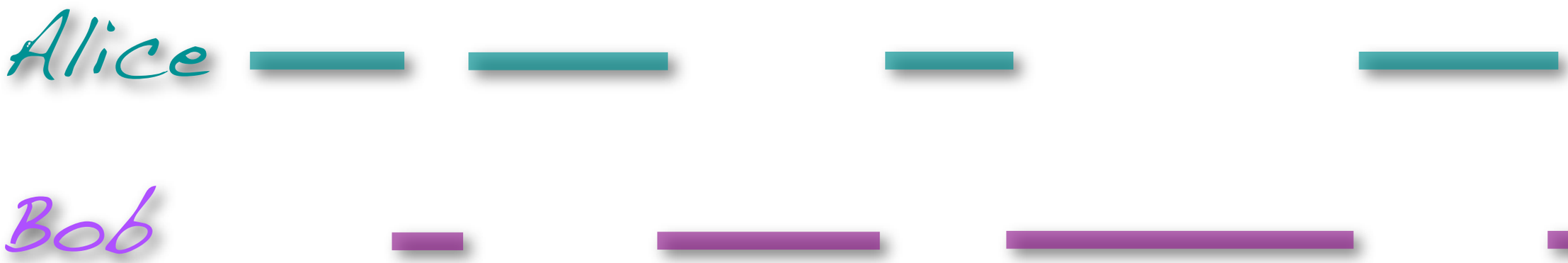
Assume BoB applies a w.s.... Hence BoB wins:



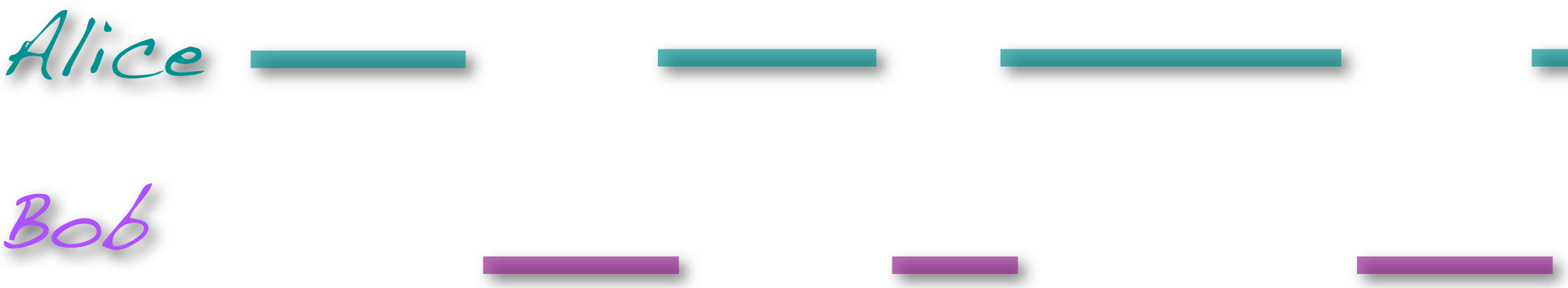
Alice plays against Bob....



Assume BoB applies a w.s.... Hence BoB wins:



Alice plays against Bob....

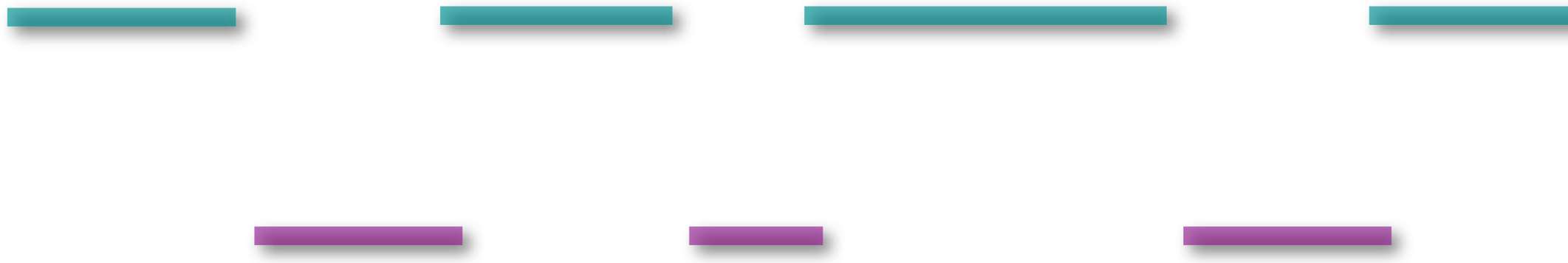


UNIL | Université de Lausanne

Assume BoB applies a w.s.... Hence BoB wins:



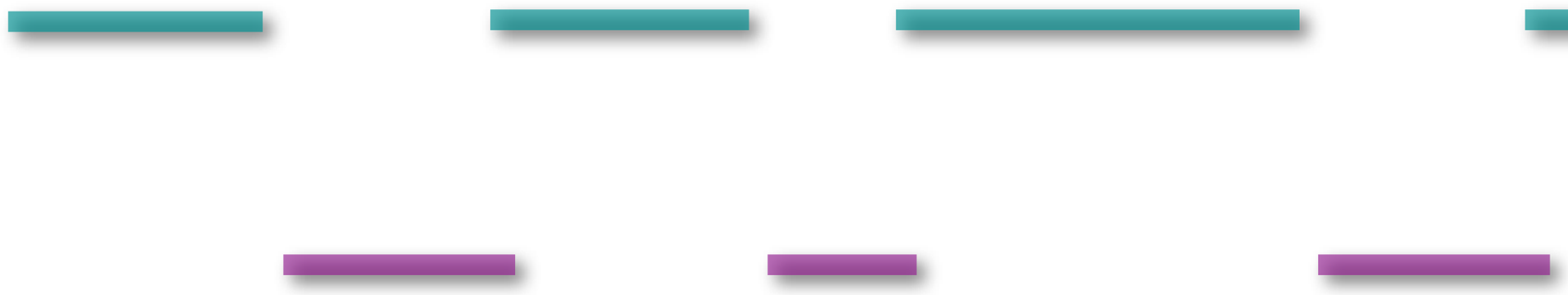
Alice plays against Bob....



Assume BoB applies a w.s.... Hence BoB wins:



Alice plays against Bob....



UNIL | Université de Lausanne

Assume BoB applies a w.s.... Hence BoB wins:

$A \nexists$

$A \exists$

Alice plays against Bob....

UNIL | Université de Lausanne

Assume BoB applies a w.s.... Hence BoB wins:

$A \nexists$ 

$A \exists$ 

Alice plays against Bob....

$A \exists$ 



Assume BoB applies a w.s.... Hence BoB wins:

$A \nexists$ 

$A \exists$ 

Alice plays against Bob....

$A \exists$ 

Alice wins !



UNIL | Université de Lausanne

UNIL | Université de Lausanne

If Alice has a w.s...

UNIL | Université de Lausanne

UNIL | Université de Lausanne

Assume Alice applies a w.s....

Alice

Bob

UNIL | Université de Lausanne

Bob plays against Alice....

Alice

Bob

Assume Alice applies a w.s....

Alice

Bob

Bob plays against Alice....

Alice

Bob

Assume Alice applies a w.s....

Alice 3

Bob

UNIL | Université de Lausanne

Bob plays against Alice....

Alice

Bob

Assume Alice applies a w.s....

Alice $3 \rangle 2$

Bob

UNIL | Université de Lausanne

Bob plays against Alice....

Alice

Bob

Assume Alice applies a w.s....

Alice 3 > 2

Bob 4

UNIL | Université de Lausanne

Bob plays against Alice....

Alice

Bob

Assume Alice applies a w.s....

Alice 3⟩

Bob 4

UNIL | Université de Lausanne

Bob plays against Alice....

Alice

Bob

Assume Alice applies a w.s....

Alice 3

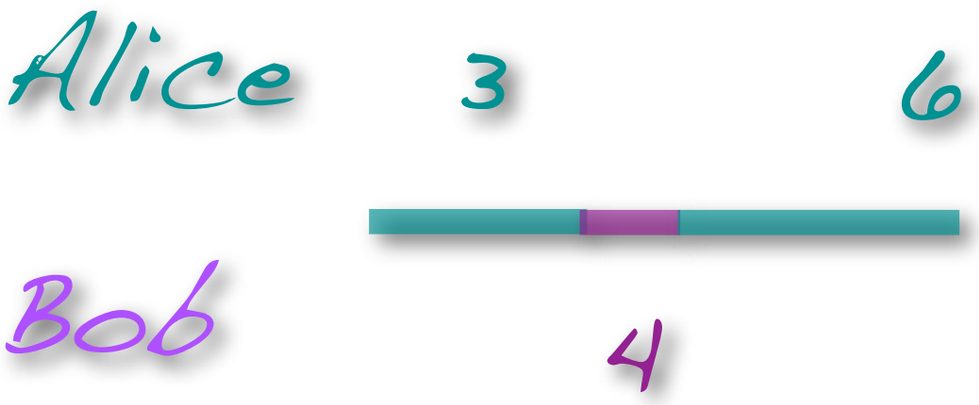
Bob 4

UNIL | Université de Lausanne

Bob plays against Alice....

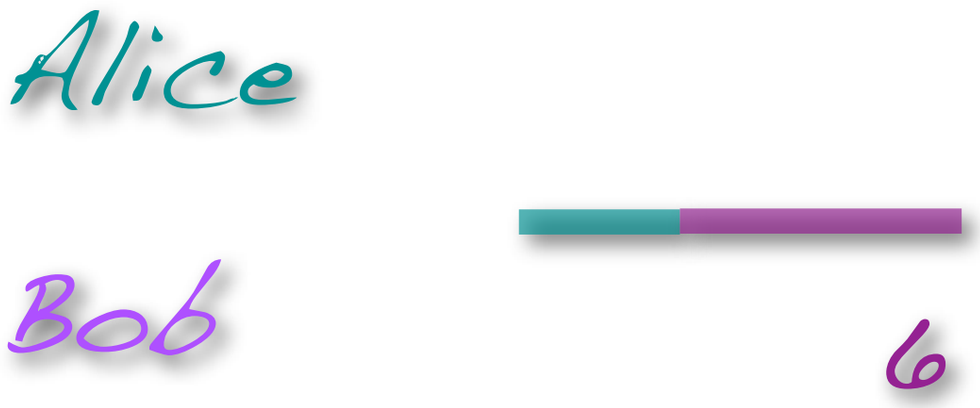


Assume Alice applies a w.s....

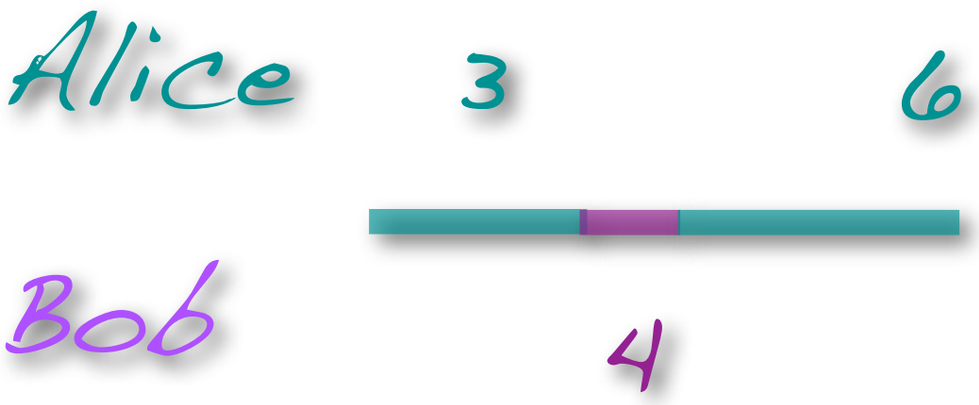


UNIL | Université de Lausanne

Bob plays against Alice....

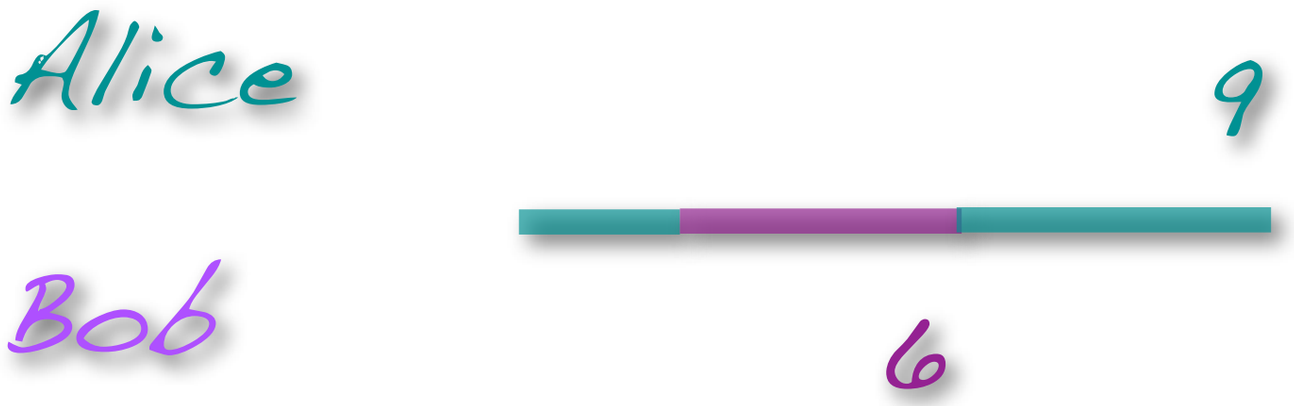


Assume Alice applies a w.s....

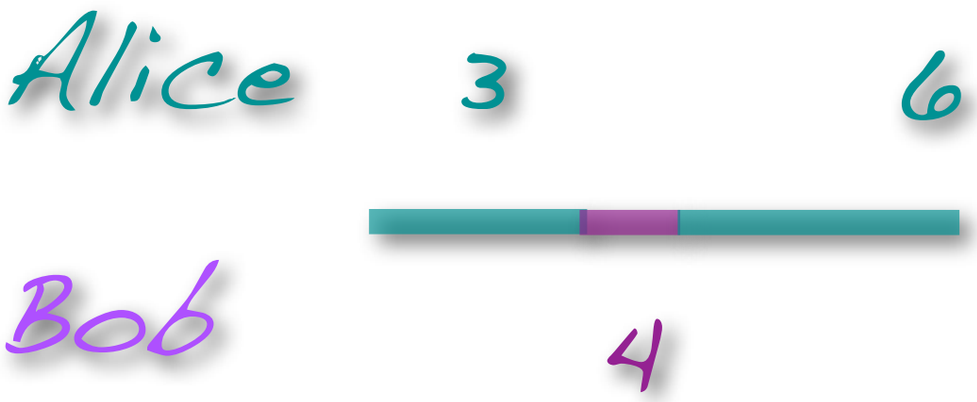


UNIL | Université de Lausanne

Bob plays against Alice....

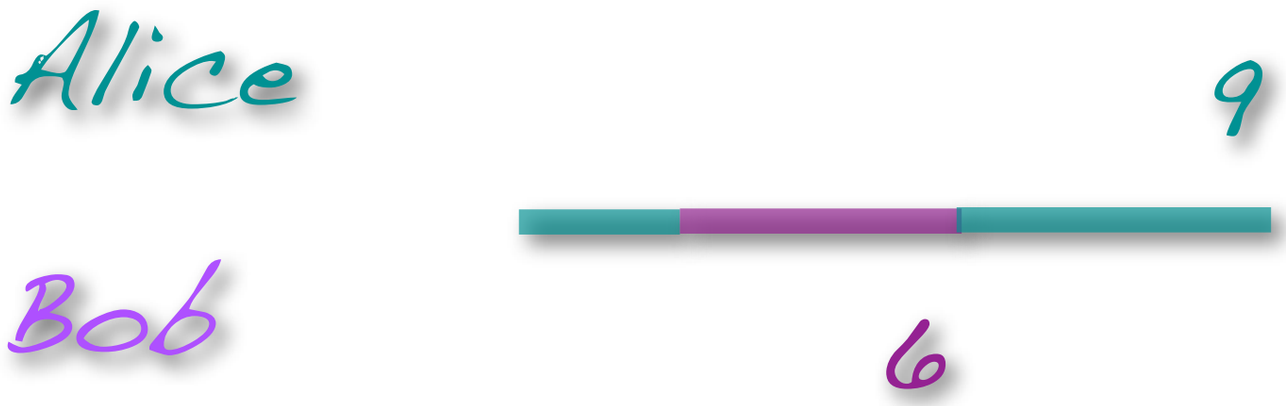


Assume Alice applies a w.s....

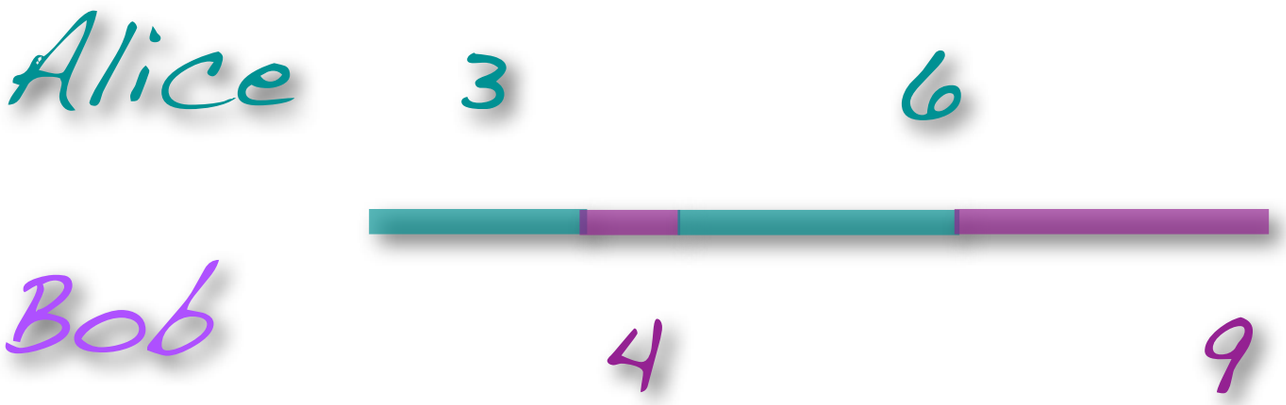


UNIL | Université de Lausanne

Bob plays against Alice....

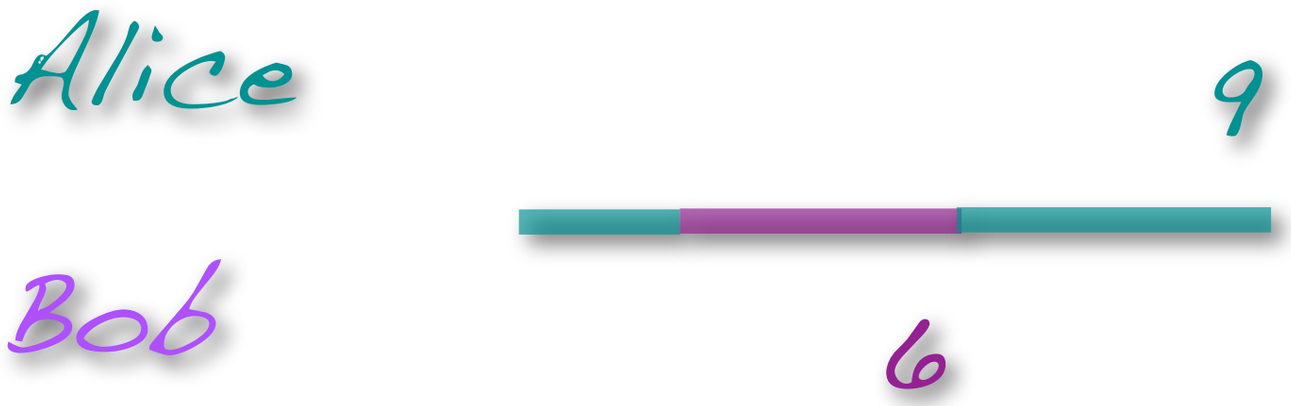


Assume Alice applies a w.s....

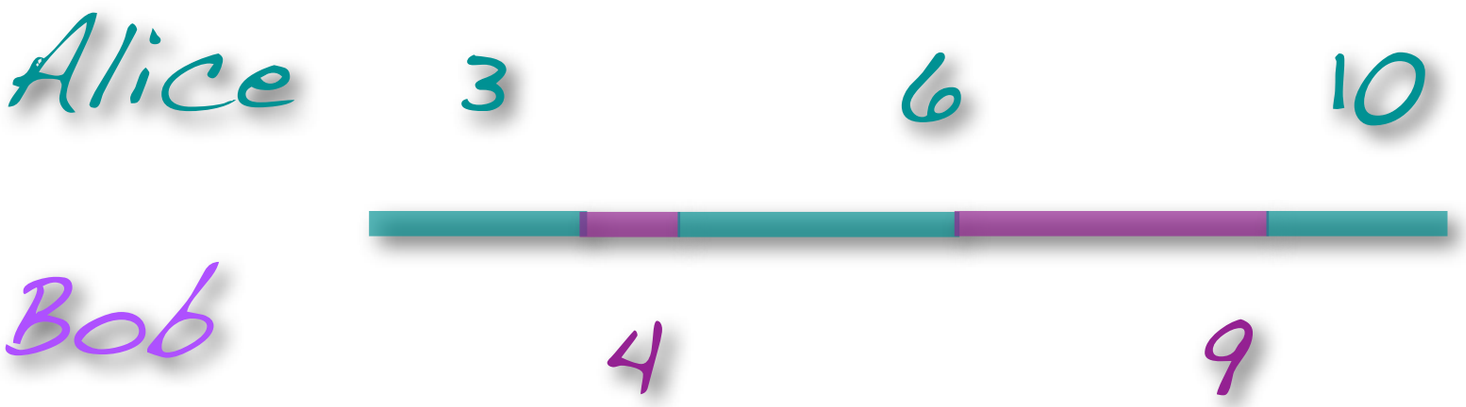


UNIL | Université de Lausanne

Bob plays against Alice....

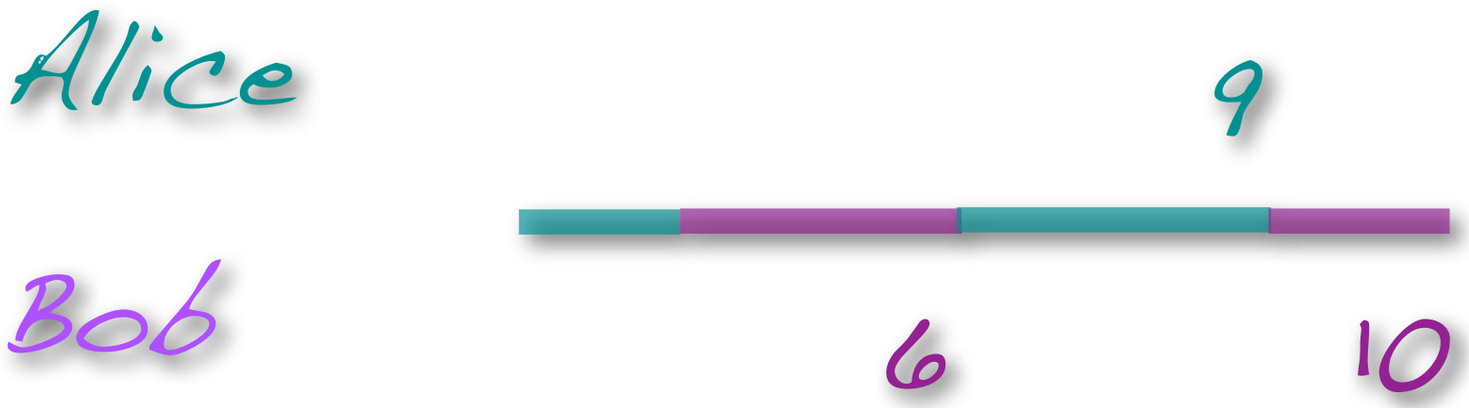


Assume Alice applies a w.s....

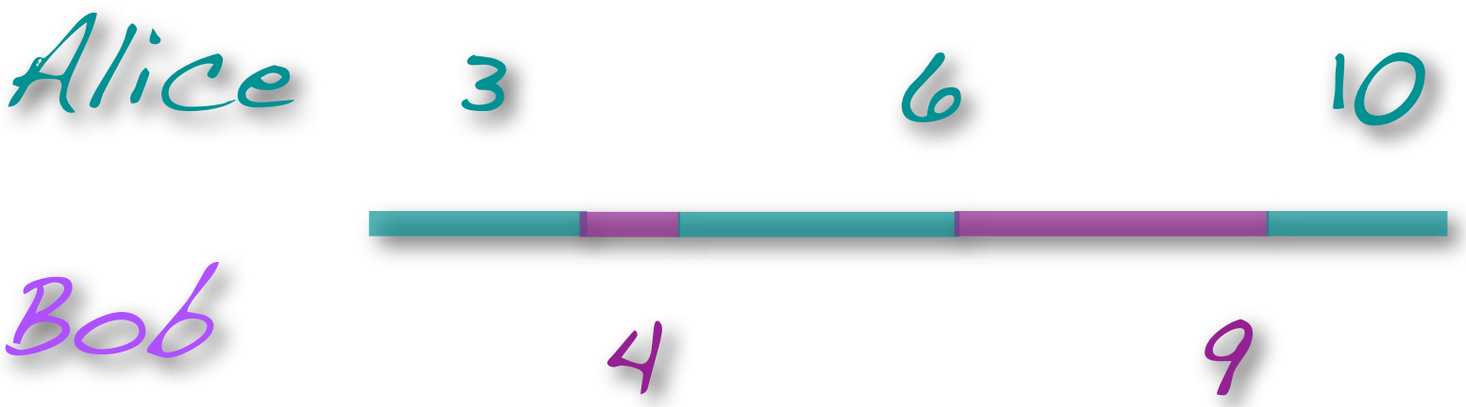


UNIL | Université de Lausanne

Bob plays against Alice....

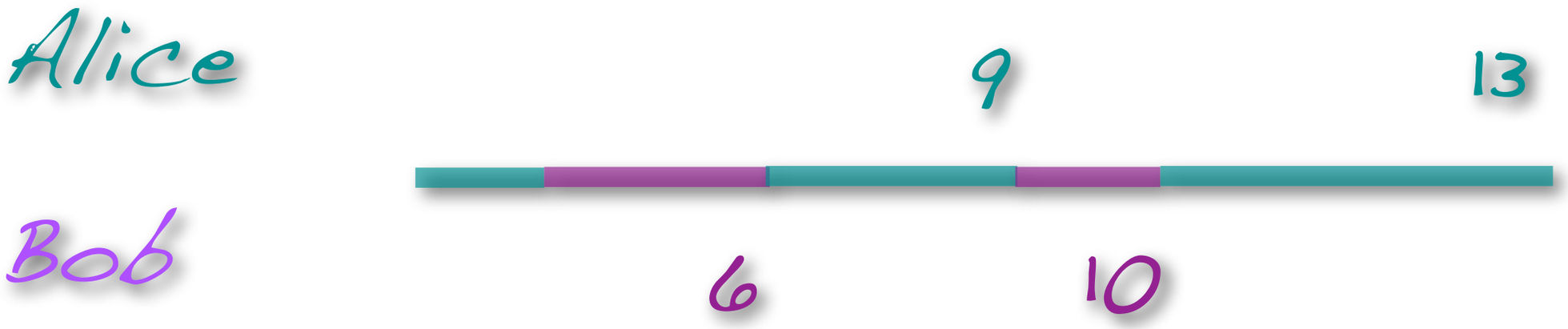


Assume Alice applies a w.s....

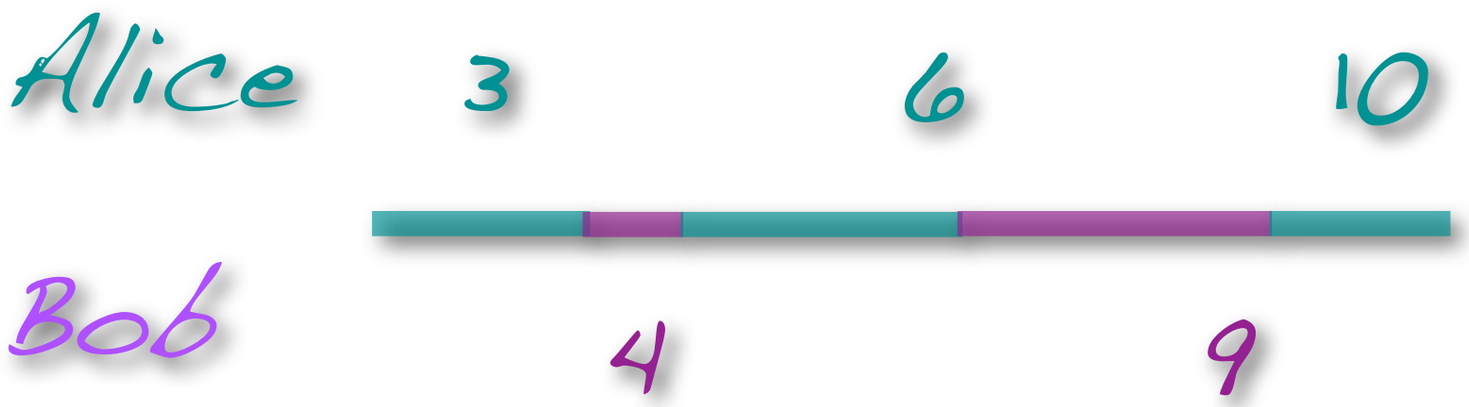


UNIL | Université de Lausanne

Bob plays against Alice....

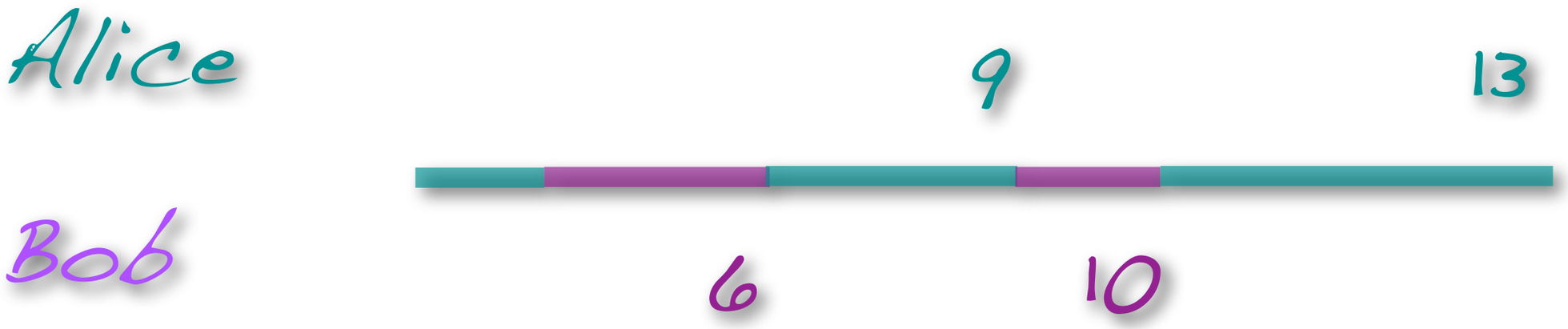


Assume Alice applies a w.s....

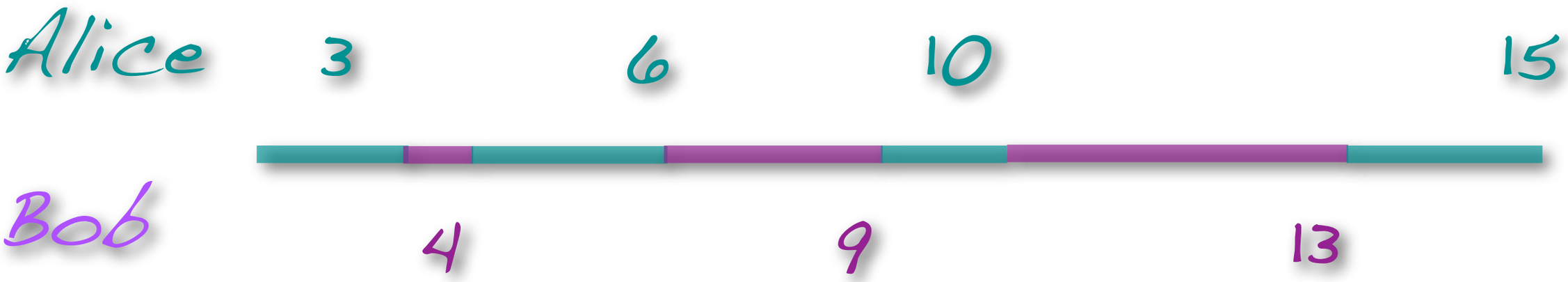


UNIL | Université de Lausanne

Bob plays against Alice....

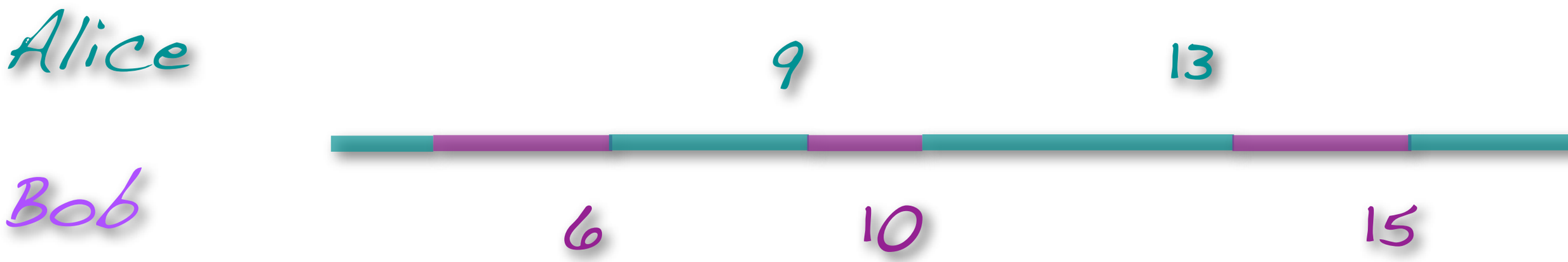


Assume Alice applies a w.s....

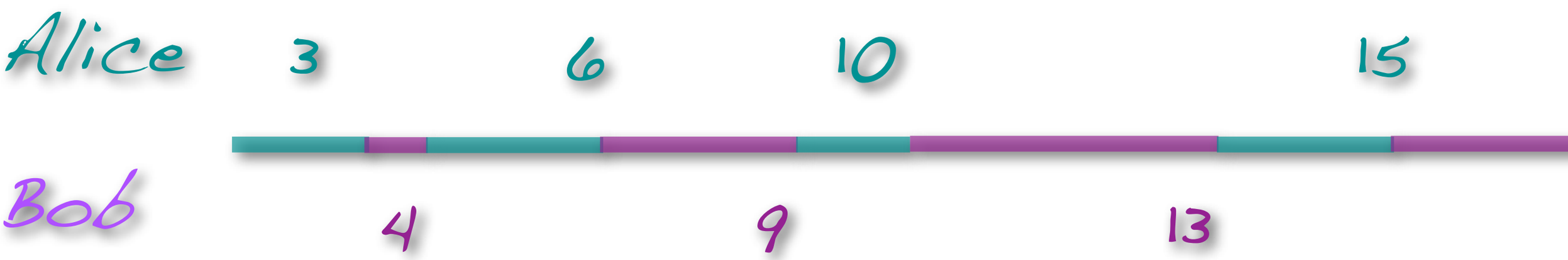


UNIL | Université de Lausanne

Bob plays against Alice....

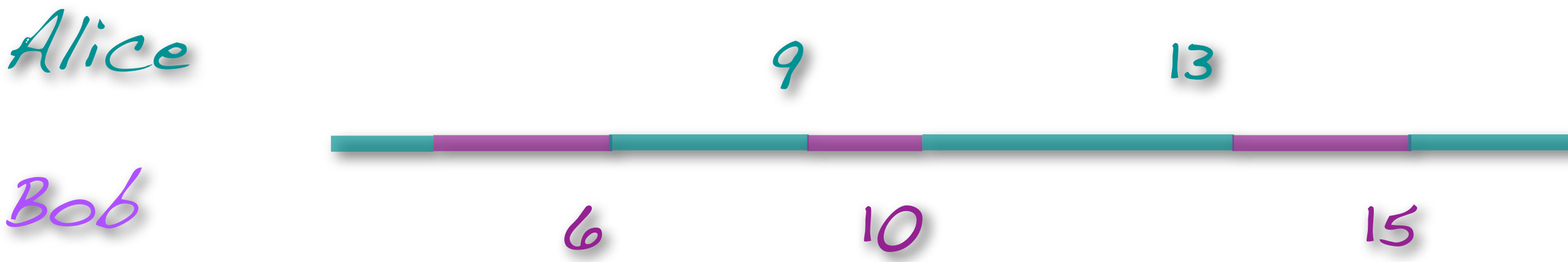


Assume Alice applies a w.s....

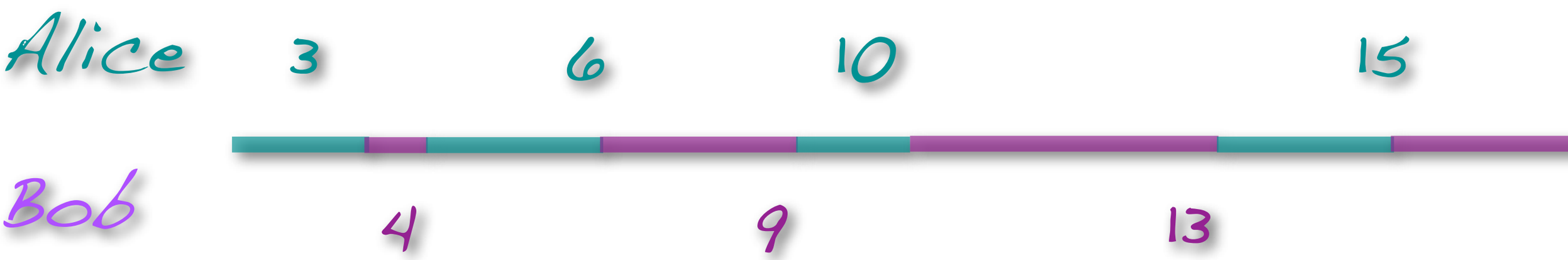


UNIL | Université de Lausanne

Bob plays against Alice....

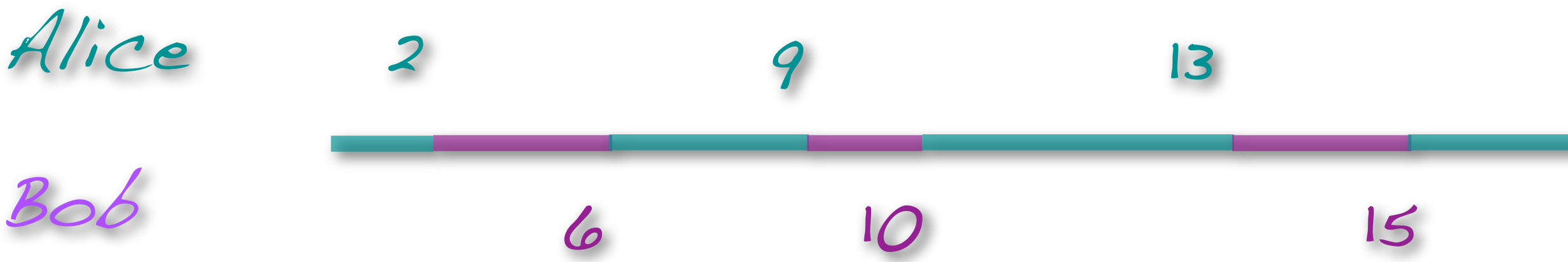


Assume Alice applies a w.s....

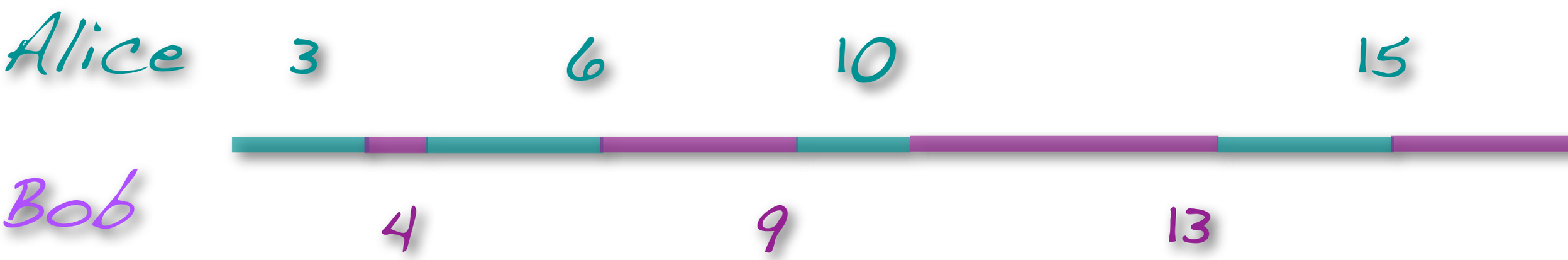


UNIL | Université de Lausanne

Bob plays against Alice....

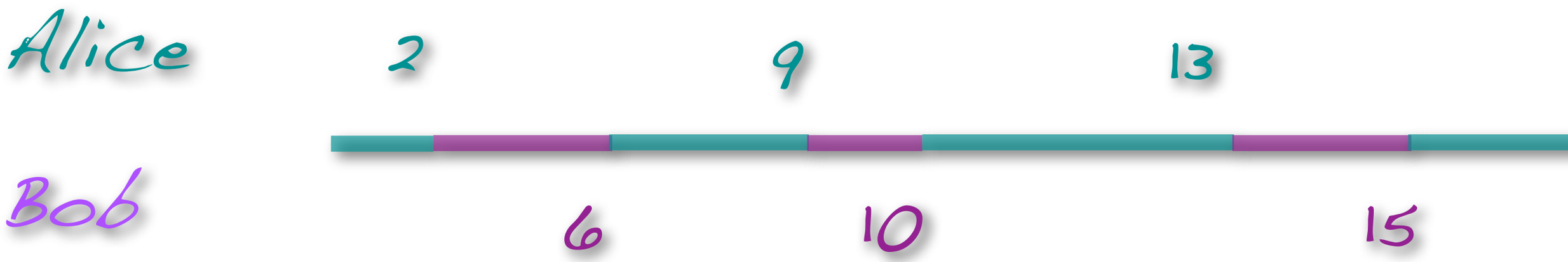


Assume Alice applies a w.s....

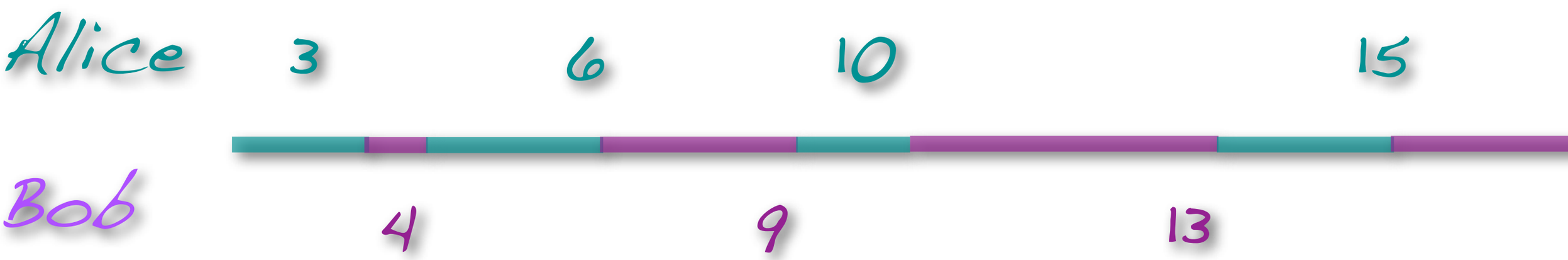


UNIL | Université de Lausanne

Bob plays against Alice....

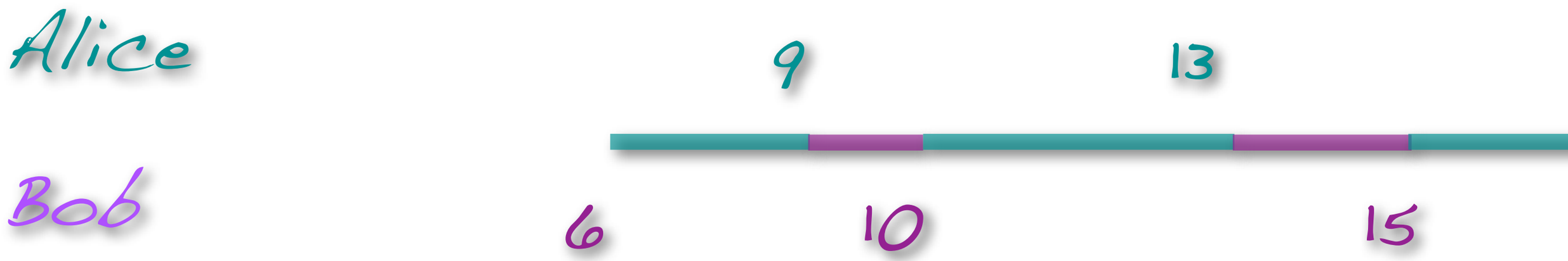


Assume Alice applies a w.s....

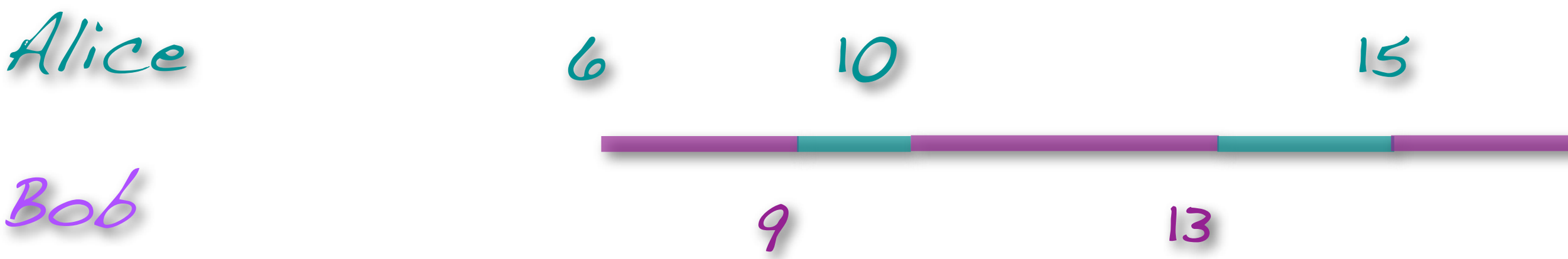


UNIL | Université de Lausanne

Bob plays against Alice....



Assume Alice applies a w.s....



UNIL | Université de Lausanne

Bob plays against Alice....

Alice



Bob

Assume Alice applies a w.s....

Alice



Bob

UNIL | Université de Lausanne

Bob plays against Alice....

Alice



Bob



Assume Alice applies a w.s....

Alice



Bob

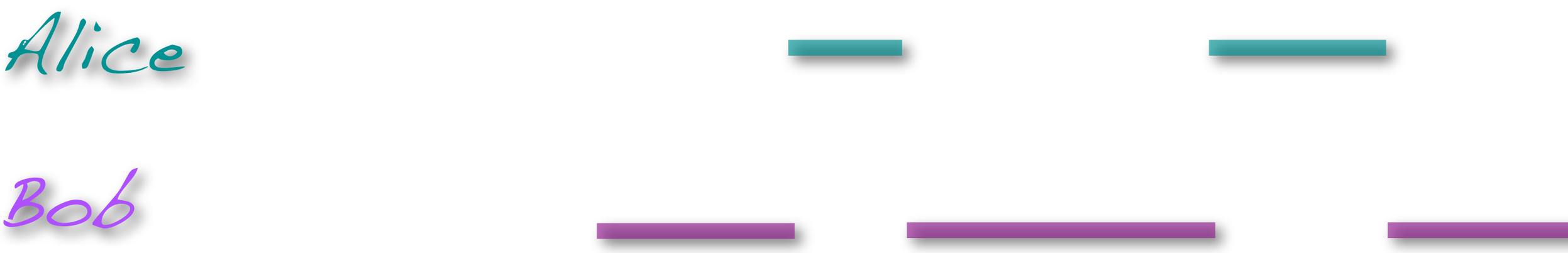


UNIL | Université de Lausanne

Bob plays against Alice....



Assume Alice applies a w.s....



UNIL | Université de Lausanne

This game is not determined !

UNIL | Université de Lausanne

UNIL | Université de Lausanne

How do we get the right A ?

(0) $\mathcal{A} \in \mathcal{P}(\mathbb{N}^\omega)$

(1) $\mathcal{A} \ni$ 

$\Leftrightarrow$

$\mathcal{A} \not\ni$ 

(2) $\mathcal{A} \ni$ 

$\Rightarrow$

$\mathcal{A} \ni$ 

$\Rightarrow$

$\mathcal{A} \ni$ 

UNIL | Université de Lausanne

(0) $\mathcal{A} \in \mathcal{P}(\mathbb{N}^\omega)$

(1) $\mathcal{A} \ni$ 

$\Leftrightarrow$

$\mathcal{A} \not\ni$ 

(2) $\mathcal{A} \ni$ 

$\Rightarrow$

$\mathcal{A} \ni$ 

$\Rightarrow$

$\mathcal{A} \ni$ 

UNIL | Université de Lausanne

(1) $\mathcal{A} \ni$ 

$\Leftrightarrow$

$\mathcal{A} \ni$ 

(2) $\mathcal{A} \ni$ 

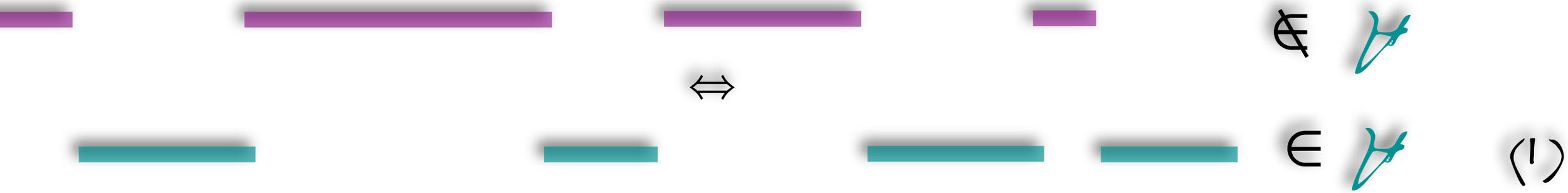
$\Rightarrow$

$\mathcal{A} \ni$ 

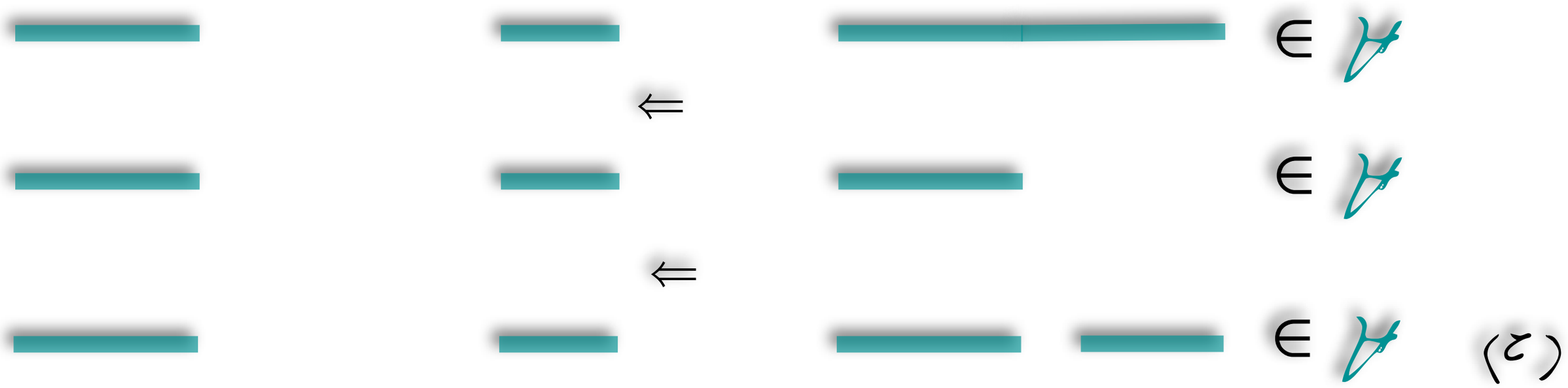
$\Rightarrow$

$\mathcal{A} \ni$ 

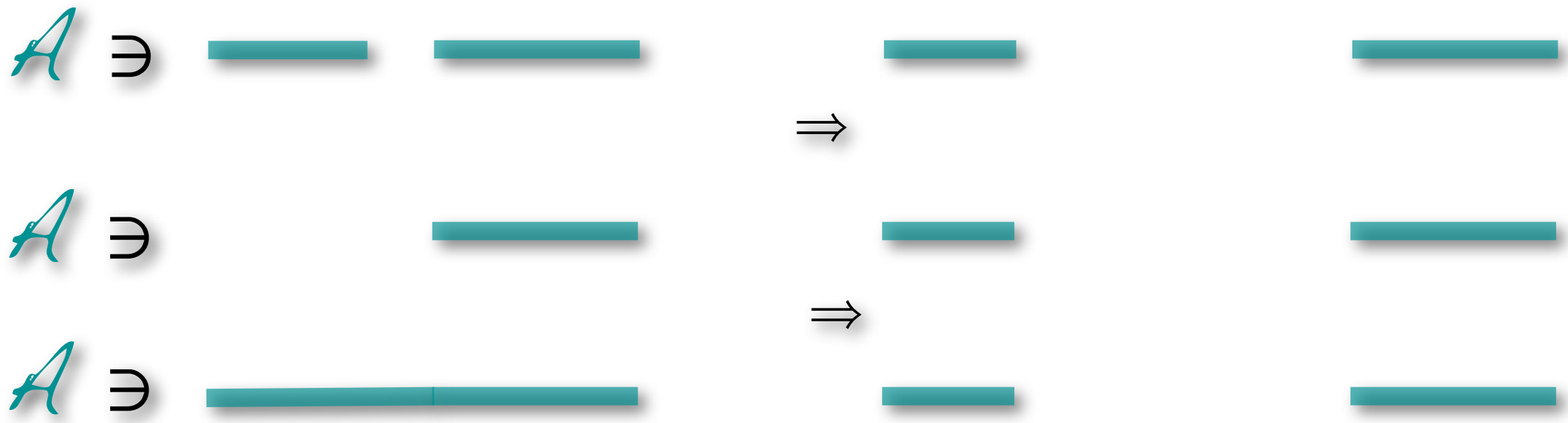
UNIL | Université de Lausanne



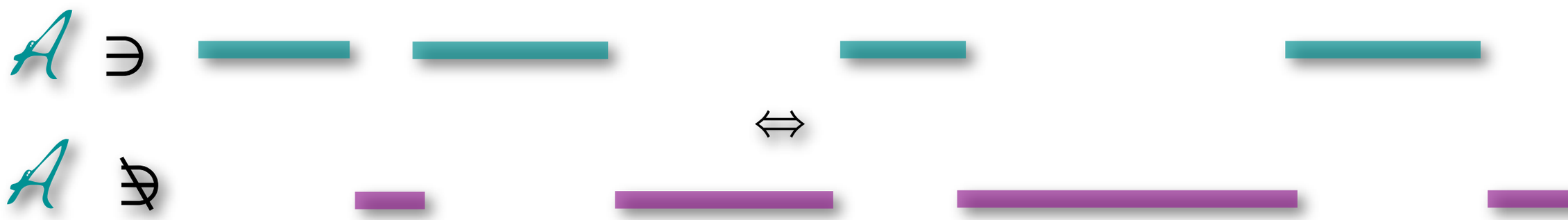
(۱)



(۲)

(2) 

$X \in \mathcal{A}$ and F finite $\Rightarrow X \cup F \in \mathcal{A}$

(1) 

(1) $\mathcal{A} \ni$ 

$\Leftrightarrow$

$\mathcal{A} \not\ni$ 

$X \in \mathcal{A}$ and F finite $\Rightarrow X \cup F \in \mathcal{A}$

UNIL | Université de Lausanne

(1) $\mathcal{A} \ni$ 

$\Leftrightarrow$

$\mathcal{A} \not\ni$ 

$$X \in \mathcal{A} \Leftrightarrow X^C \notin \mathcal{A}$$

$$X \in \mathcal{A} \text{ and } F \text{ finite} \Rightarrow X \cup F \in \mathcal{A}$$

UNIL | Université de Lausanne

$X \in \mathcal{A}$ and F finite $\Rightarrow X \cup F \in \mathcal{A}$

$$X \in \mathcal{A} \text{ and } F \text{ finite} \Rightarrow X \cup F \in \mathcal{A}$$

$$F_{frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

$X \in \mathcal{A}$ and F finite $\Rightarrow X \cup F \in \mathcal{A}$

$\mathcal{F}_{Frechet} = \{X \subseteq \mathbb{N} : X^c \text{ is finite}\}$



$$X \in \mathcal{A} \text{ and } F \text{ finite} \Rightarrow X \cup F \in \mathcal{A}$$

$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

$$X \in \mathcal{A} \Leftrightarrow X^C \notin \mathcal{A}$$



$$X \in \mathcal{A} \text{ and } F \text{ finite} \Rightarrow X \cup F \in \mathcal{A}$$

$$F_{rechet} = \{X \subseteq \mathbb{N} : X^c \text{ is finite}\}$$

$$X \in \mathcal{A} \Leftrightarrow X^c \notin \mathcal{A}$$



Frechet is a Filter...

....extend it into an Ultrafilter !

$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

$$\mathcal{F}_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

$$\mathcal{F}_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

☞ $\emptyset \notin \mathcal{F}_{Frechet}$

$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

- $\emptyset \notin F_{Frechet}$
- $X, Y \in F_{Frechet} \Rightarrow X \cap Y \in F_{Frechet}$

$$\mathcal{F}_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

- $\emptyset \notin \mathcal{F}_{Frechet}$
- $X, Y \in \mathcal{F}_{Frechet} \Rightarrow X \cap Y \in \mathcal{F}_{Frechet}$
- $X \in \mathcal{F}_{Frechet}$ and $X \subset Y \Rightarrow Y \in \mathcal{F}_{Frechet}$

$$\mathcal{F}_{Frechet}=\{X\subseteq\mathbb{N}:\ X^C \text{ is finite}\}$$

- $\emptyset\notin\mathcal{F}_{Frechet}$
- $X,Y\in\mathcal{F}_{Frechet}\Rightarrow X\cap Y\in\mathcal{F}_{Frechet}$
- $X\in\mathcal{F}_{Frechet}$ and $X\subset Y\Rightarrow Y\in\mathcal{F}_{Frechet}$

} Filter

$$F_{frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

- $\emptyset \notin F_{frechet}$
- $X, Y \in F_{frechet} \Rightarrow X \cap Y \in F_{frechet}$
- $X \in F_{frechet}$ and $X \subset Y \Rightarrow Y \in F_{frechet}$

} Filter

LES SHADOKS' logic

$$F_{Frechet}=\{X\subseteq\mathbb{N}:\ X^C \text{ is finite}\}$$

- $\emptyset\notin F_{Frechet}$
- $X,Y\in F_{Frechet}\Rightarrow X\cap Y\in F_{Frechet}$
- $X\in F_{Frechet}$ and $X\subset Y\Rightarrow Y\in F_{Frechet}$

} Filter

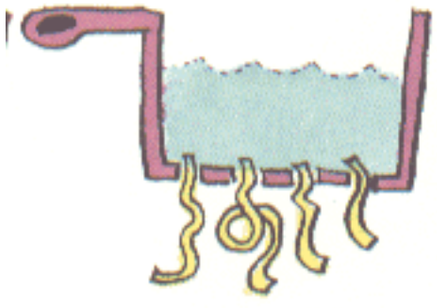
strainers in LES SHADOKS' logic

$$F_{Frechet}=\{X\subseteq\mathbb{N}:\ X^C \text{ is finite}\}$$

- $\emptyset\notin F_{Frechet}$
- $X,Y\in F_{Frechet}\Rightarrow X\cap Y\in F_{Frechet}$
- $X\in F_{Frechet}$ and $X\subset Y\Rightarrow Y\in F_{Frechet}$

} Filter

strainers in LES SHADOKS' logic



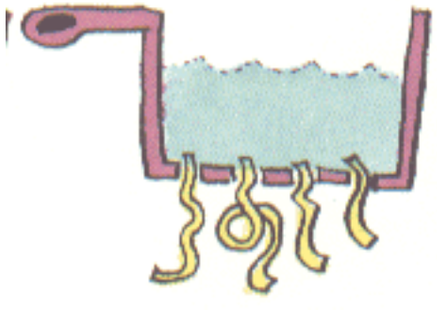
$$F_{Frechet}=\{X\subseteq\mathbb{N}:\ X^C \text{ is finite}\}$$

- $\emptyset\notin F_{Frechet}$
- $X,Y\in F_{Frechet}\Rightarrow X\cap Y\in F_{Frechet}$
- $X\in F_{Frechet}$ and $X\subset Y\Rightarrow Y\in F_{Frechet}$

} Filter

strainers in LES SHADOKS' logic



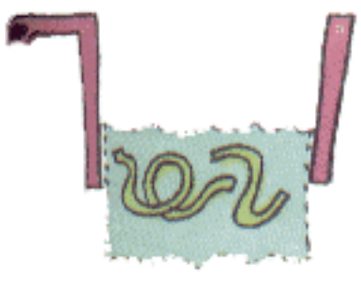


$$F_{Frechet}=\{X\subseteq\mathbb{N}:\ X^C \text{ is finite}\}$$

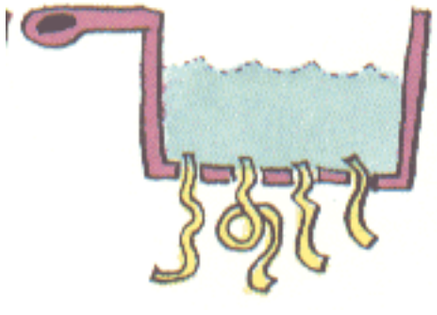
- $\emptyset\notin F_{Frechet}$
- $X,Y\in F_{Frechet}\Rightarrow X\cap Y\in F_{Frechet}$
- $X\in F_{Frechet}$ and $X\subset Y\Rightarrow Y\in F_{Frechet}$

} Filter

strainers in LES SHADOKS' logic







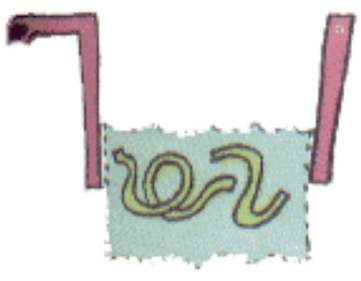
$$F_{frechet}=\{X\subseteq\mathbb{N}:\ X^C \text{ is finite}\}$$

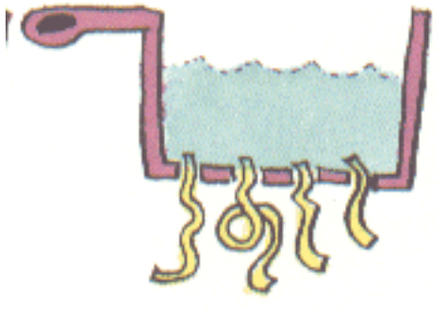
- $\emptyset\notin F_{frechet}$
- $X,Y\in F_{frechet}\Rightarrow X\cap Y\in F_{frechet}$
- $X\in F_{frechet}$ and $X\subset Y\Rightarrow Y\in F_{frechet}$

} Filter

strainers in Les Shadoks' logic

Ideal

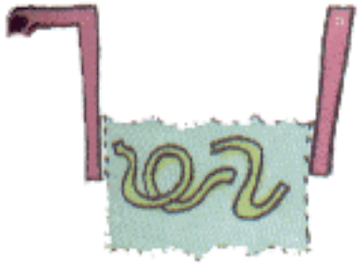





$$F_{frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

- $\emptyset \notin F_{frechet}$
- $X, Y \in F_{frechet} \Rightarrow X \cap Y \in F_{frechet}$
- $X \in F_{frechet}$ and $X \subset Y \Rightarrow Y \in F_{frechet}$

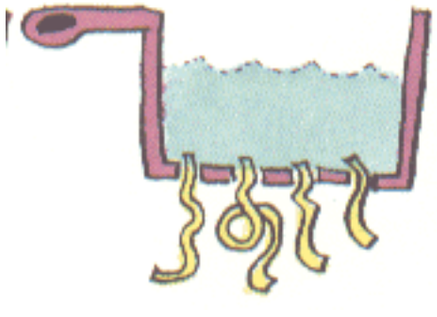




Filter



Ideal



$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

- $\emptyset \notin F_{Frechet}$
- $X, Y \in F_{Frechet} \Rightarrow X \cap Y \in F_{Frechet}$
- $X \in F_{Frechet}$ and $X \subset Y \Rightarrow Y \in F_{Frechet}$

} Filter

$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

- $\emptyset \notin F_{Frechet}$
- $X, Y \in F_{Frechet} \Rightarrow X \cap Y \in F_{Frechet}$
- $X \in F_{Frechet}$ and $X \subset Y \Rightarrow Y \in F_{Frechet}$

} Filter

BPIT

every Filter can be extended into an Ultrafilter

UNIL | Université de Lausanne

$$\mathcal{F}_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

☞ Take $\mathcal{U}$ any Ultrafilter that extends $F_{Frechet}$

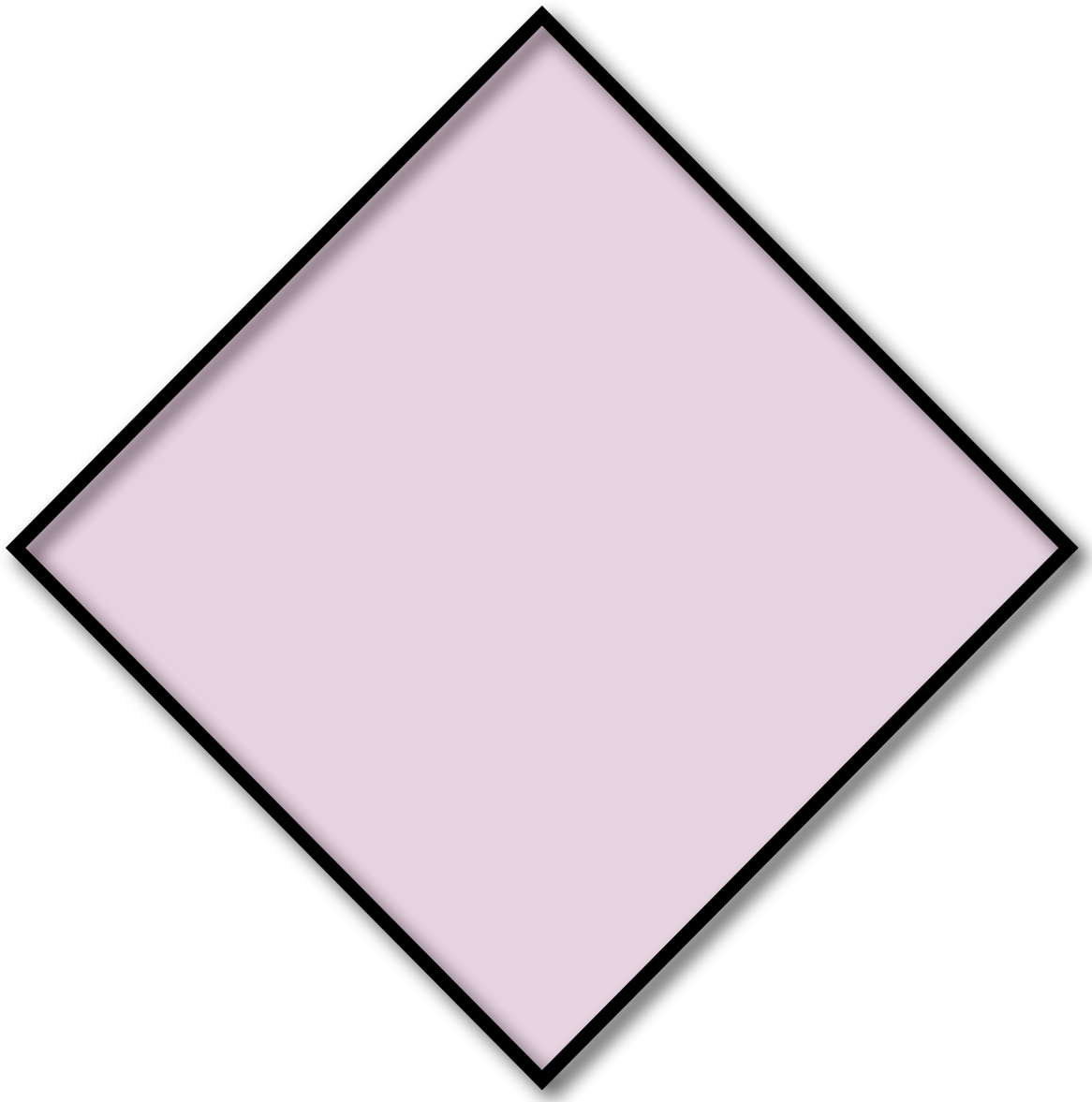
$$F_{Frechet} = \{X \subseteq \mathbb{N} : X^C \text{ is finite}\}$$

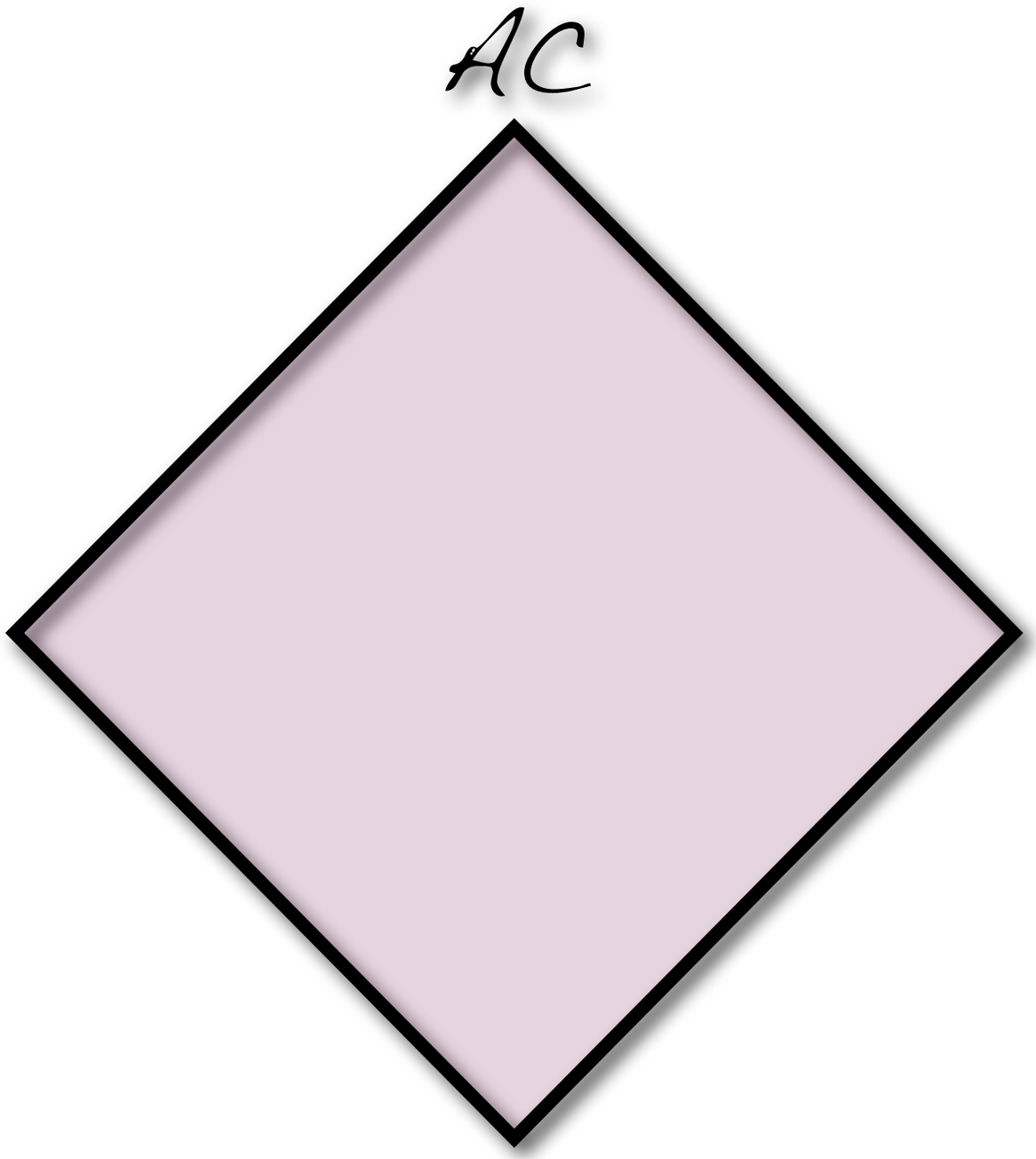
☞ Take $\mathcal{U}$ any Ultrafilter that extends $F_{Frechet}$

☞ Set A s.t. any legal play induces Alice's set $\in \mathcal{U}$

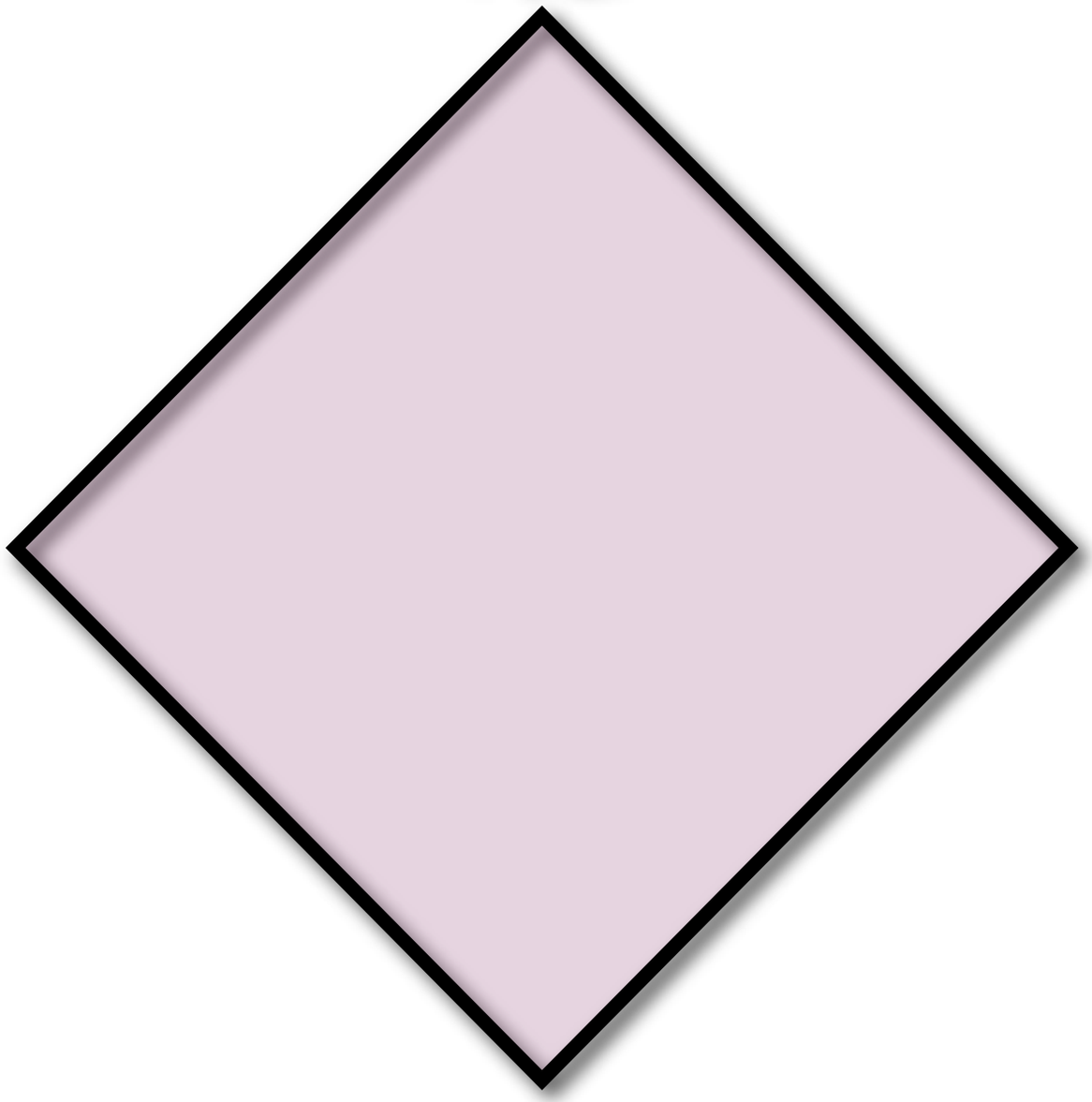
UNIL | Université de Lausanne

UNIL | Université de Lausanne



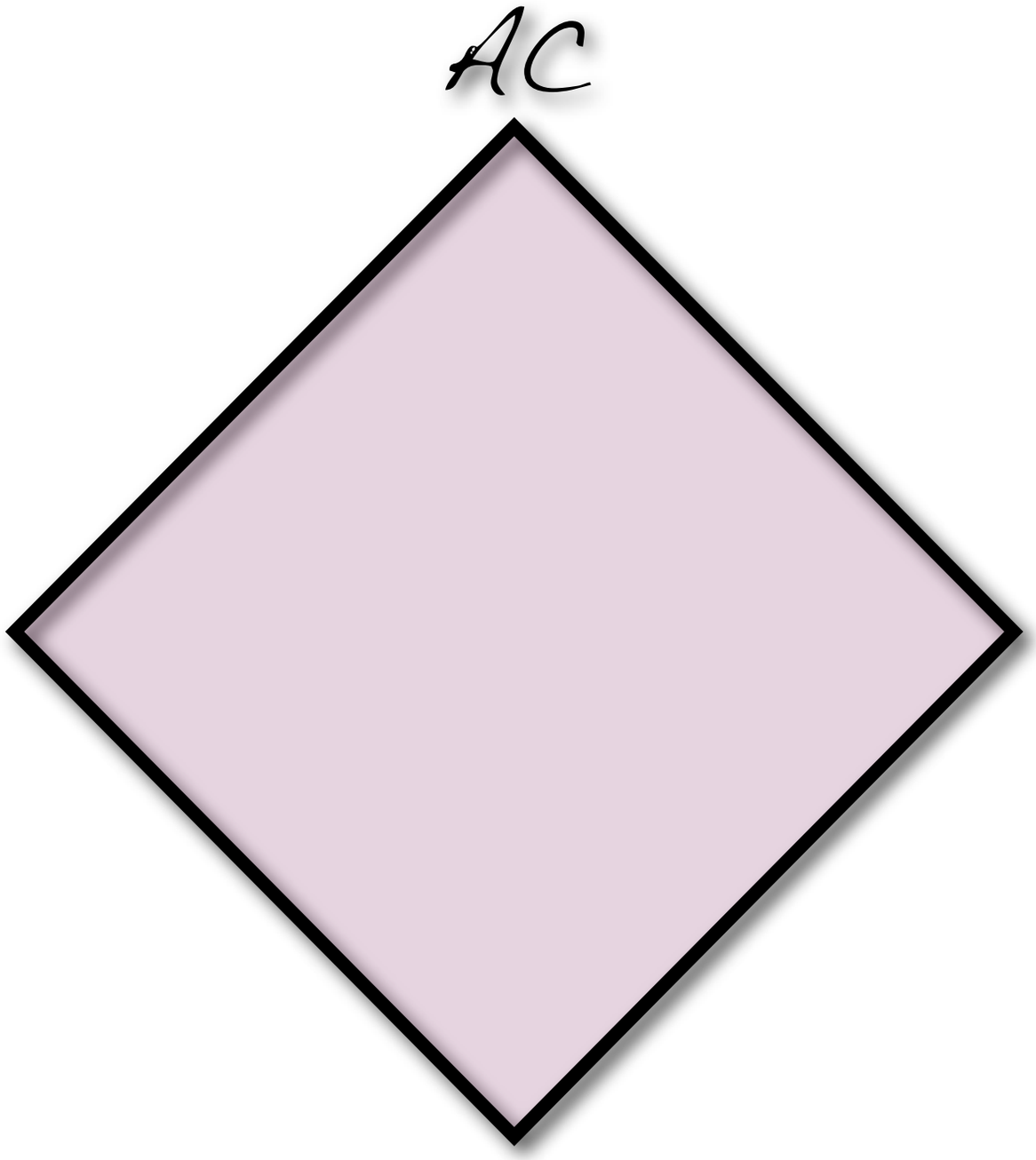

AC

AC



CC

UNIL | Université de Lausanne



BPIT

every Filter can be extended into an Ultrafilter

UNIL | Université de Lausanne

AC

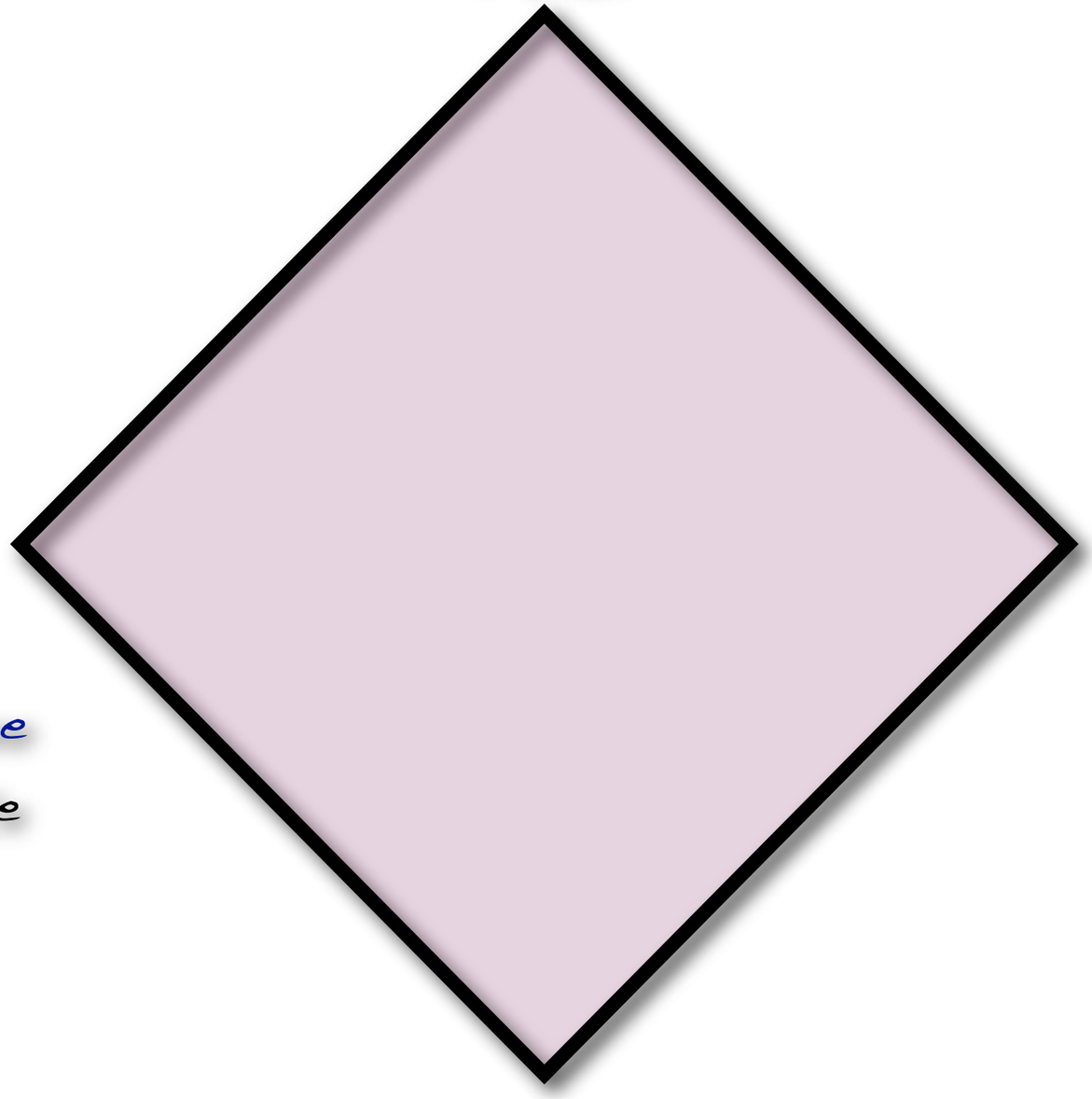
DC

every n.e.p tree has an infinite branch

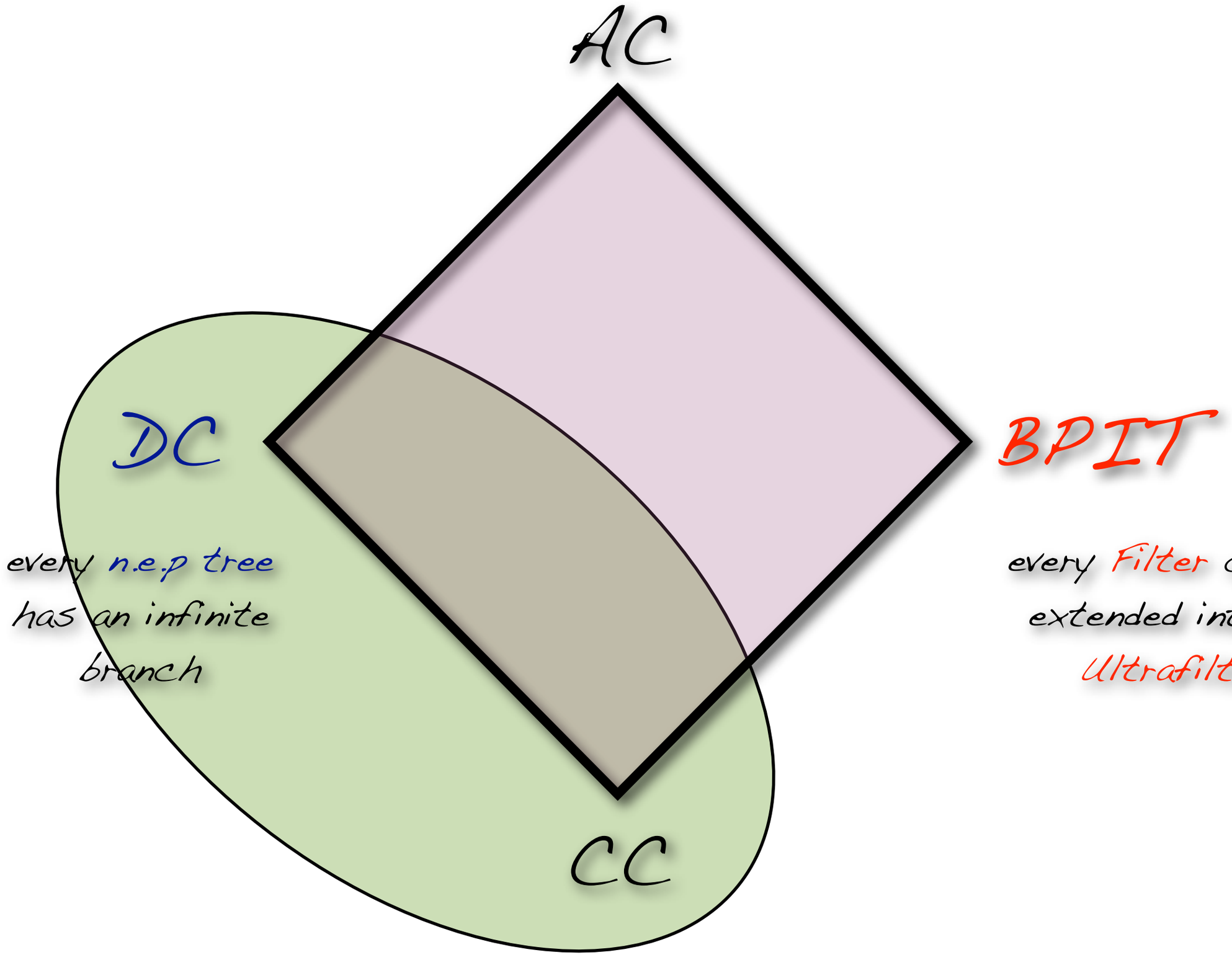
BPIT

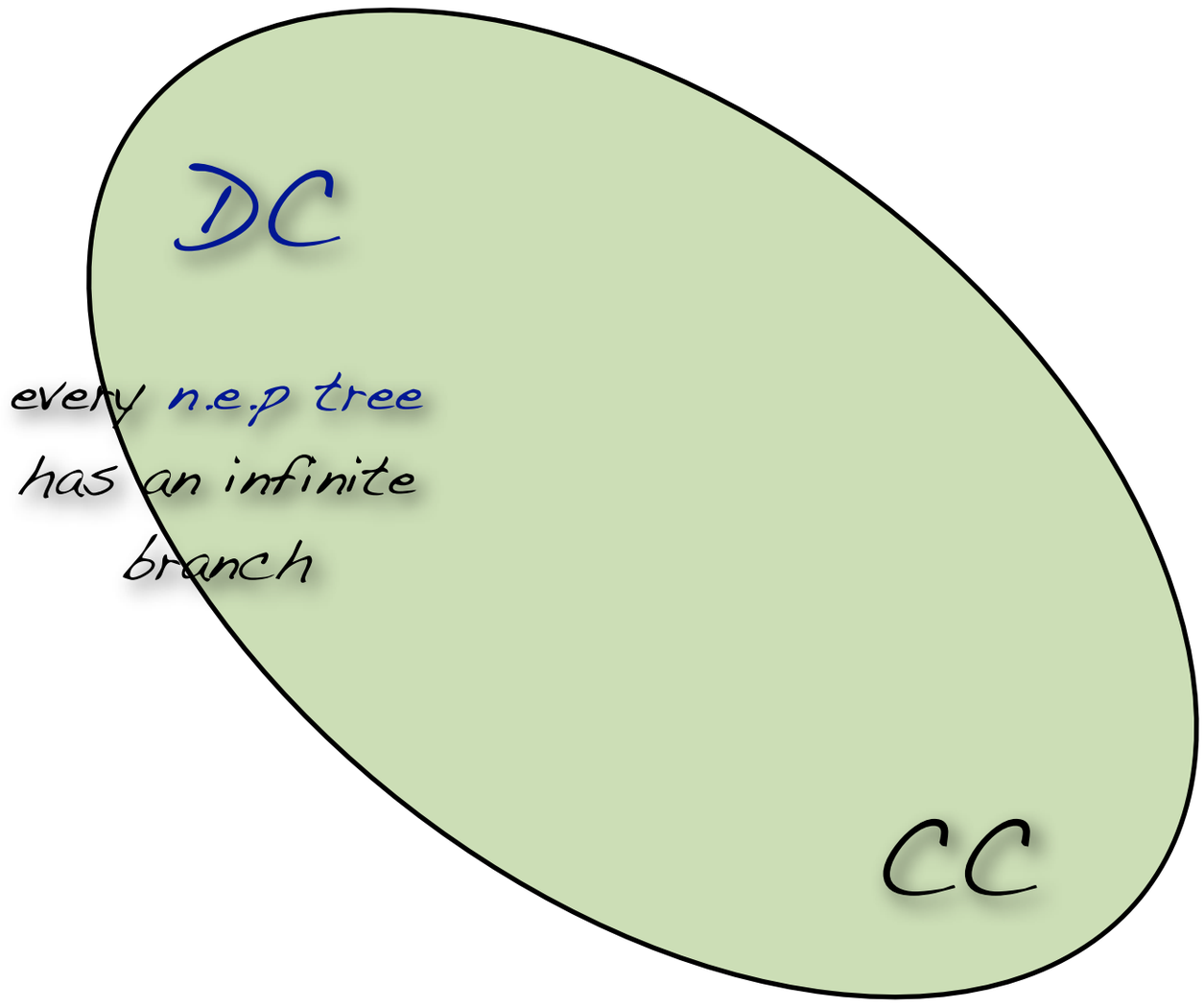
every Filter can be extended into an Ultrafilter

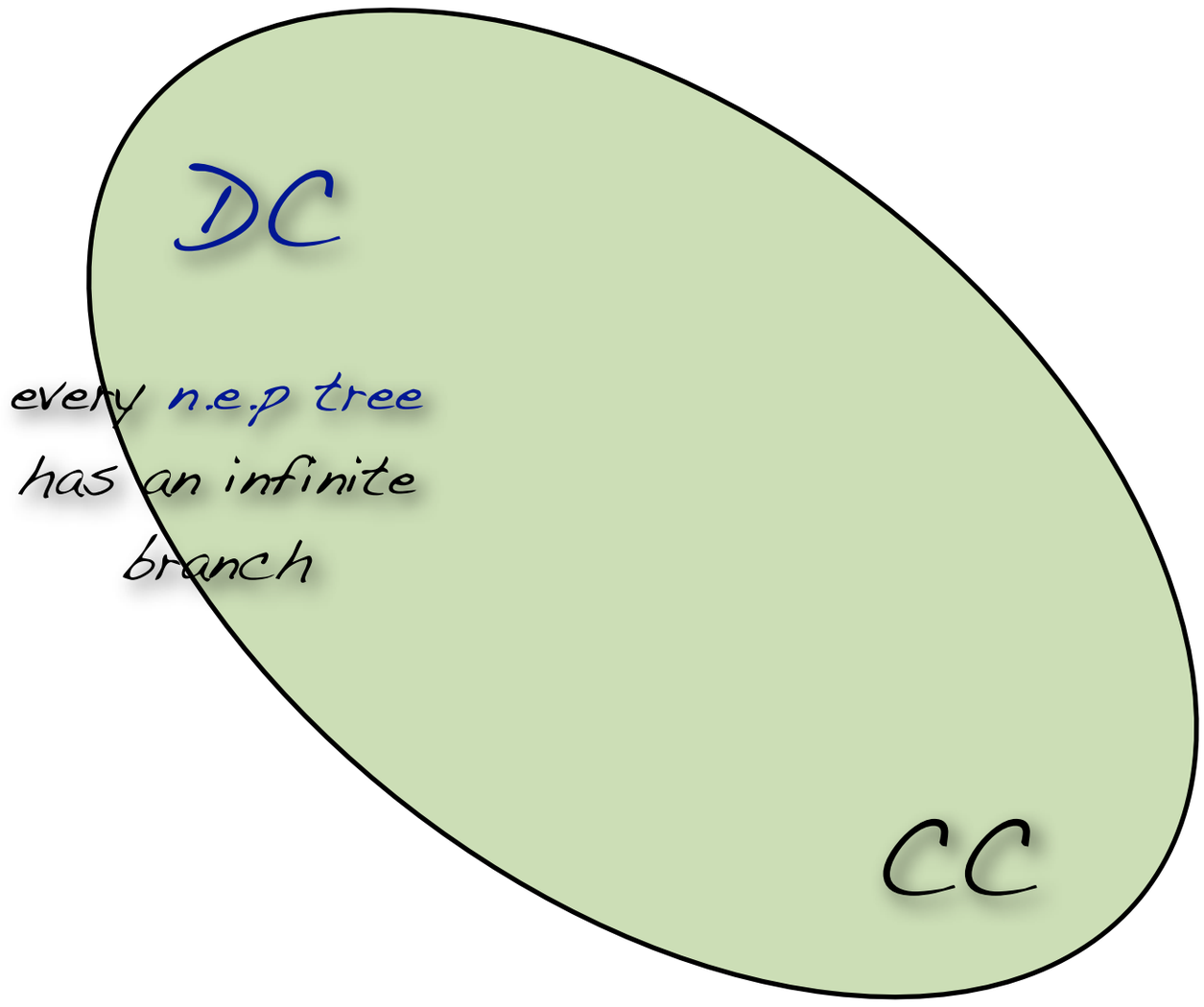
CC

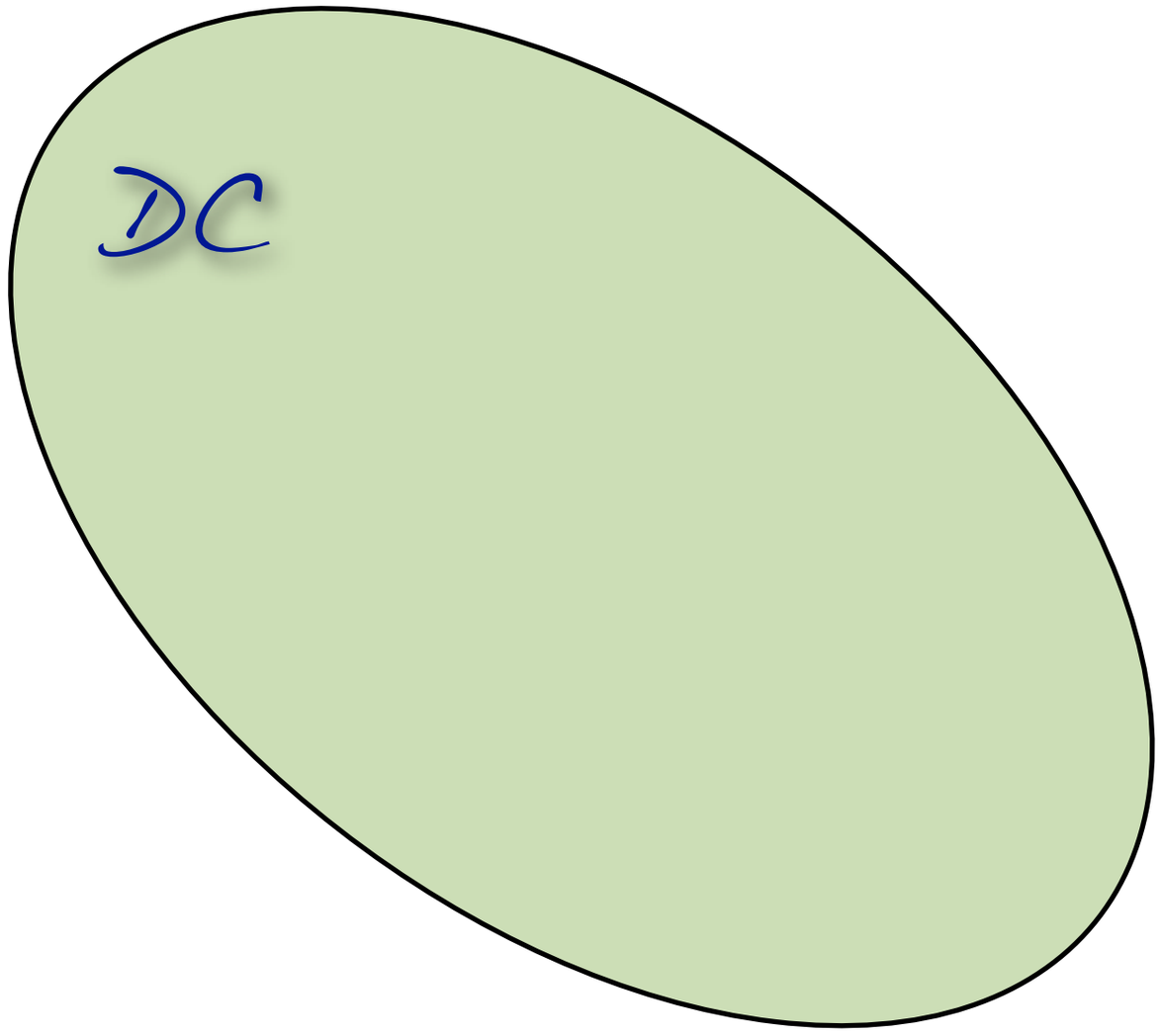


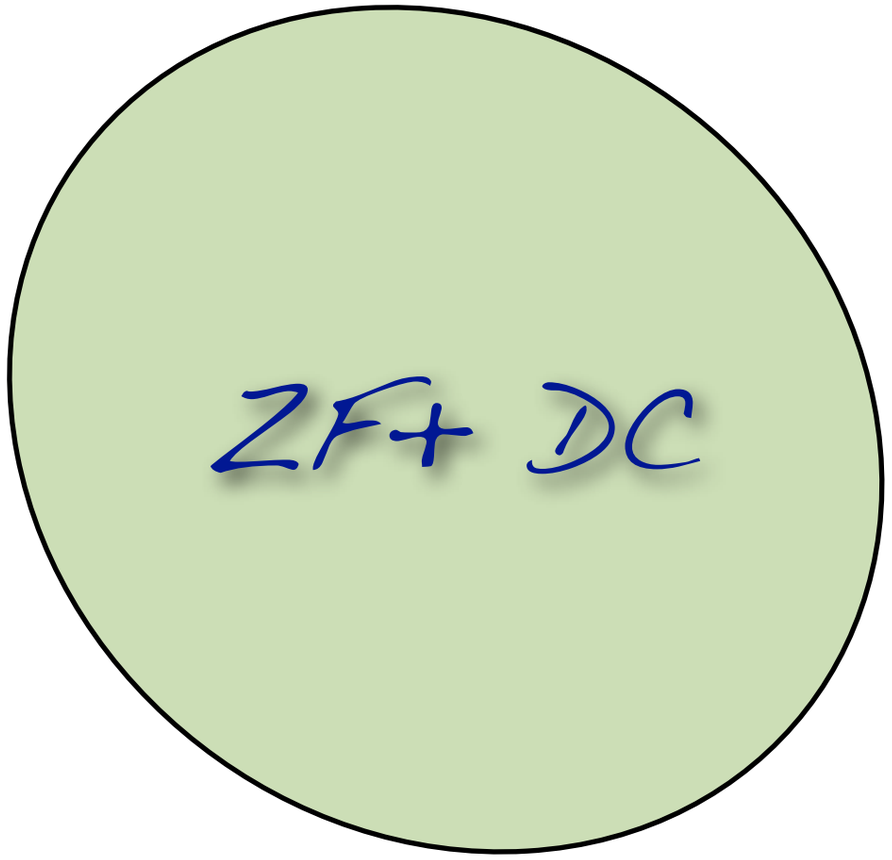
UNIL | Université de Lausanne


AC
DC
BPIT
every n.e.p tree
has an infinite
branch
every Filter can be
extended into an
Ultrafilter
CC


DC
every n.e.p tree
has an infinite
branch
CC


DC
every n.e.p tree
has an infinite
branch
CC


DC


ZF+ DC

There exists a non determined game

@ different approaches

(1)
- enumerate all strategies for Alice ($2^{\aleph_0}$ many)
- enumerate all strategies for Bob ($2^{\aleph_0}$ many)
- enumerate all $x \in \mathbb{N}^\omega$ ($2^{\aleph_0}$ many)
- by transfinite induction and a diagonal argument, construct payoff A s.t. neither Bob nor Alice has a w.s.

(2)
- similar to the swiss chocolate-game ?

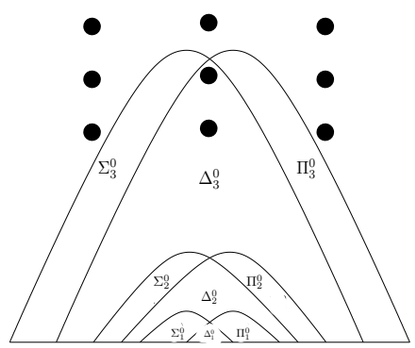
There exists a non determined game

(2) - similar to the swiss chocolate-game ?

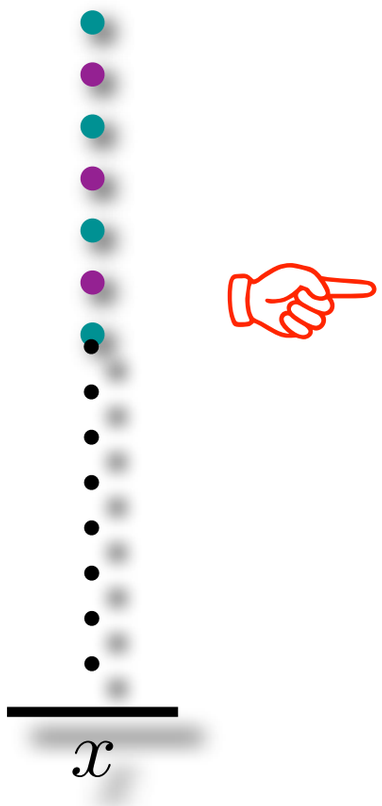


An example: evaluation game for 1st order logic

non determined game

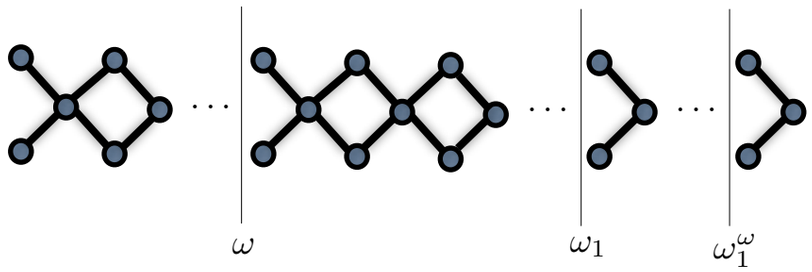


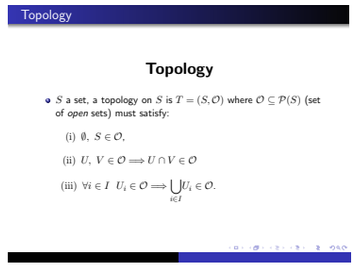
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

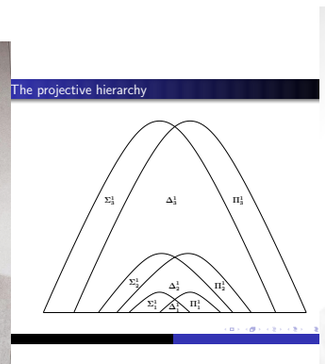
Games that construct mappings



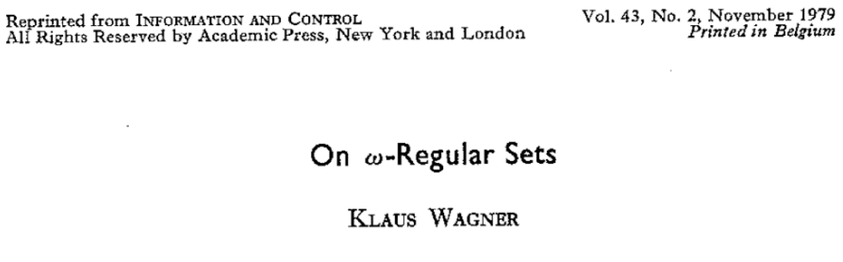










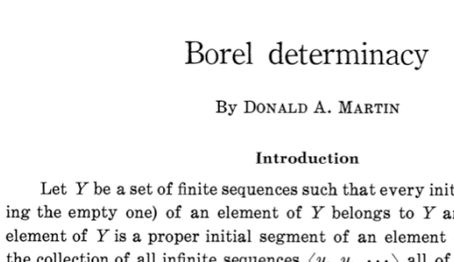


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of



UNIL | Université de Lausanne

- Determinacy for Open and Closed sets

- What about larger classes ?



- Open and closed ⇨ Topology ⇨ Measure Theory ⇨ Lebesgue Measure

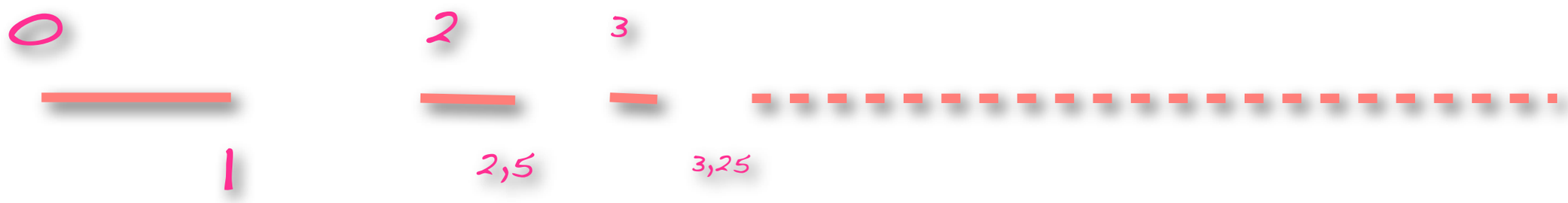
UNIL | Université de Lausanne



- Determinacy for Open and Closed sets

- What about larger classes ?

- Open and closed ⇨ Topology ⇨ Measure Theory ⇨ Lebesgue Measure



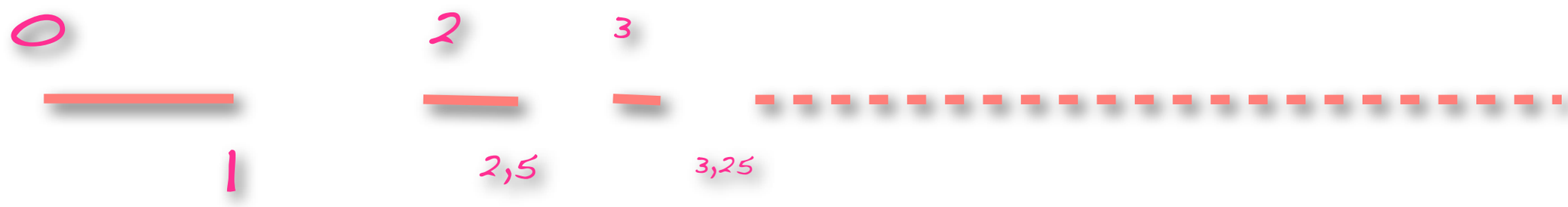


UNIL | Université de Lausanne

@ Determinacy for Open and Closed sets

@ What about larger classes ?

@ Open and closed ⇨ Topology ⇨ Measure Theory ⇨ Lebesgue Measure



$\mu(\text{—} + \text{—} + \text{—} + \ldots\ldots\ldots) = 1$

UNIL | Université de Lausanne



UNIL | Université de Lausanne

☑ Open (and Closed) sets

☑ Open (and Closed) sets

☑ closed under countable union

- ☑ Open (and Closed) sets
- ☑ closed under countable union
- ☑ closed under complementation

- [x] Open (and C
- [x] closed unde
- [x] closed unde



UNIL | Université de Lausanne

☑ Open (and Closed) sets

☑ closed under countable union

☑ closed under complementation

} Borel Sets

☑ Open (and Closed) sets

☑ closed under countable union

☑ closed under complementation

} Borel Sets

☼ Set up in a nice hierarchy by Baire ☼



UNIL | Université de Lausanne

- ☑ Open (and Closed) sets
- ☑ closed under countable union
- ☑ closed under complementation

} Borel Sets

☼ Set up in a nice hierarchy by Baire ☼

Unil
UNIL | Université de Lausanne

☑ Open (and Closed) sets

☑ closed under countable intersection

☑ closed under complementation

} Borel Sets

☼ Set up in a nice hierarchy by Baire ☼

☑ Open (and Closed) sets

☑ closed under countable intersection

☑ closed under complementation

} Borel Sets

☼ Set up in a nice hierarchy by Baire ☼

- ☑ Open and Closed sets
- ☑ closed under countable intersection
- ☑ closed under countable union

} Borel Sets

☀ Set up in a nice hierarchy by Baire ☀

☑ Open and Closed sets

☑ closed under countable intersection

☑ closed under countable union

} Borel Sets

☀ Set up in a nice hierarchy by Baire ☀

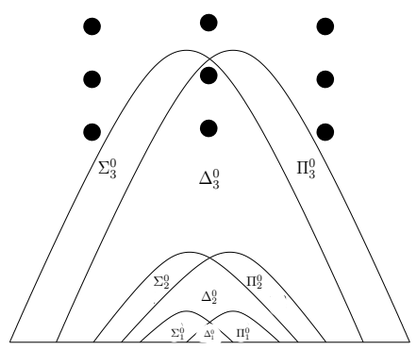
- ☑ Open and Closed sets
- ☑ closed under countable intersection
- ☑ closed under countable union



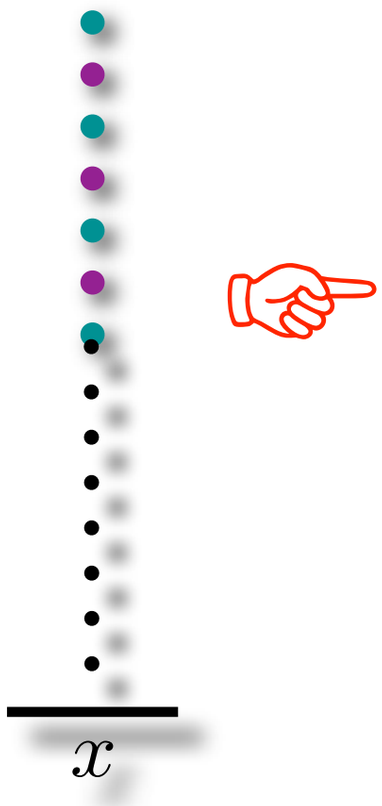


An example: evaluation game for 1st order logic

non determined game

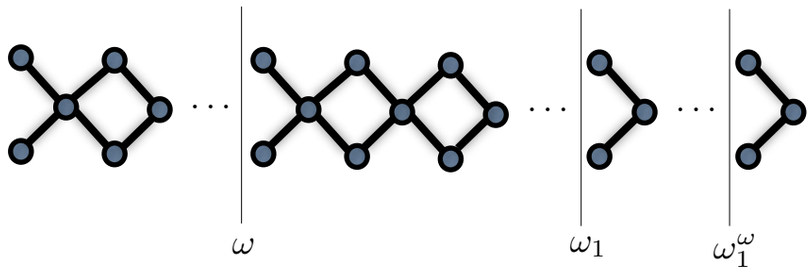


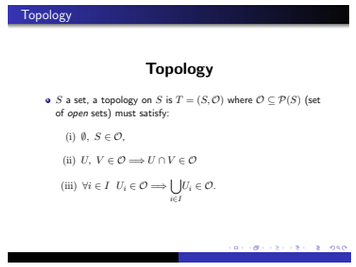
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

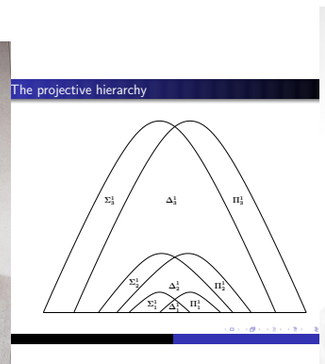
Games that construct mappings



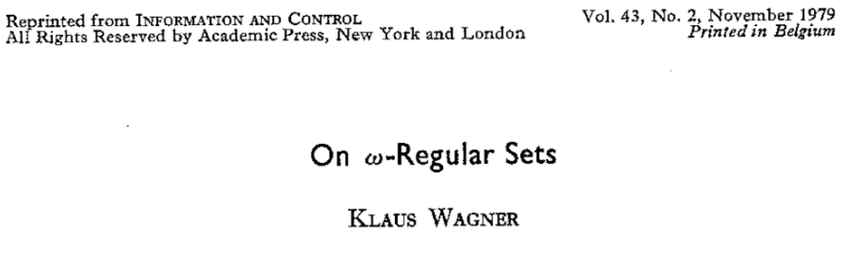










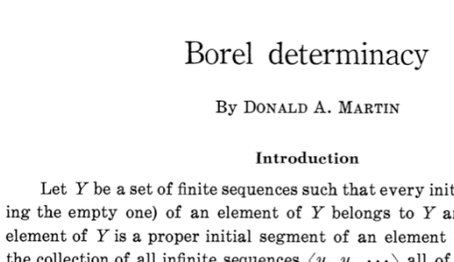


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



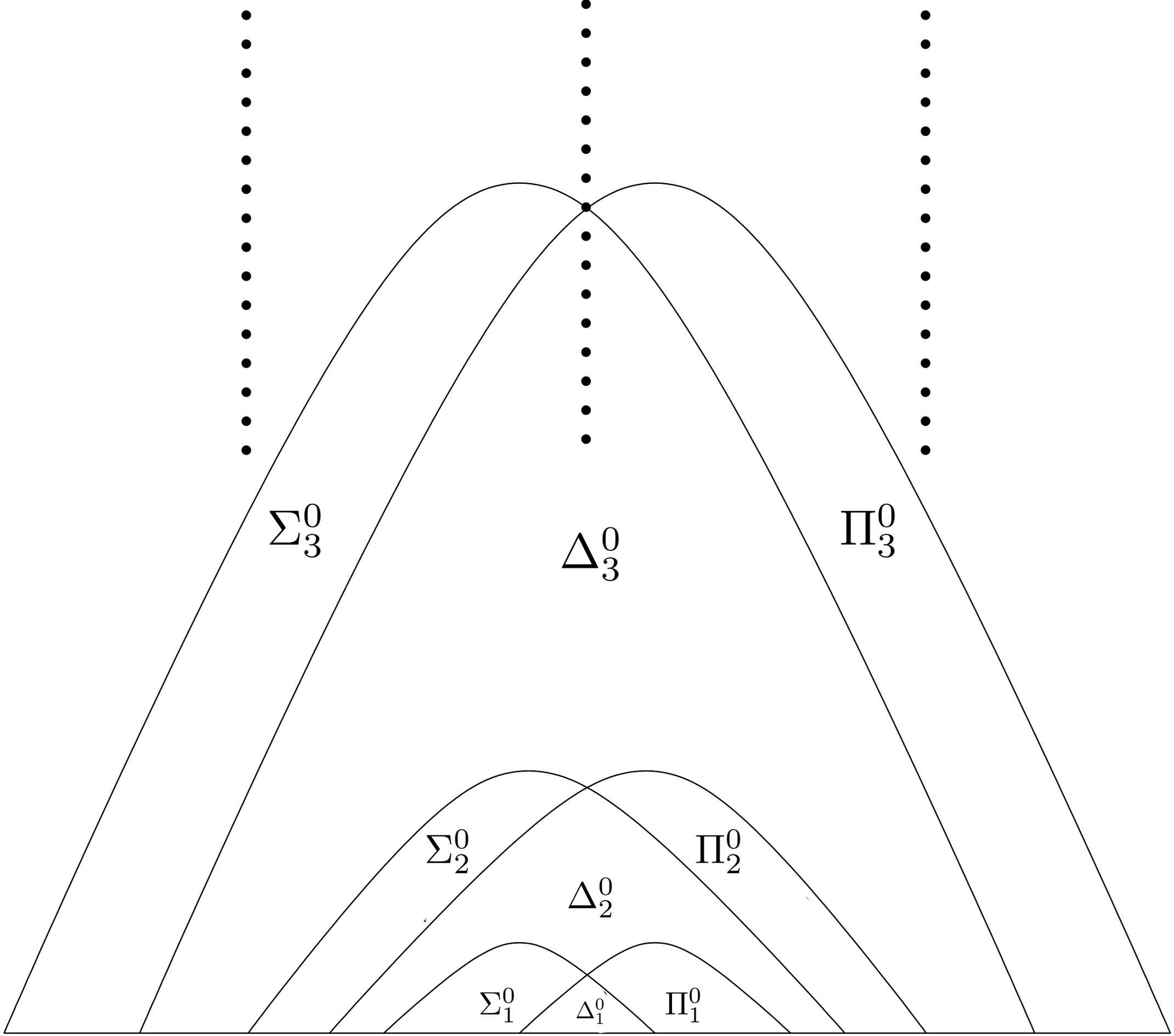
Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of


Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0

☑ closed under countable union

☑ closed under countable intersection

☑ Open and Closed sets

UNIL | Université de Lausanne

☑ closed under countable union



quantifying existentially over integers ($\exists n \ldots \Phi[\ldots,n,\ldots]$)

☑ closed under countable intersection

☑ Open and Closed sets

UNIL | Université de Lausanne

☑ closed under countable union

✎ quantifying existentially over integers ($\exists n \ldots . \Phi[\ldots, n, \ldots .]$)

☑ closed under countable intersection

☑ Open and Closed sets

✎ quantifying universally over integers ($\forall n \ldots . \Phi[\ldots, n, \ldots .]$)

☑ closed under countable union

✎ quantifying existentially over integers ($\exists n \ldots \Phi[\ldots,n,\ldots]$)

☑ closed under countable intersection

✎ quantifying universally over integers ($\forall n \ldots \Phi[\ldots,n,\ldots]$)

☑ Open and Closed sets

✎ recursively enumerable (co-recursively enumerable)

☑ closed under $\exists n$ ....

☑ closed under $\forall n$ ....

☑ recursively enumerable (co-recursively enumerable)

☑ recursively enumerable (co-recursively enumerable)

☑ closed under $\exists n$ ....

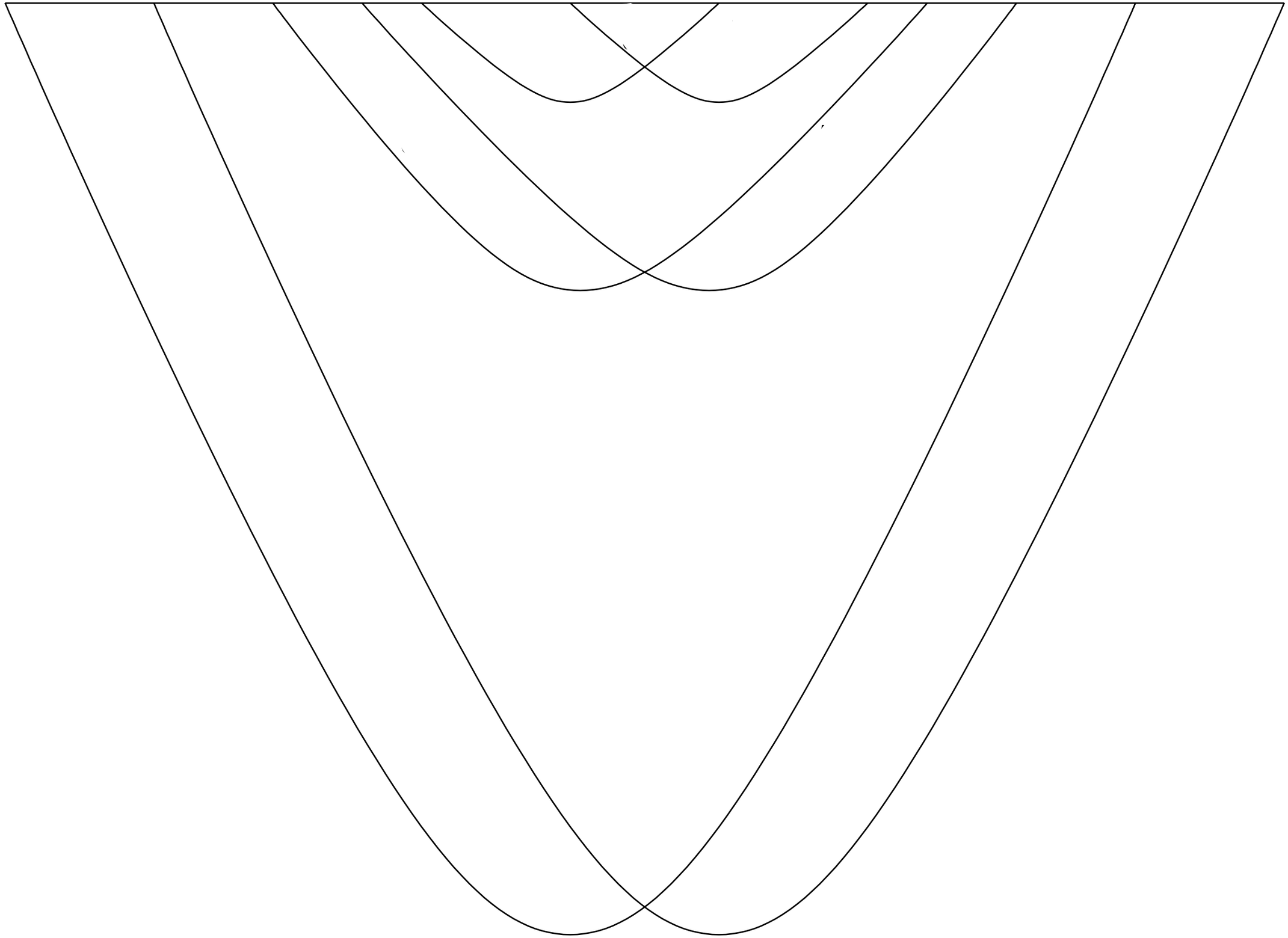
☑ closed under $\forall n$ ....

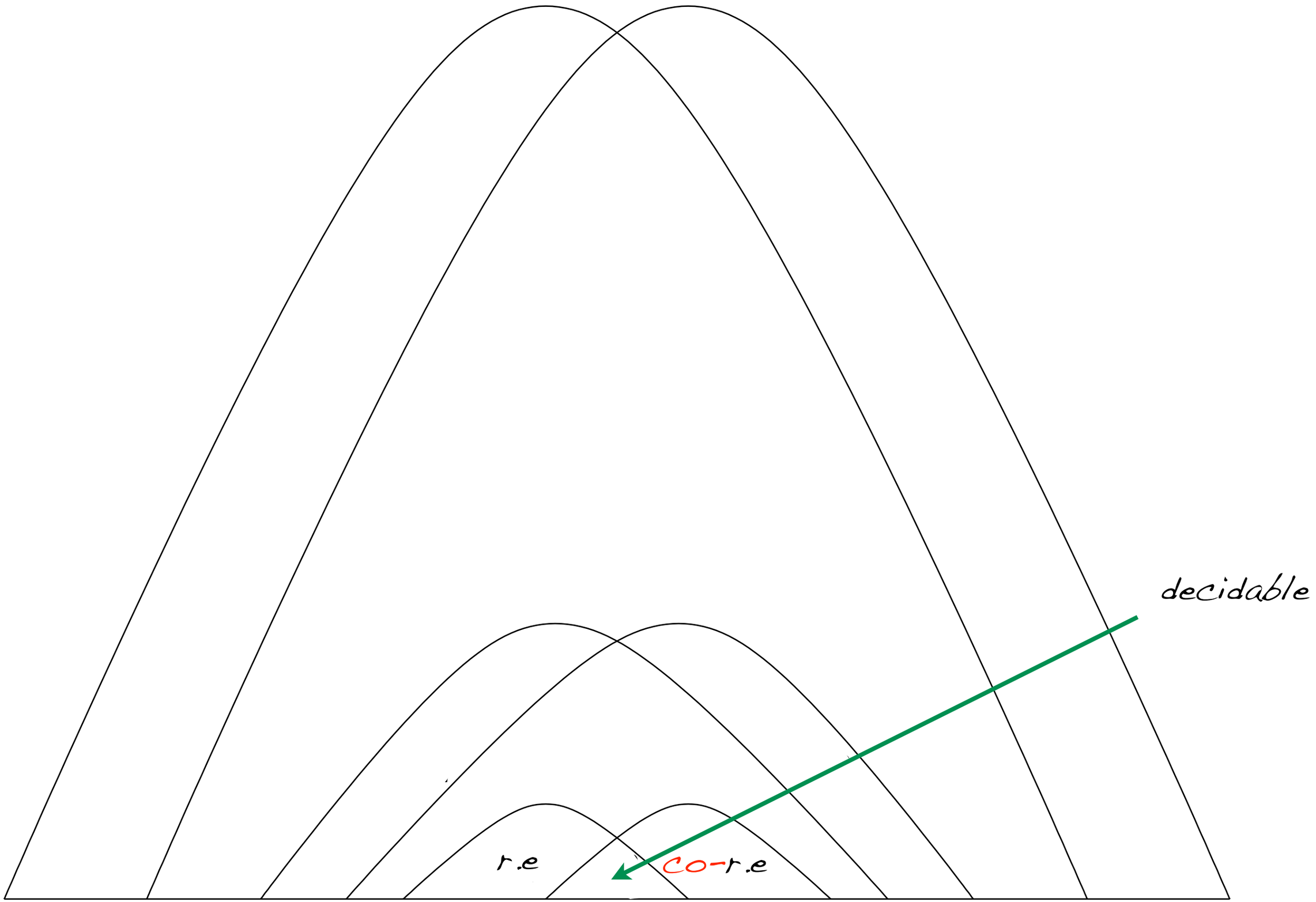
☑ recursively enumerable

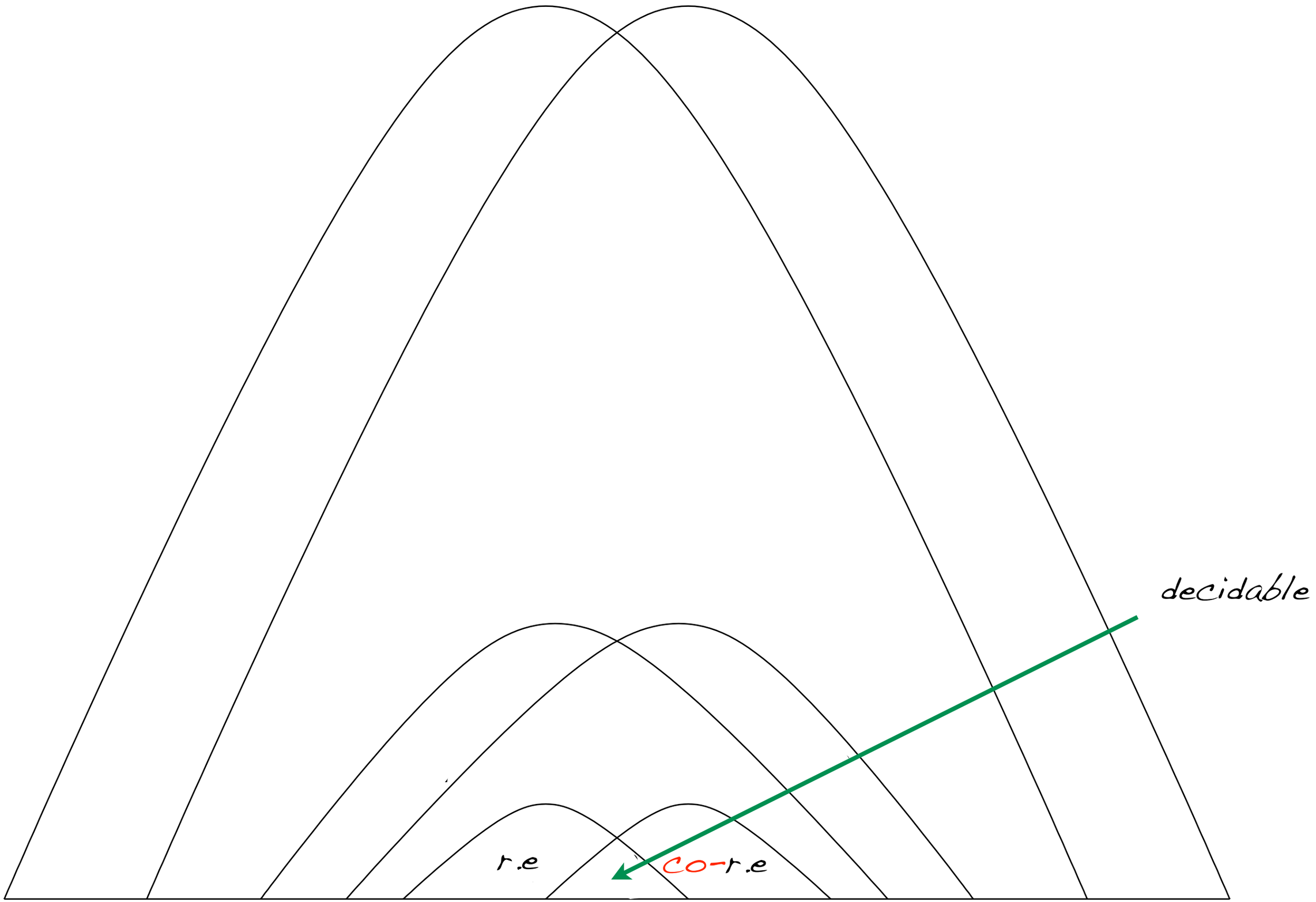
☑ closed under $\exists n$ ....

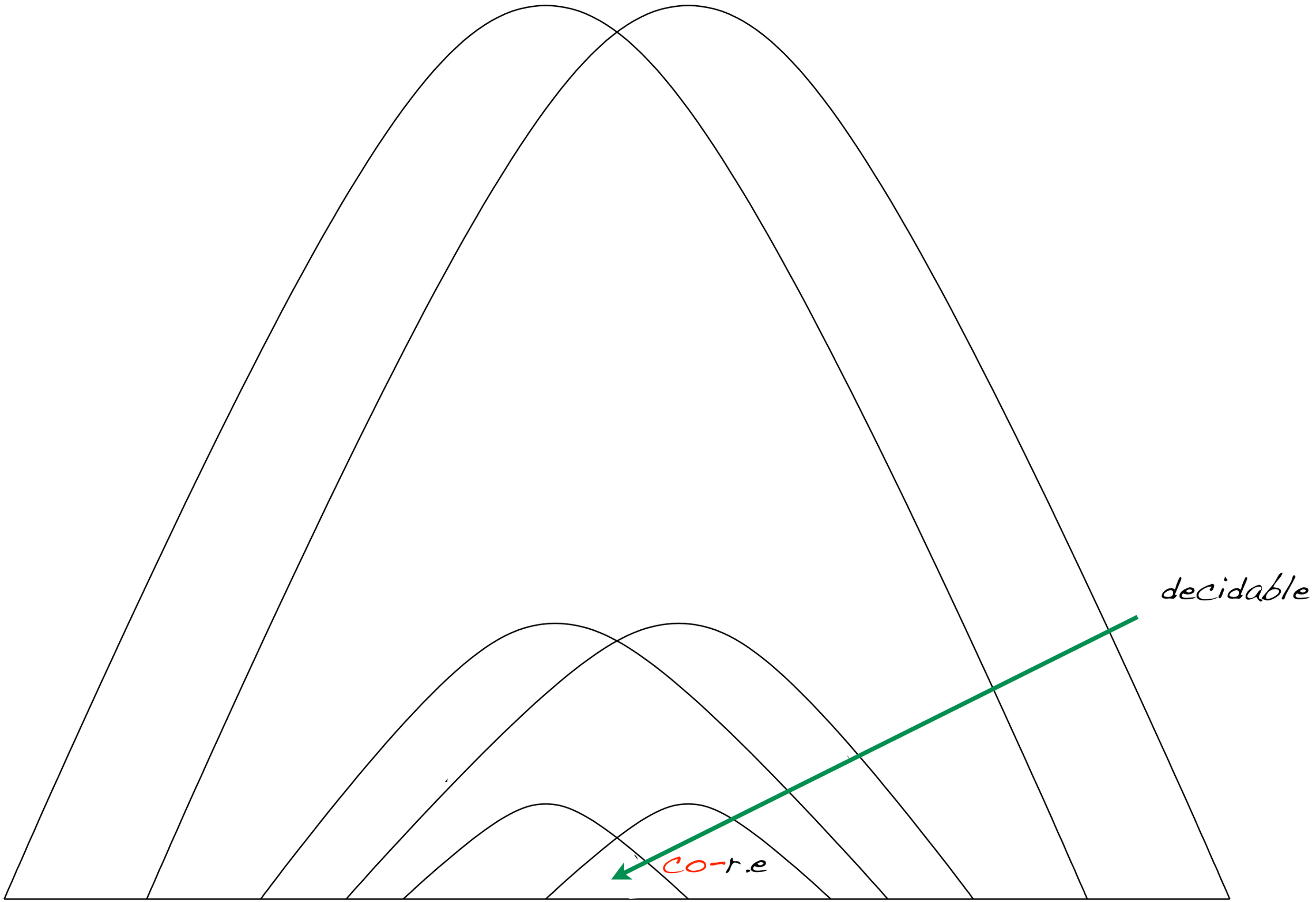
☑ closed under $\neg$

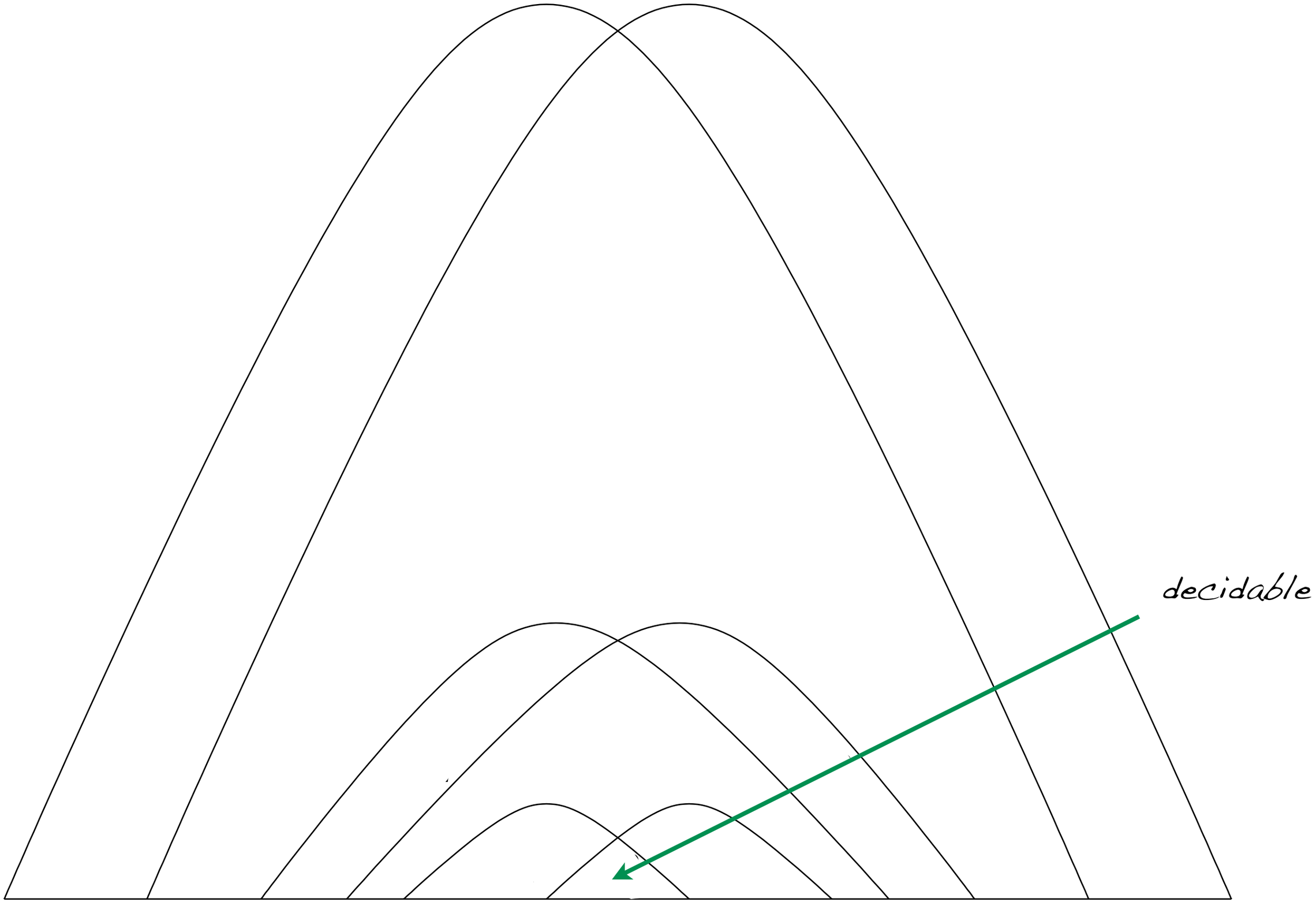
UNIL | Université de Lausanne

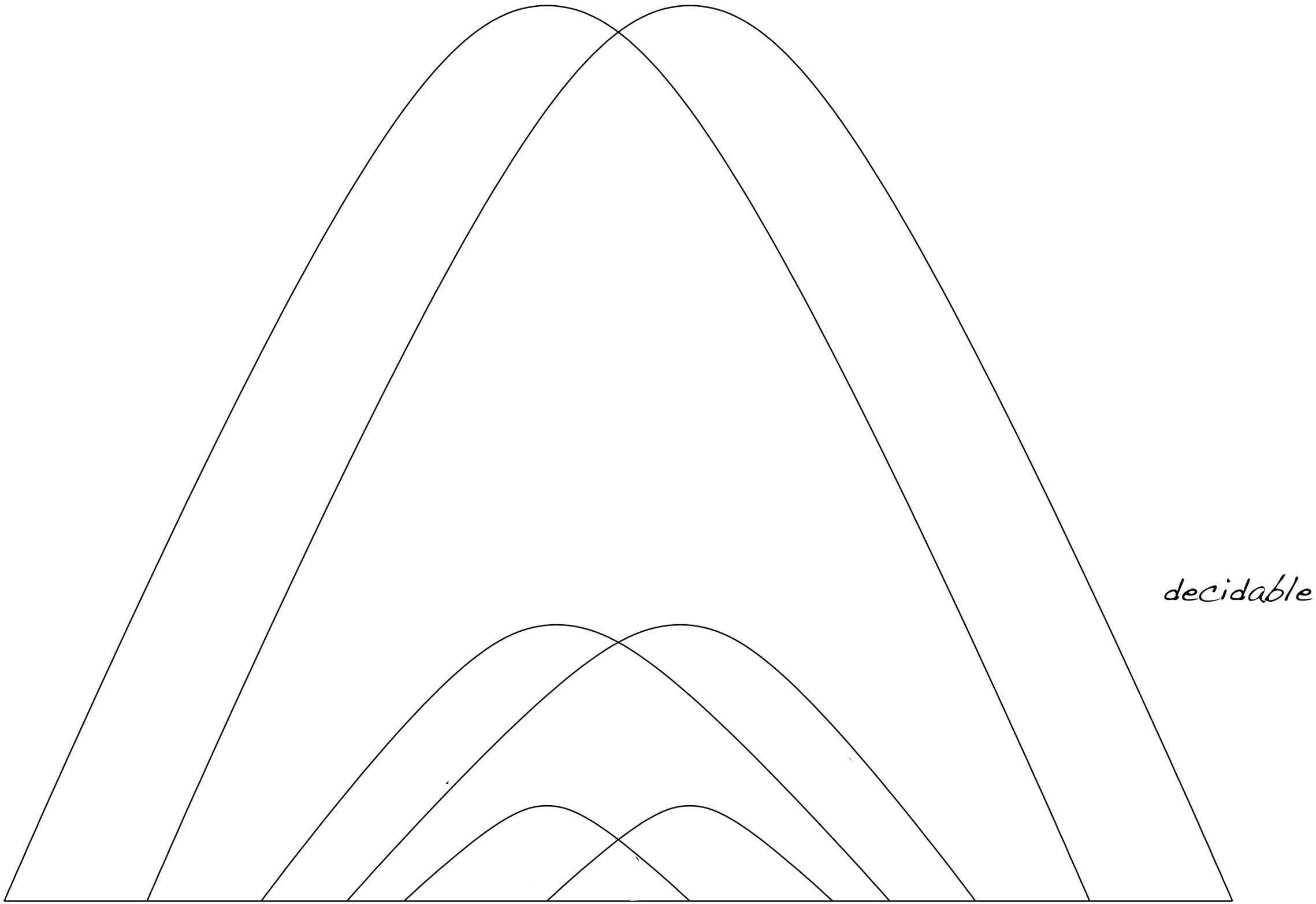


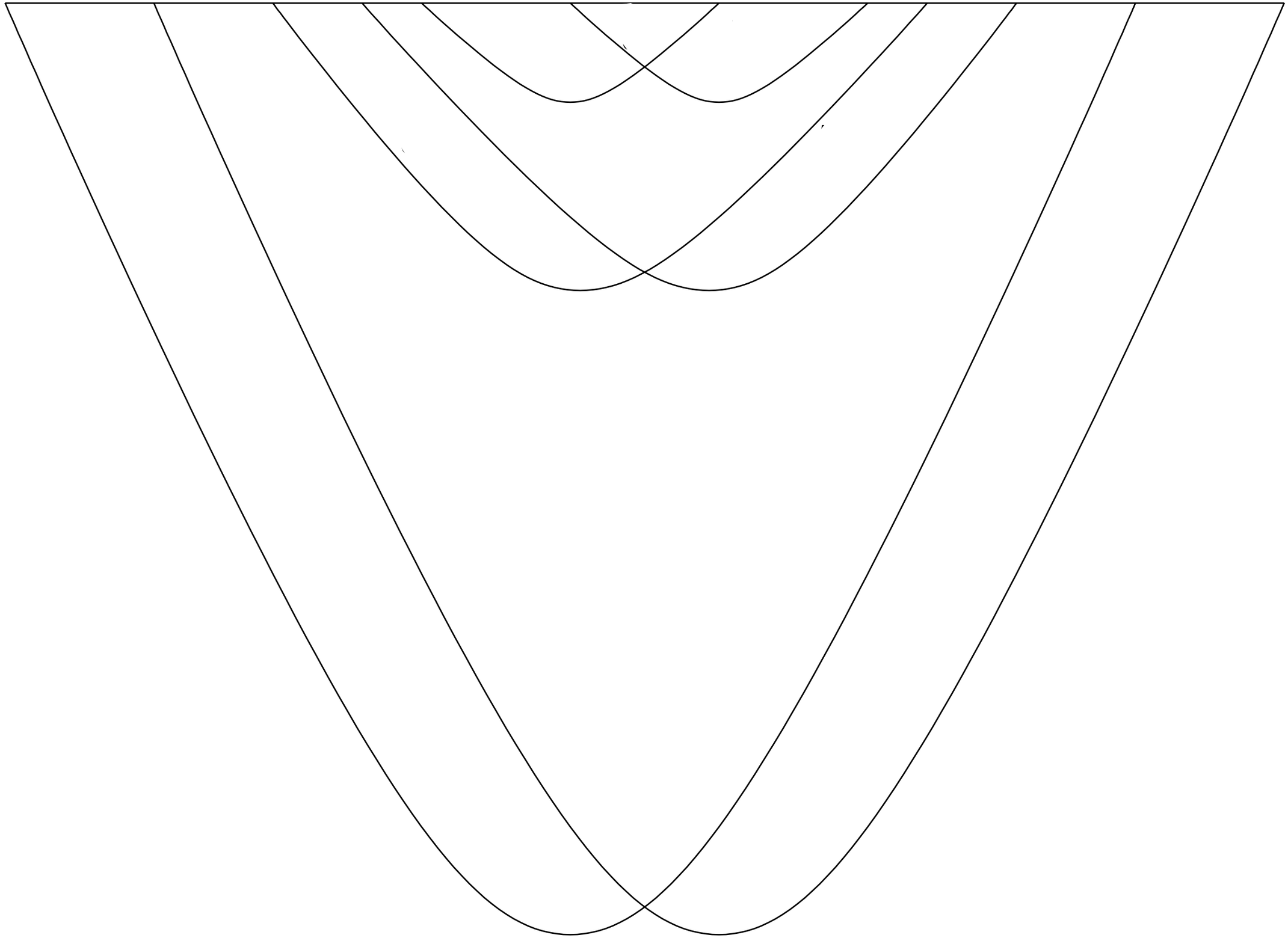

decidable
r.e
co-r.e

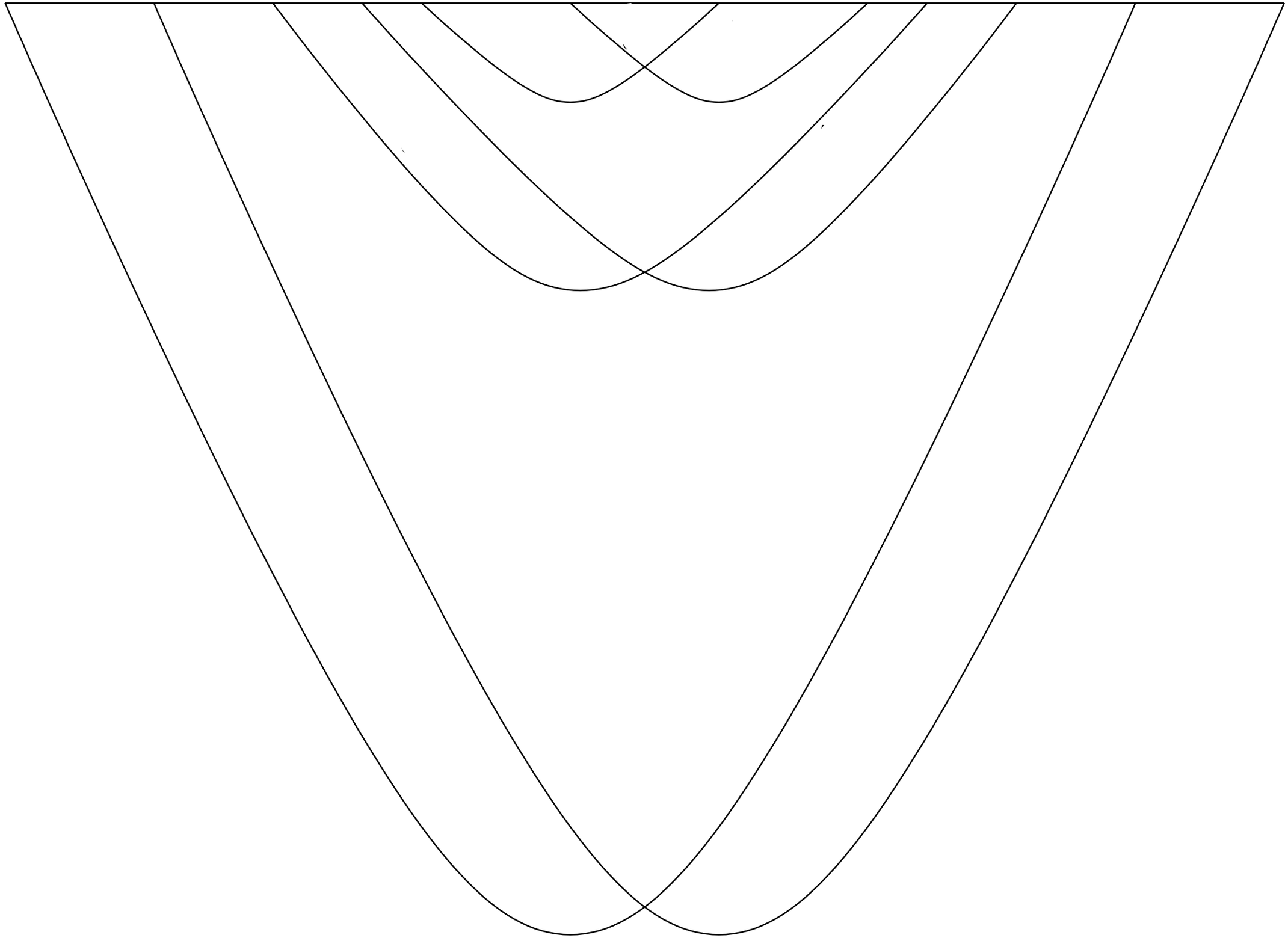

decidable
r.e
co-r.e

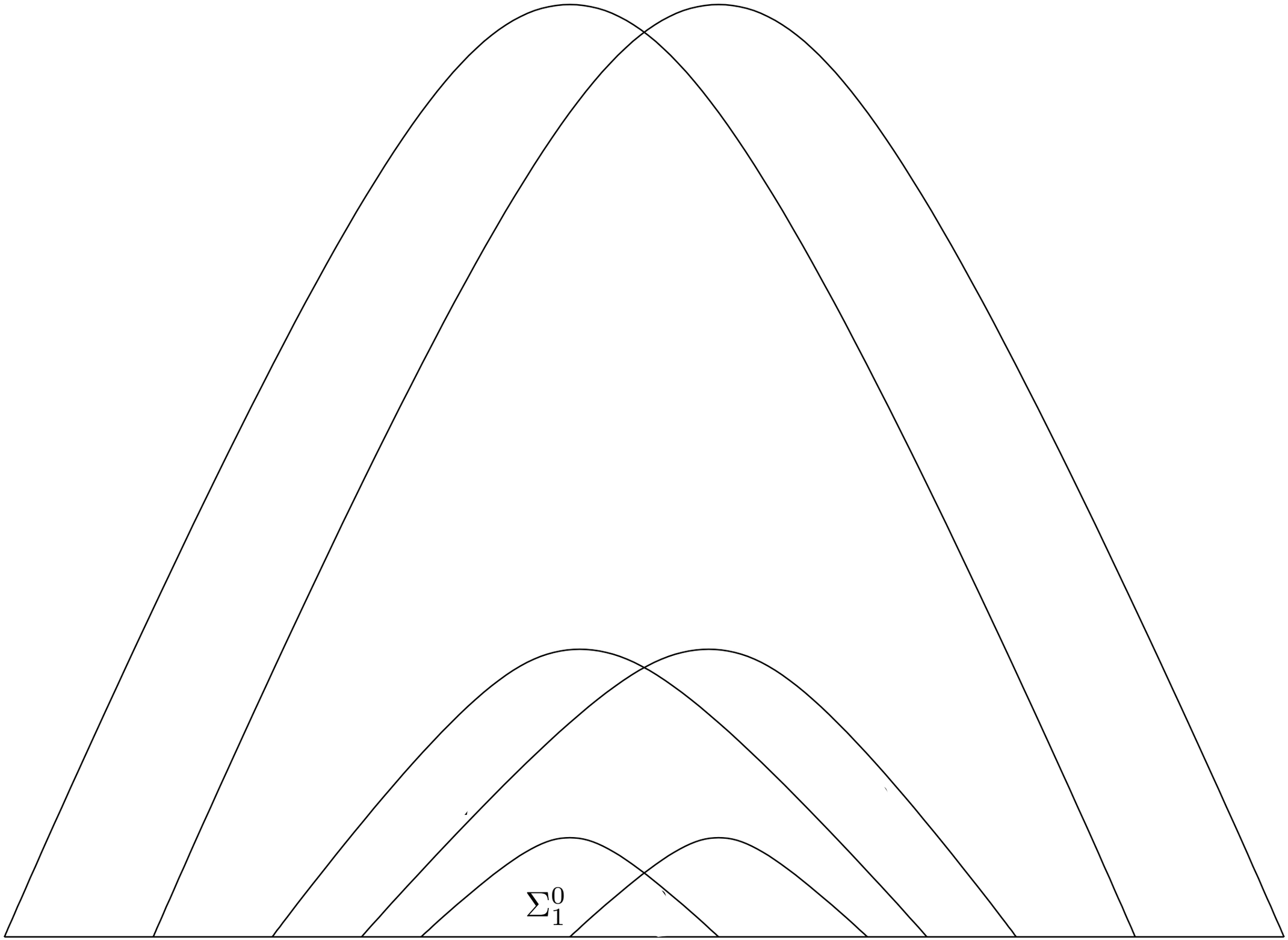

decidable
co-r.e

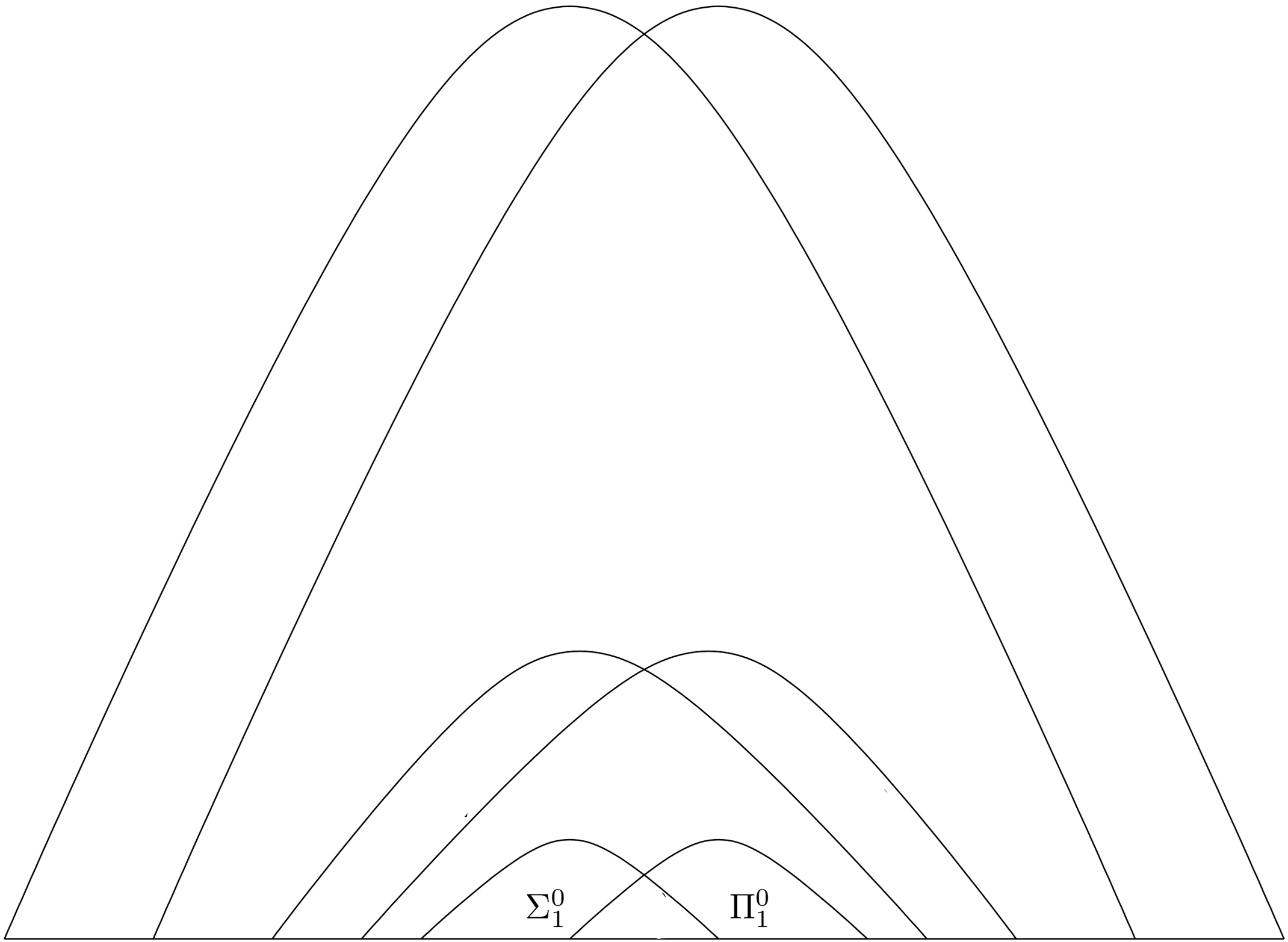

decidable

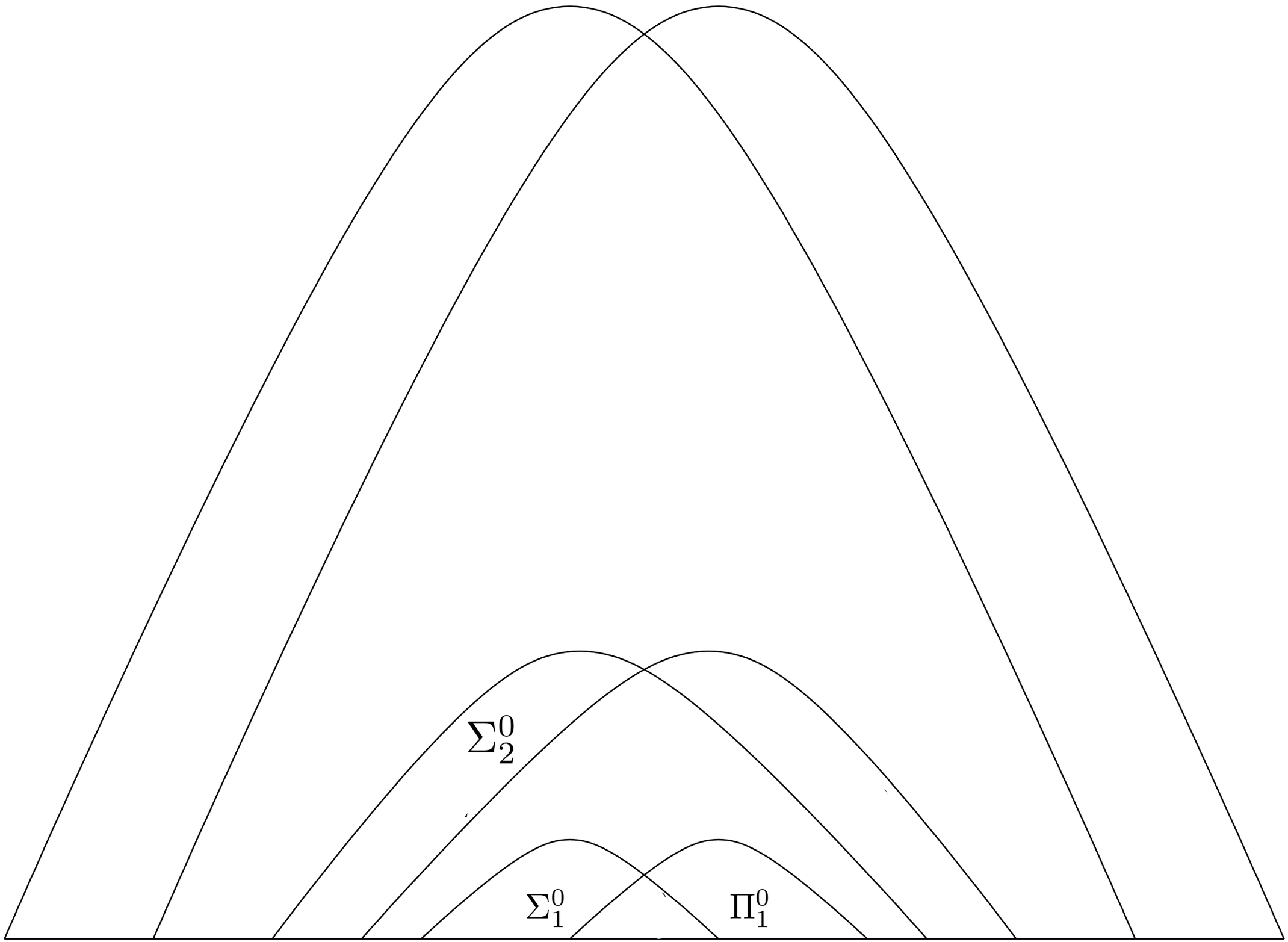

decidable

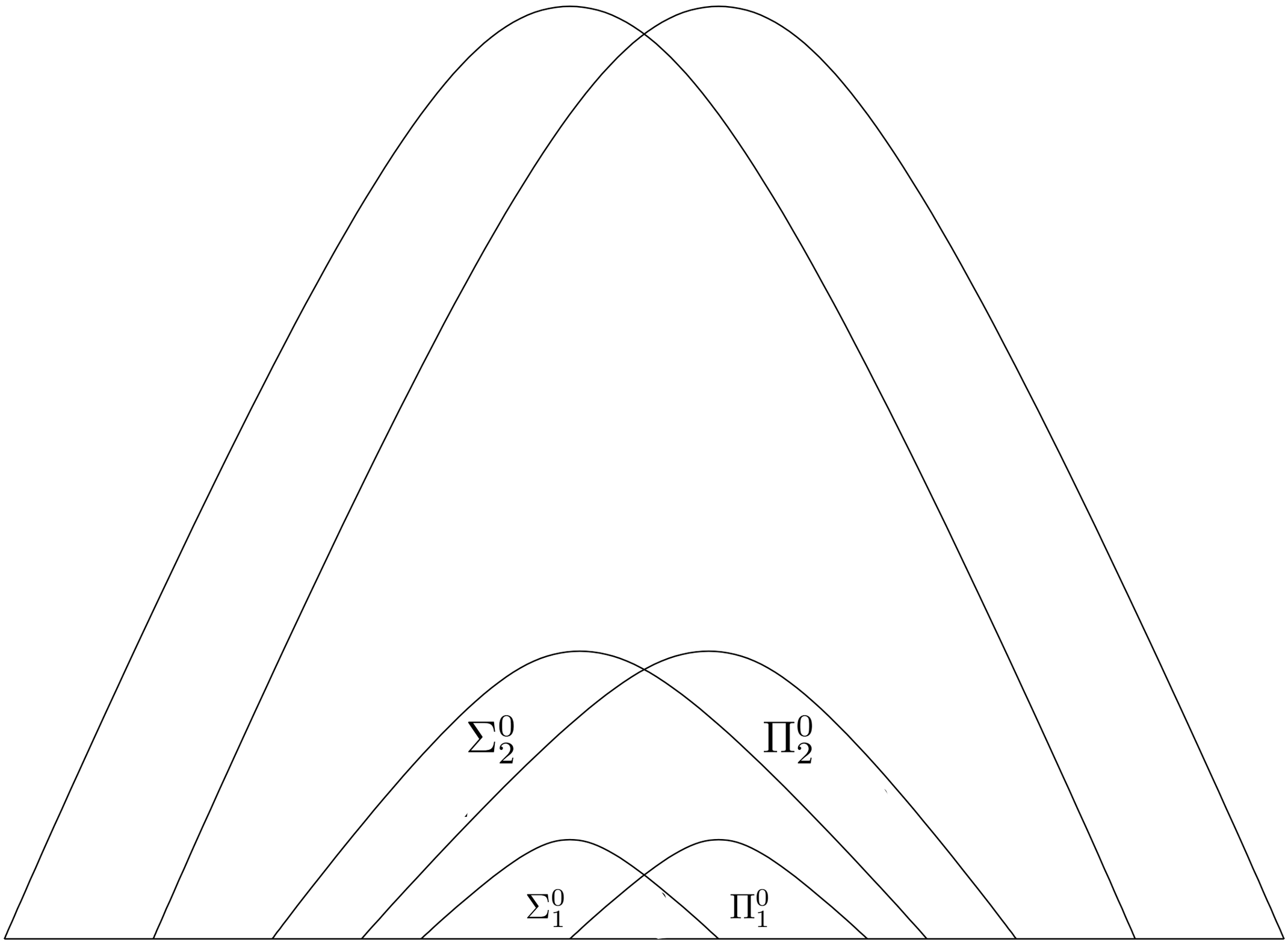


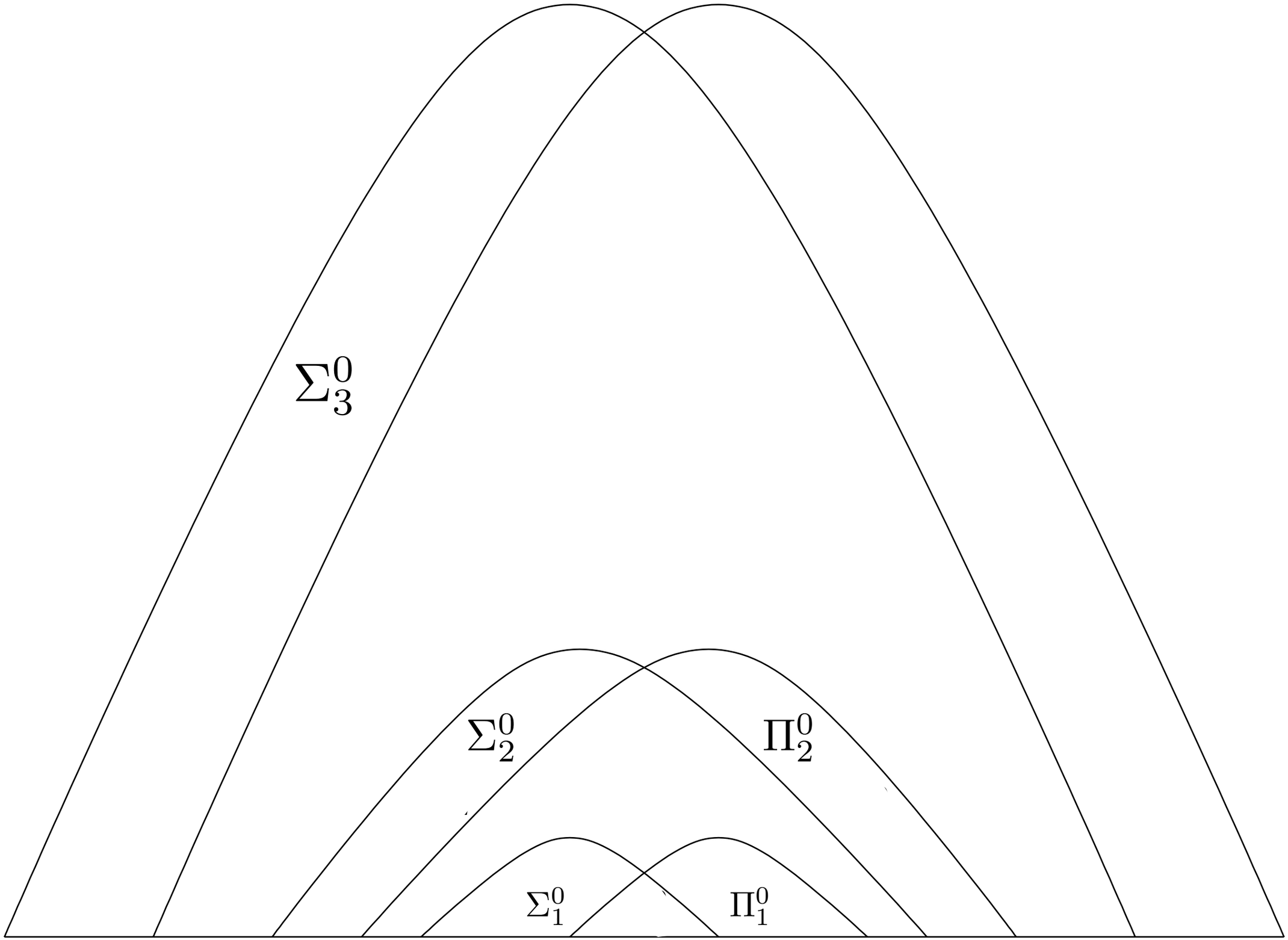


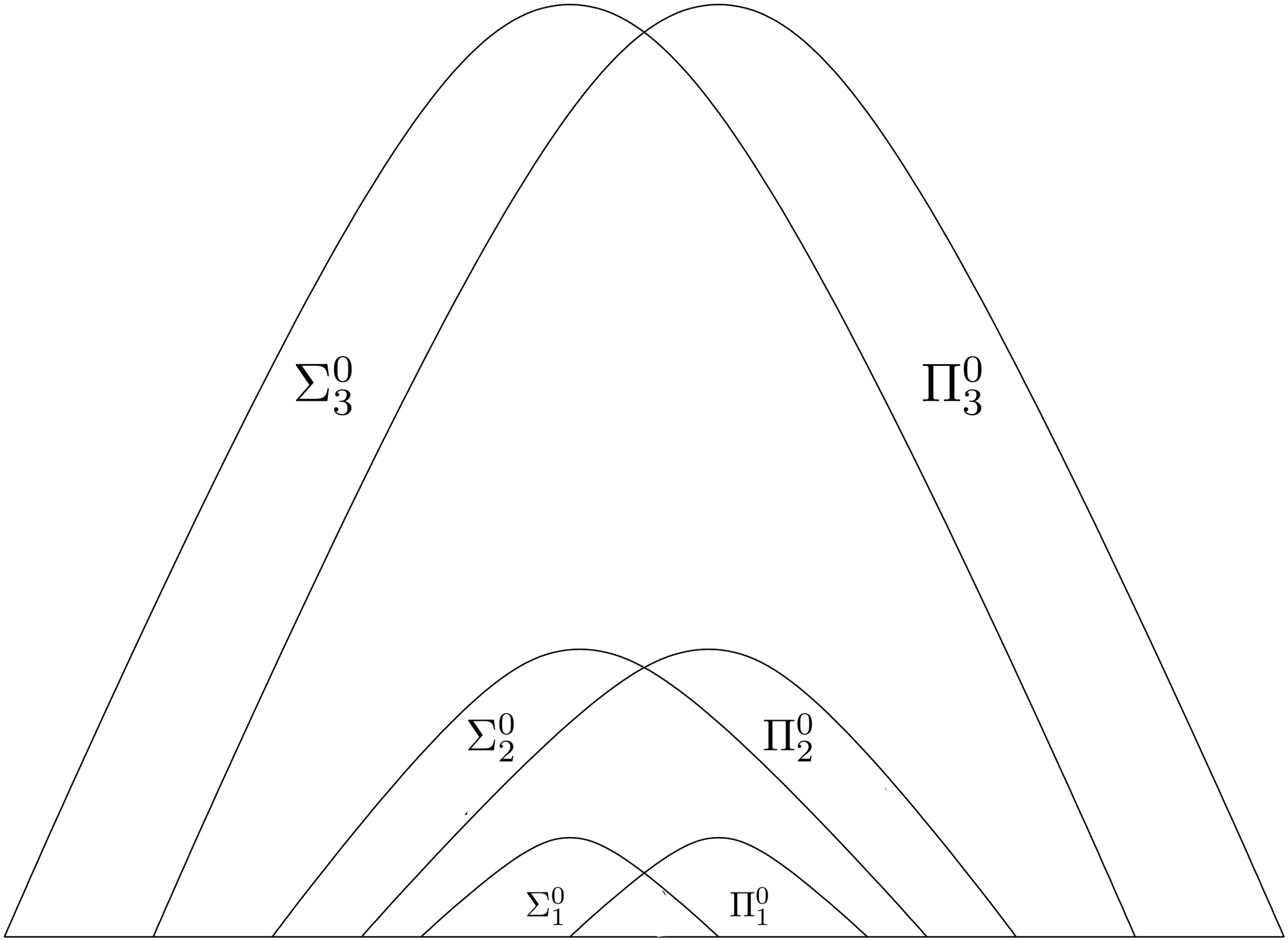

Σ_1^0

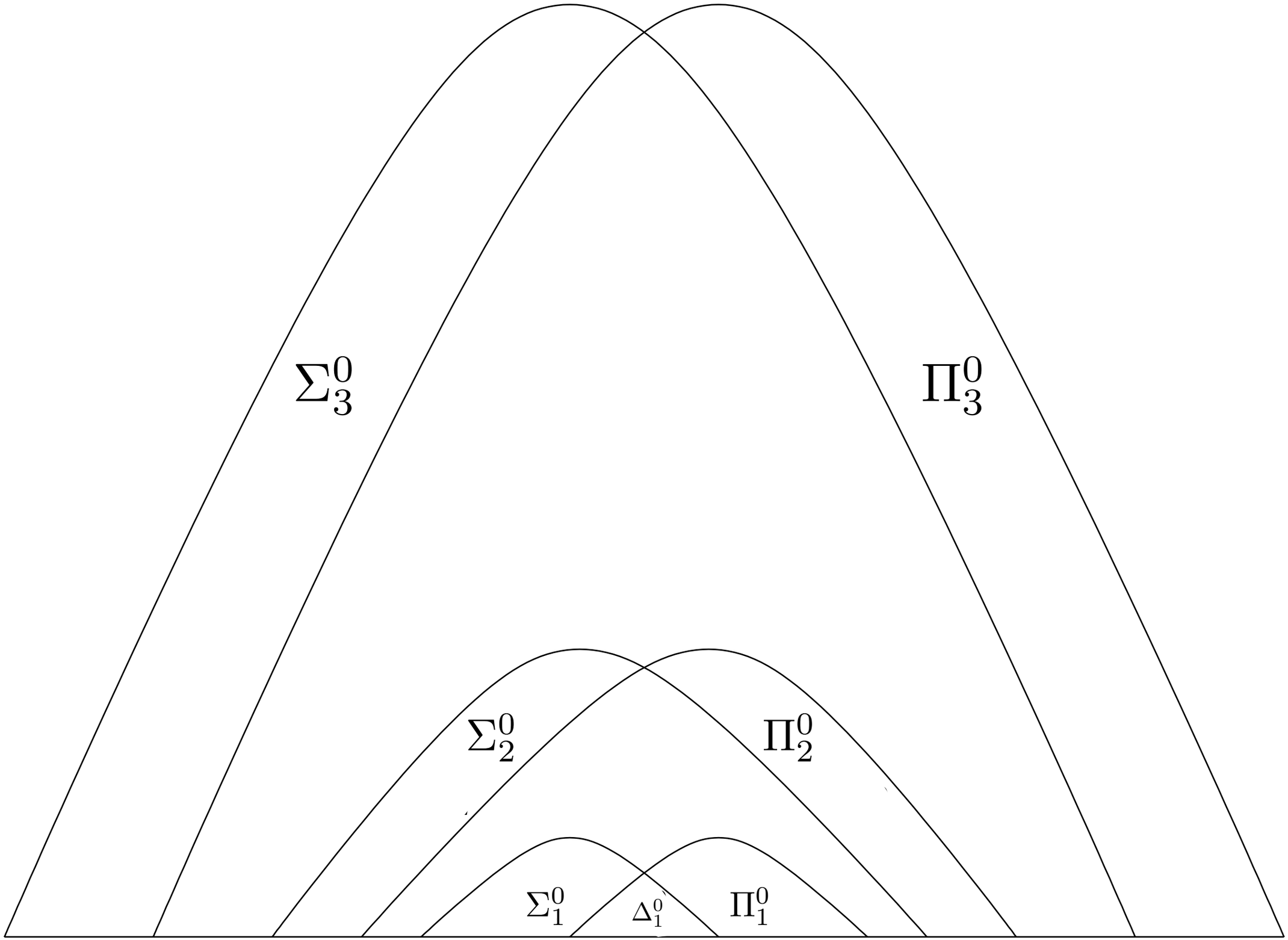

Σ_1^0
Π_1^0

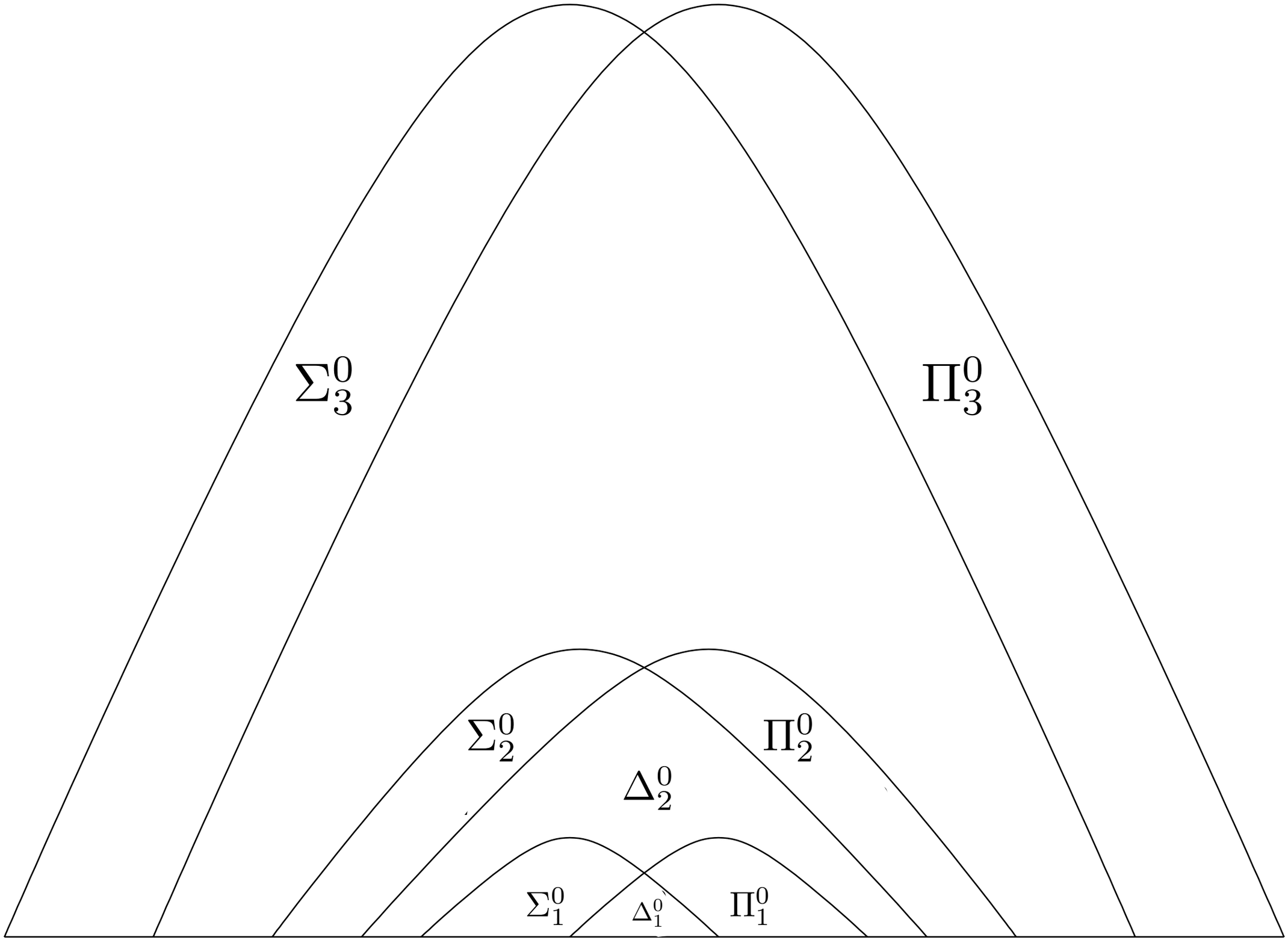

Σ_2^0
Σ_1^0
Π_1^0

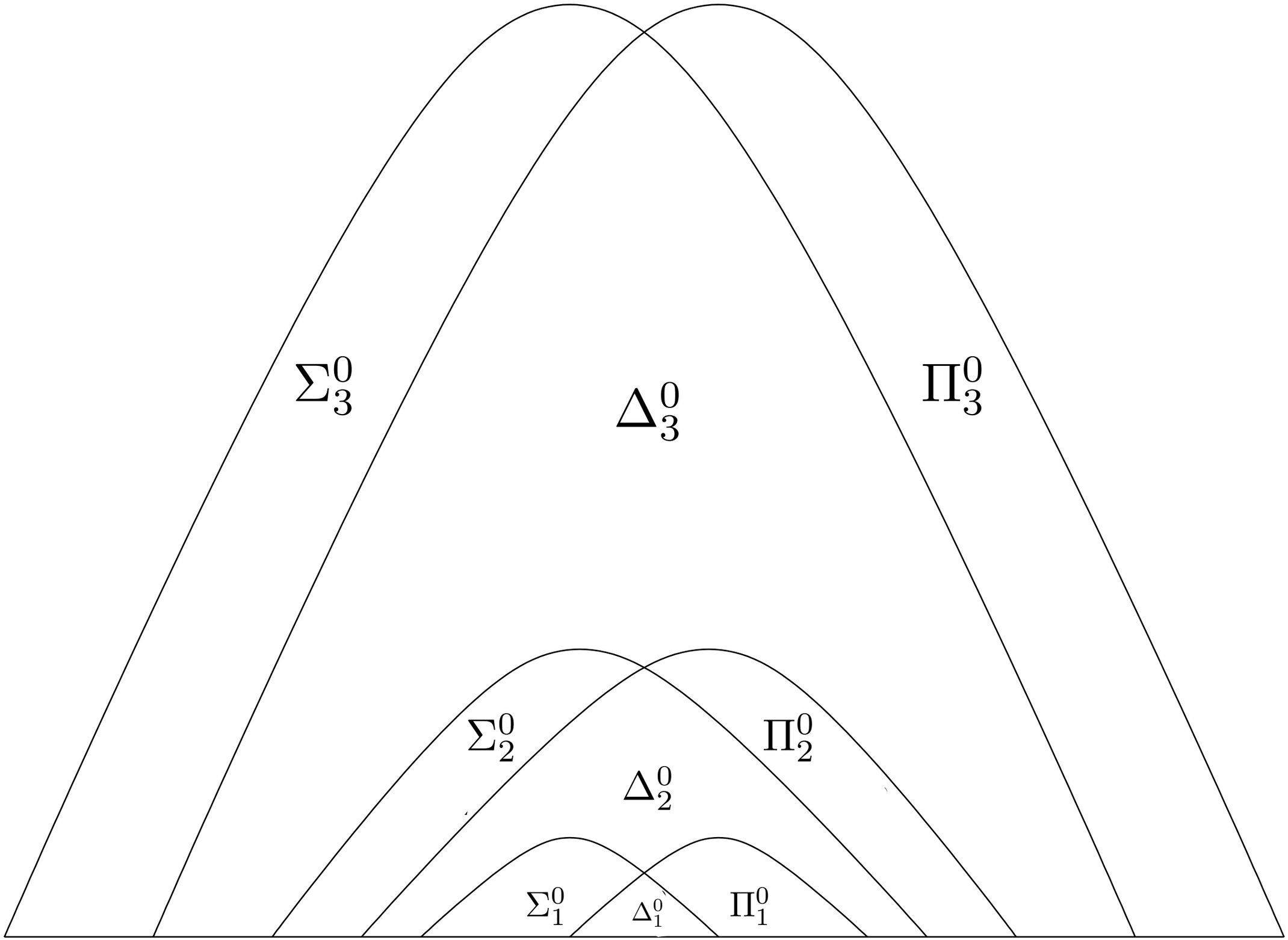

Σ_2^0
Π_2^0
Σ_1^0
Π_1^0

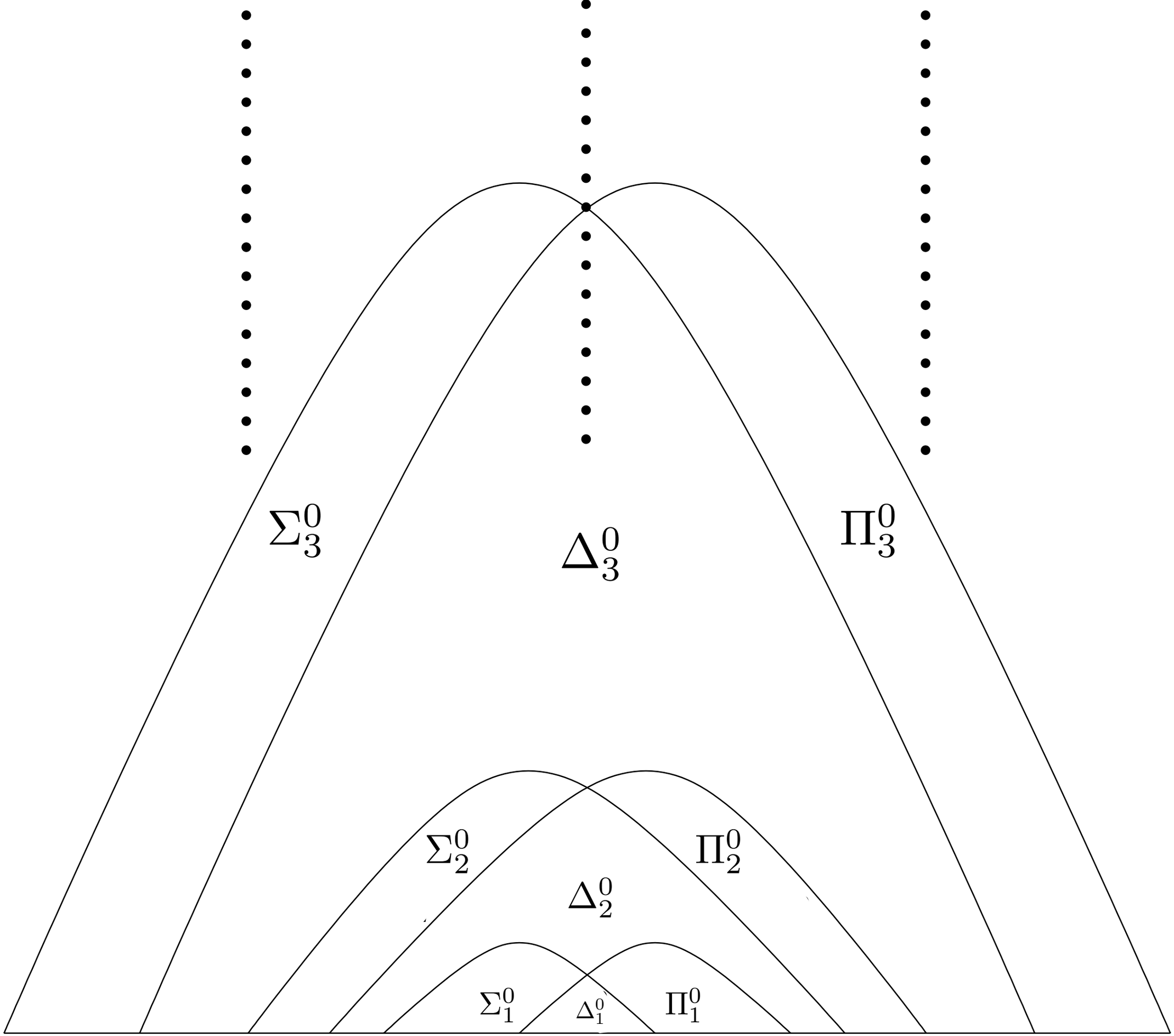

Σ_3^0
Σ_2^0
Π_2^0
Σ_1^0
Π_1^0

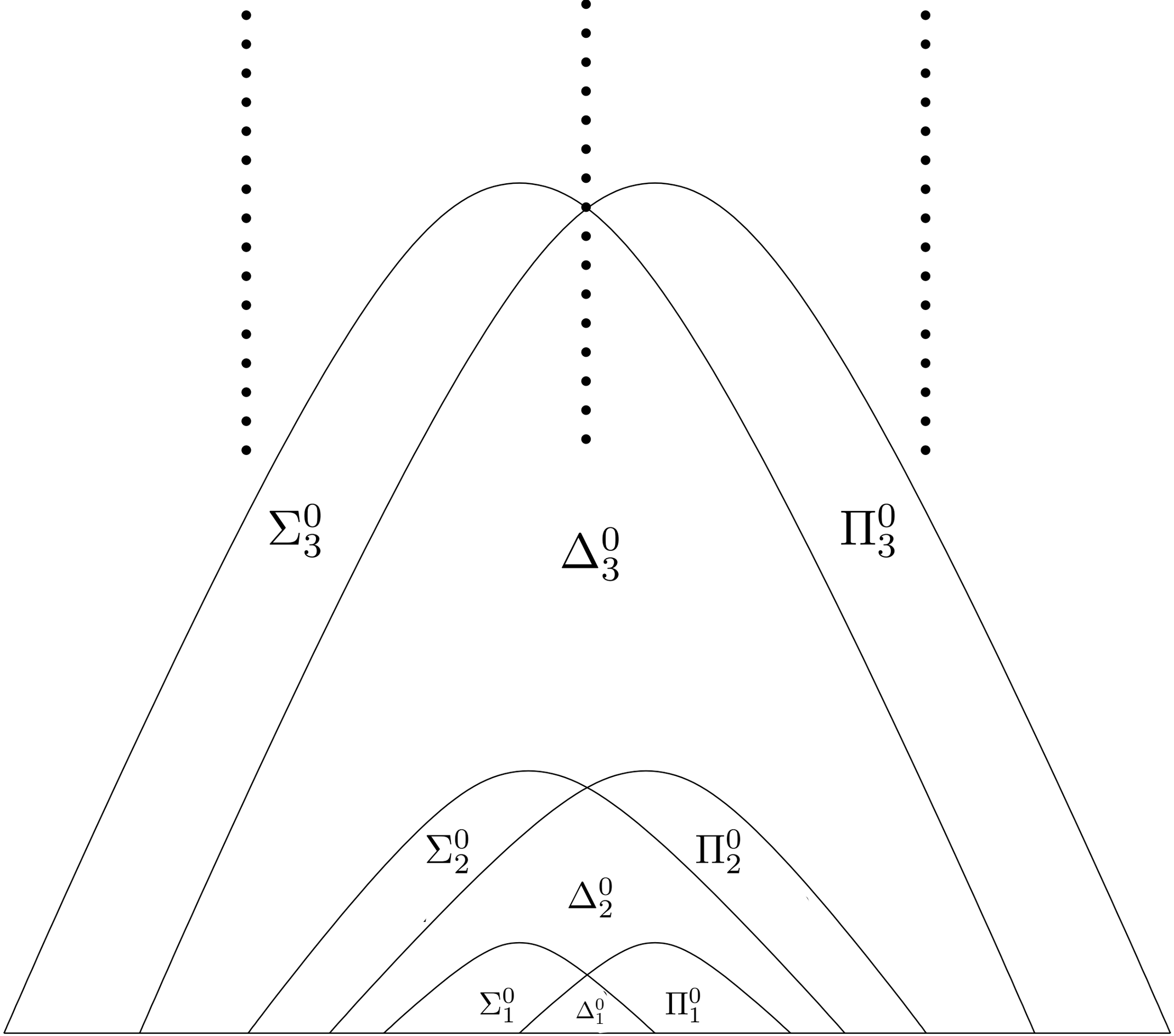

Σ_3^0
Π_3^0
Σ_2^0
Π_2^0
Σ_1^0
Π_1^0

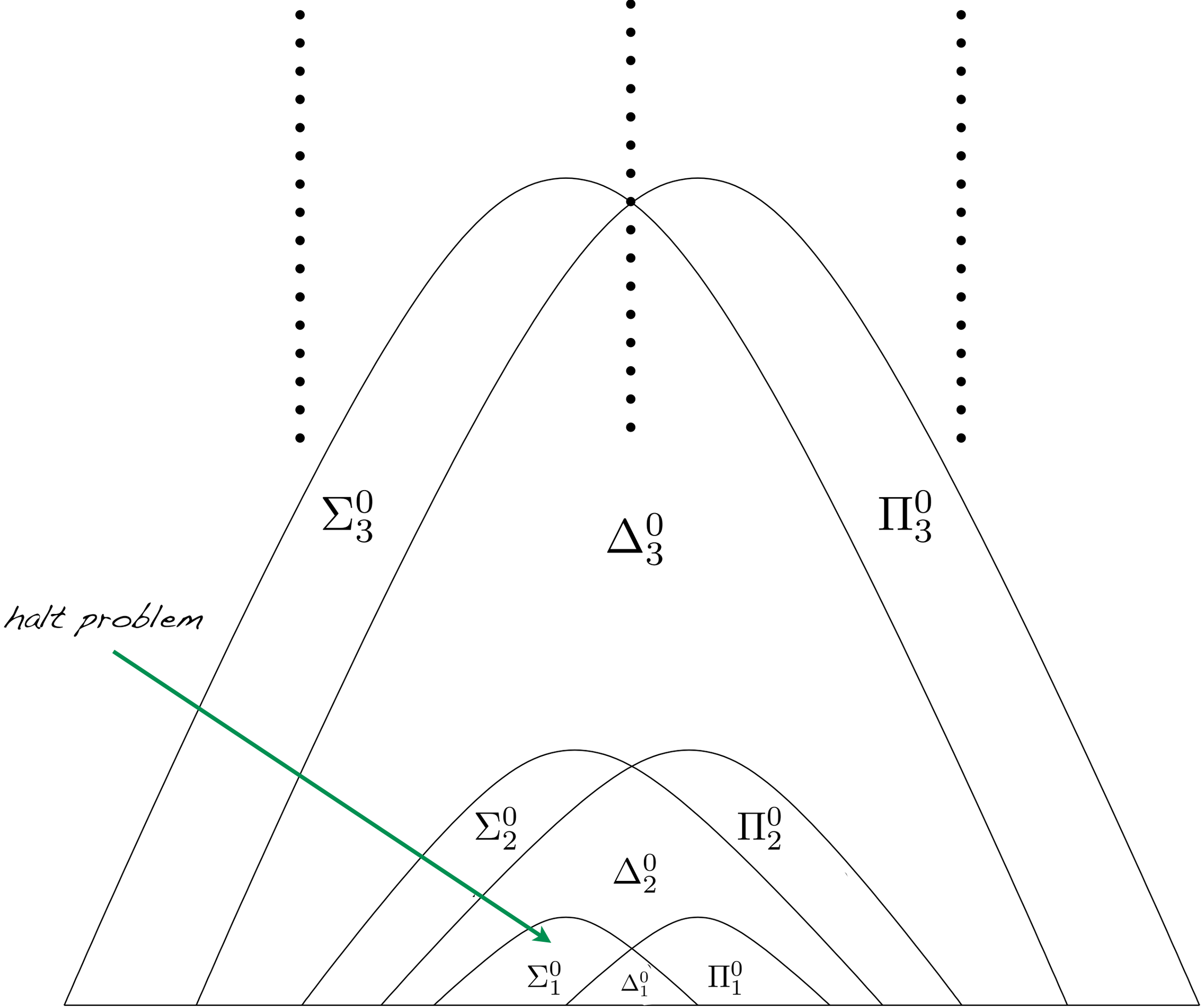

Σ_3^0
Π_3^0
Σ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0

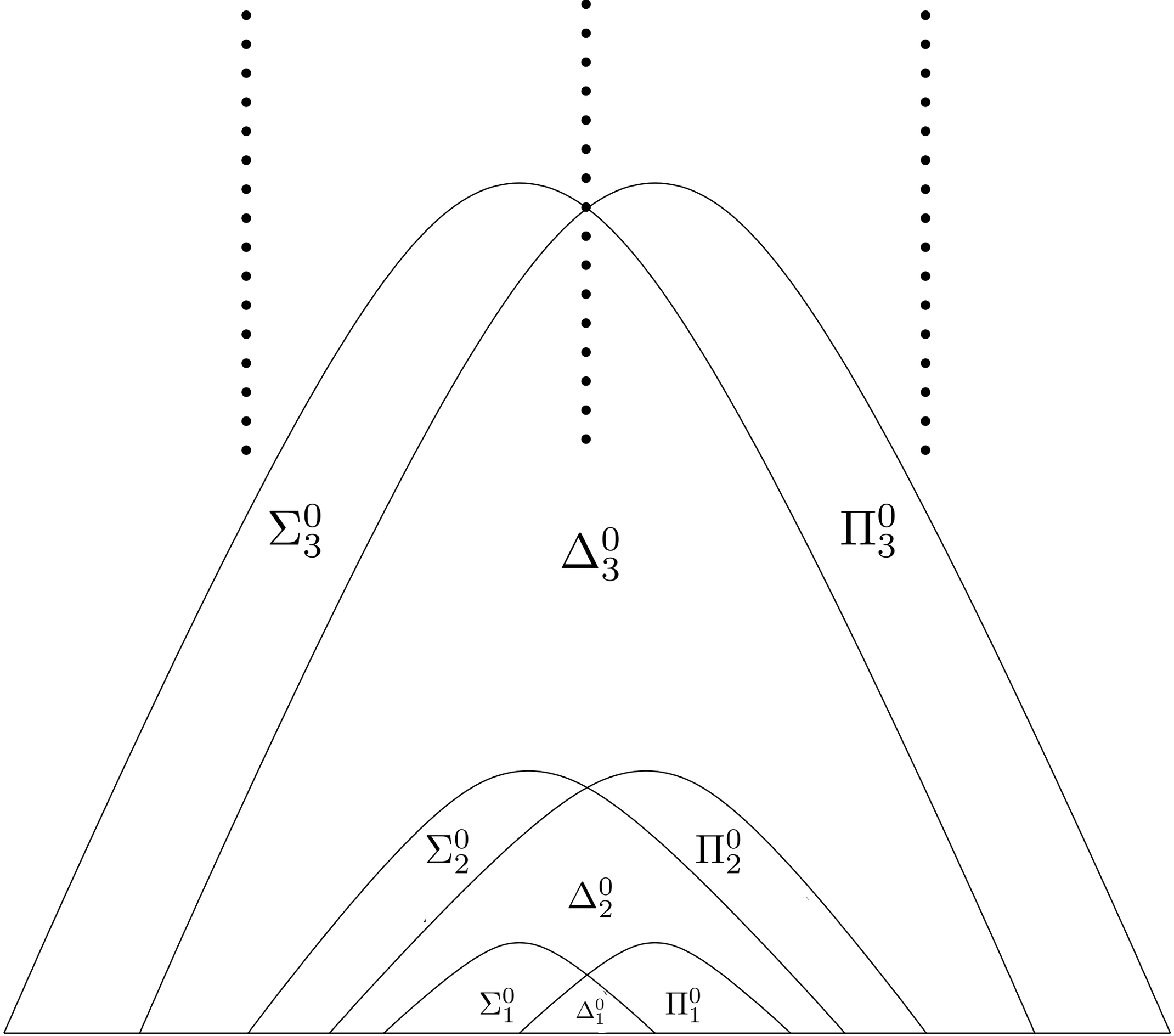

Σ_3^0
Π_3^0
Σ_2^0
Π_2^0
Δ_2^0
Σ_1^0
Δ_1^0
Π_1^0

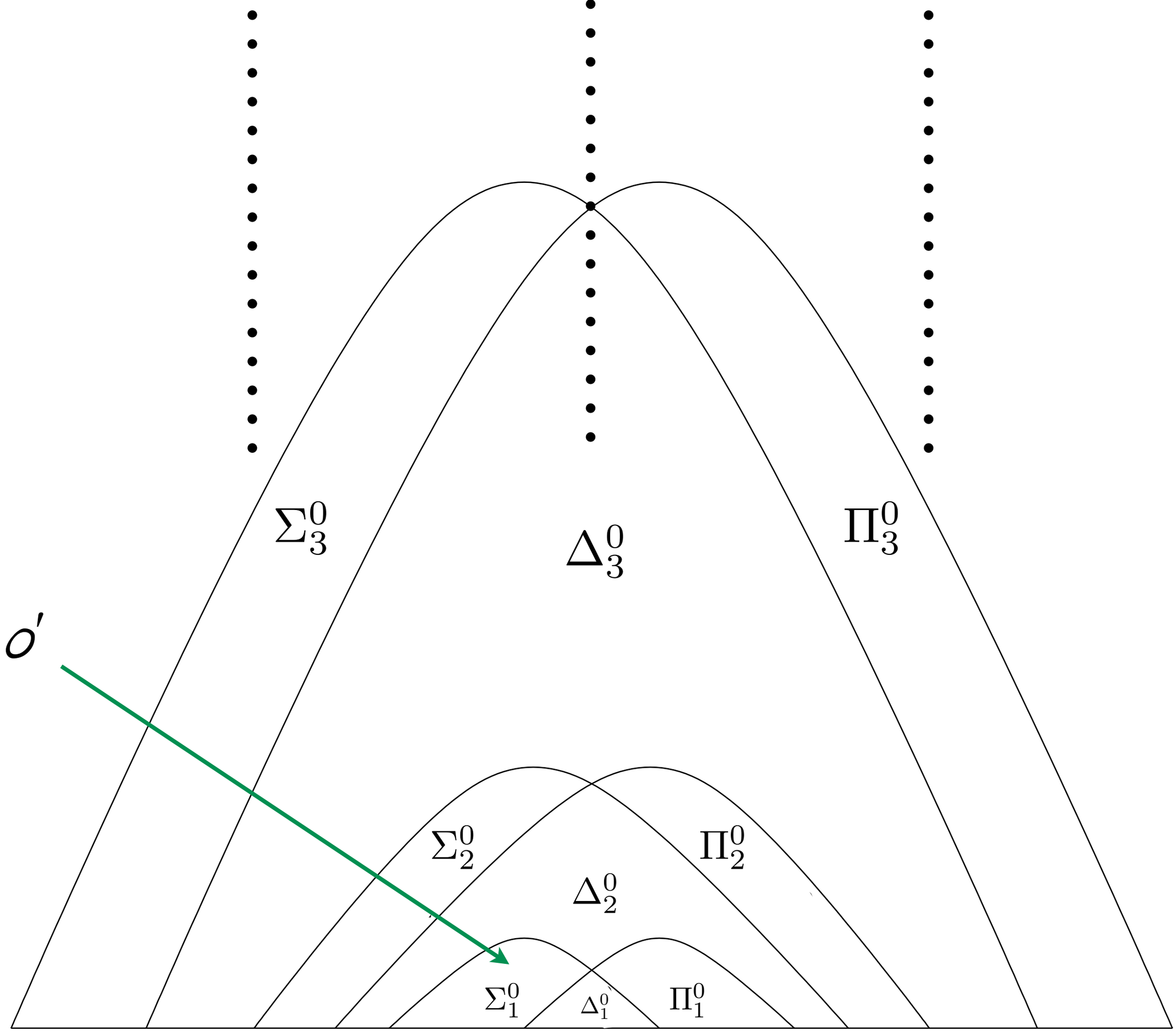

Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Π_2^0
Δ_2^0
Σ_1^0
Δ_1^0
Π_1^0

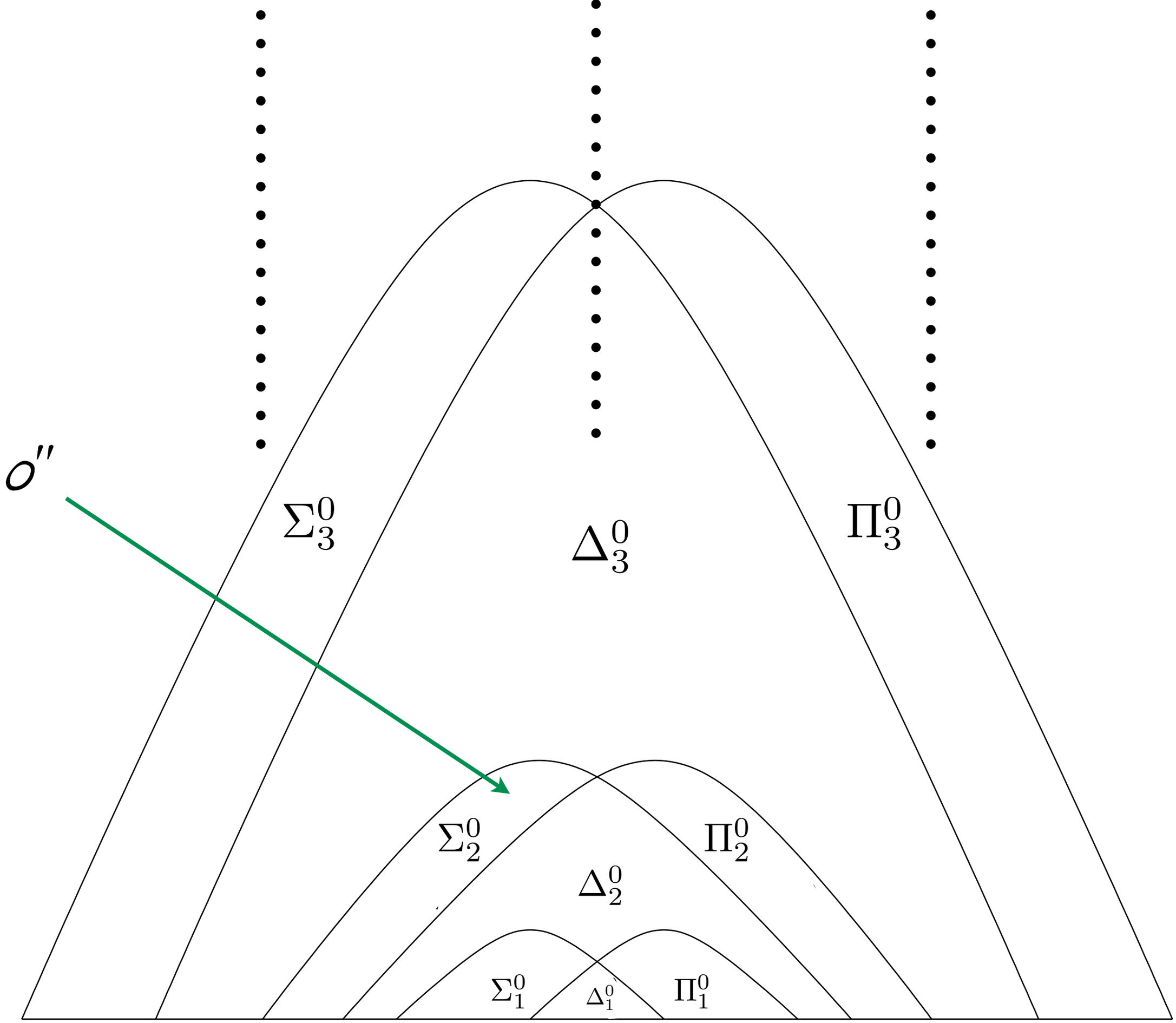

Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


Σ_3^0
Δ_3^0
Π_3^0
halt problem
Σ_2^0
Π_2^0
Δ_2^0
Σ_1^0
Δ_1^0
Π_1^0


Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


Σ_3^0
Δ_3^0
Π_3^0
O'
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


o''
Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0

O'''

Σ_3^0

Δ_3^0

Π_3^0

Σ_2^0

Δ_2^0

Π_2^0

Σ_1^0

Δ_1^0

Π_1^0

O'''

Σ_3^0

Δ_3^0

Π_3^0

Σ_2^0

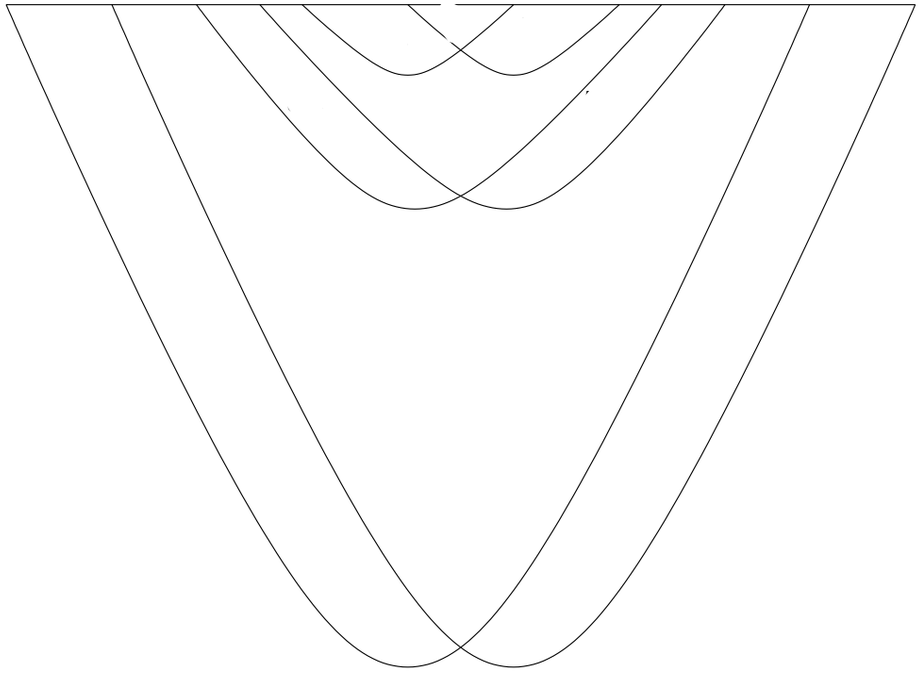
Δ_2^0

Π_2^0

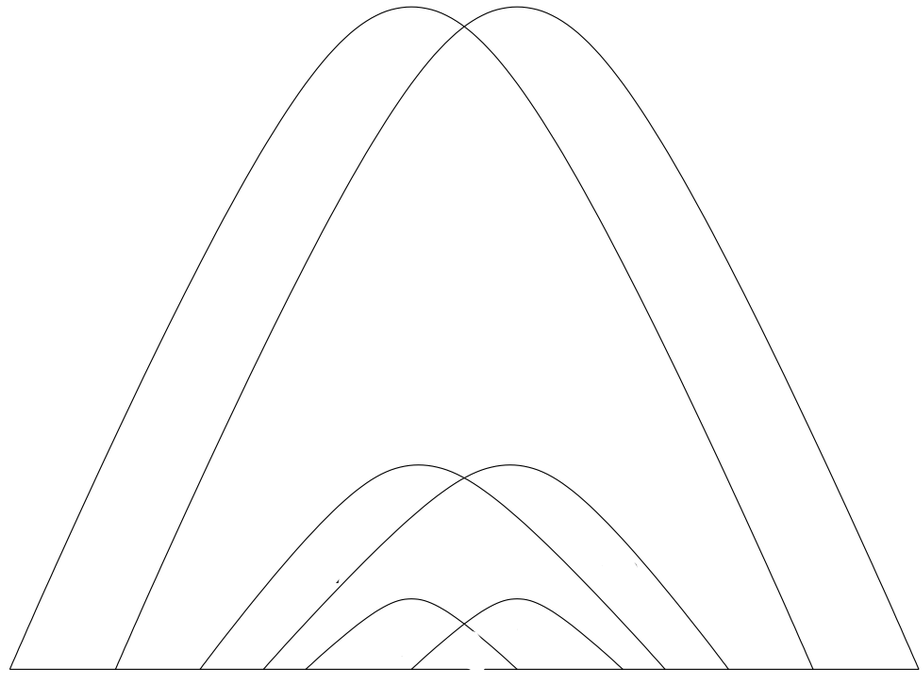
Σ_1^0

Δ_1^0

Π_1^0



The Arithmetical Hierarchy



The Arithmetical Hierarchy

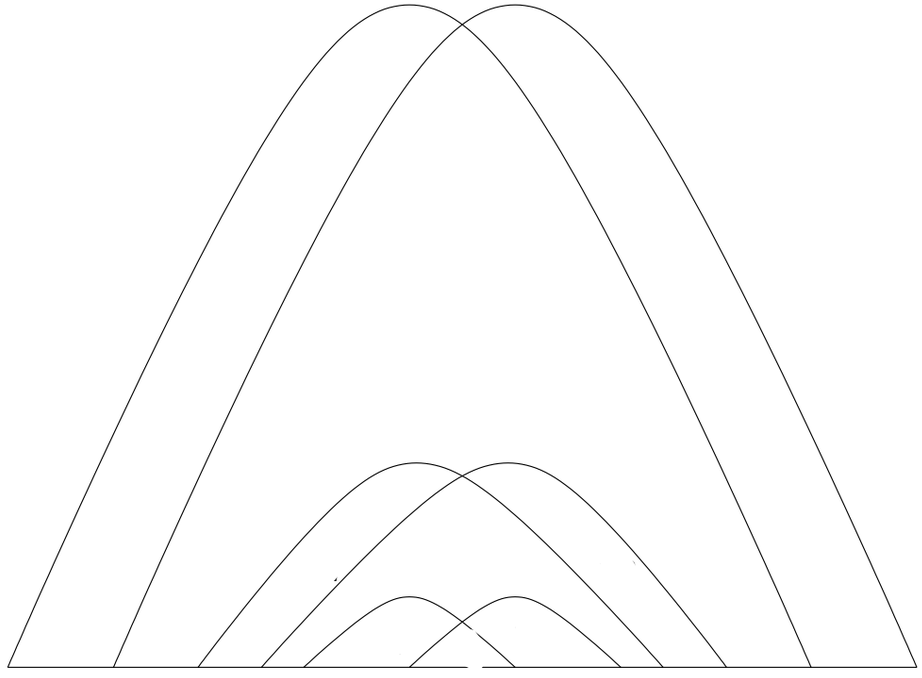
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

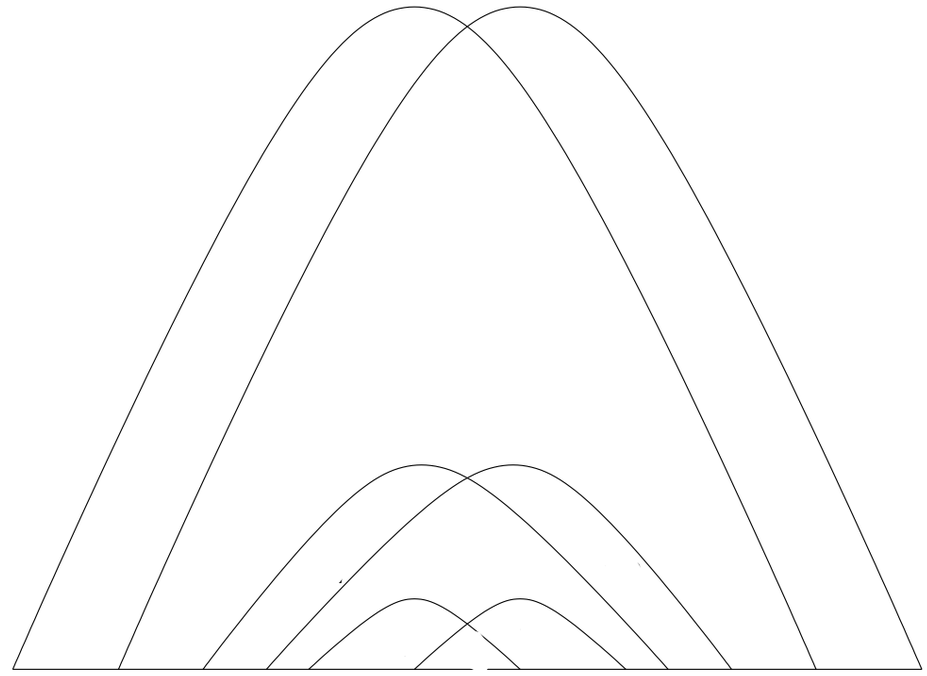
- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

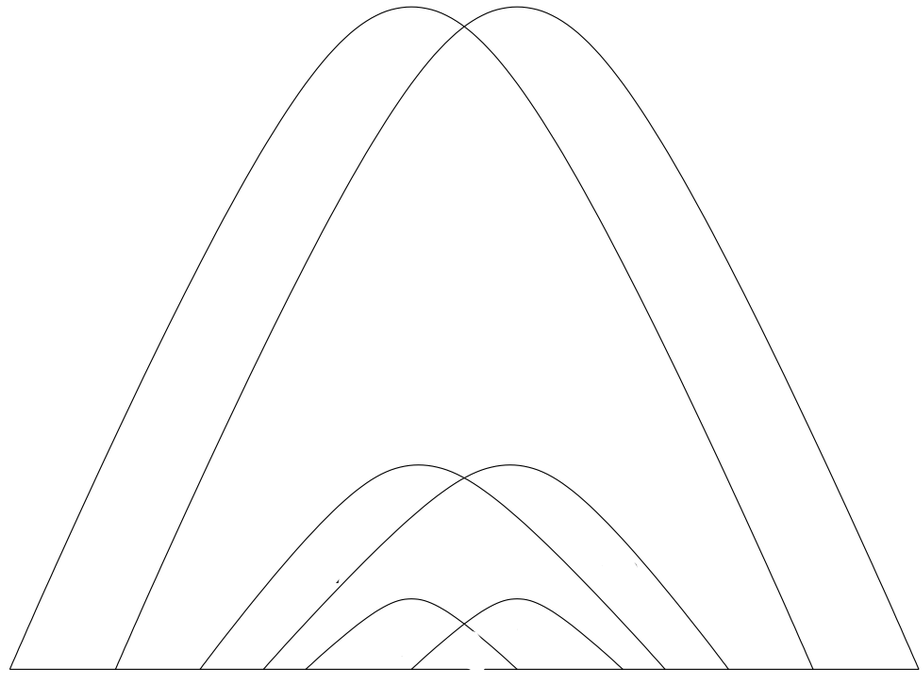
The Borel Hierarchy



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

The Borel Hierarchy

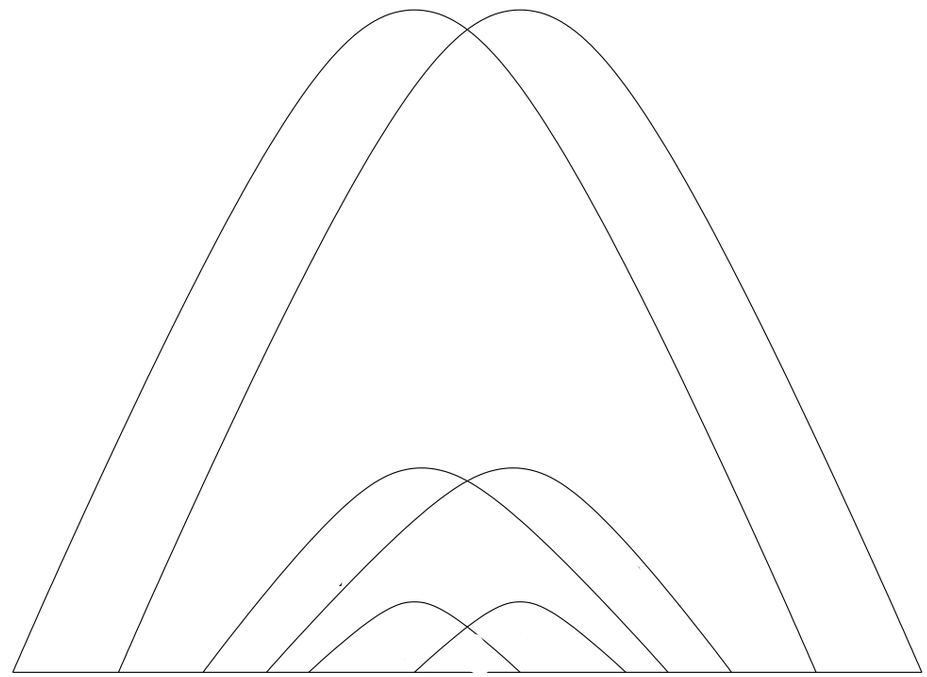


The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$



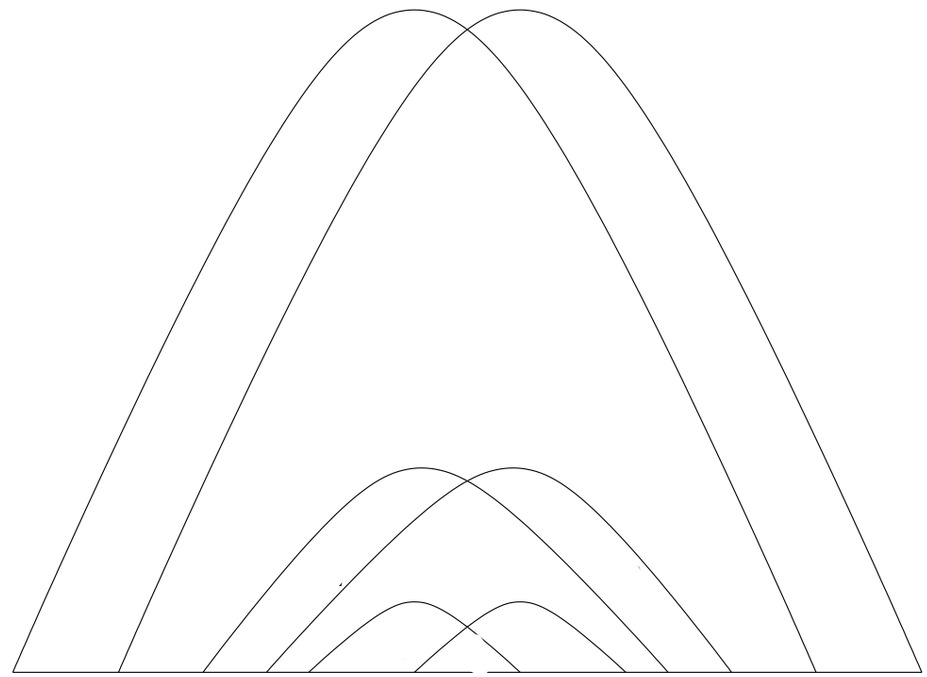
The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$



The Arithmetical Hierarchy

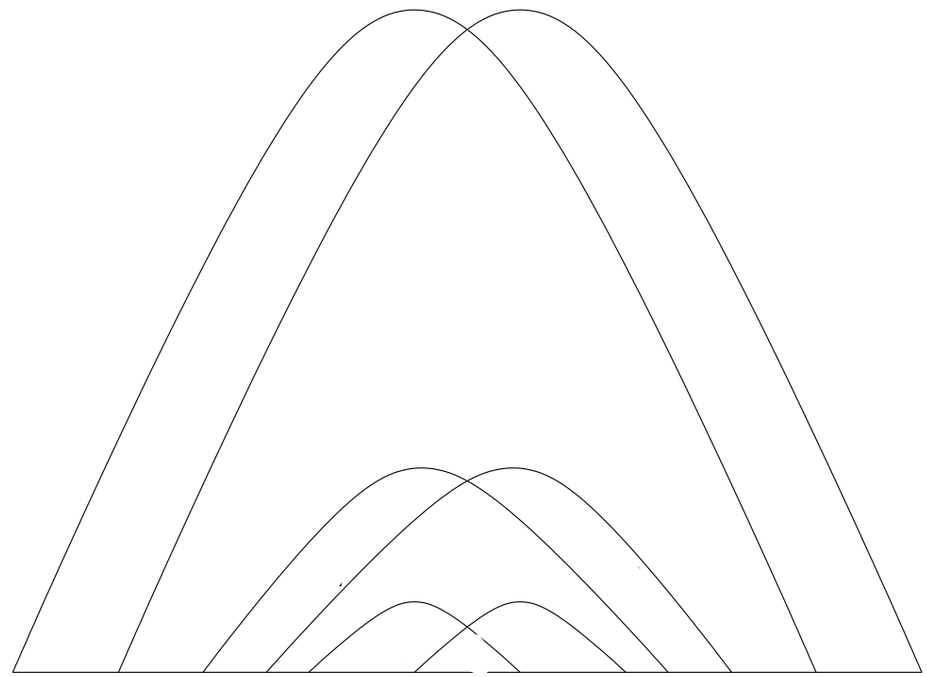
Subsets of $\mathbb{N}$

defined by formulas

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

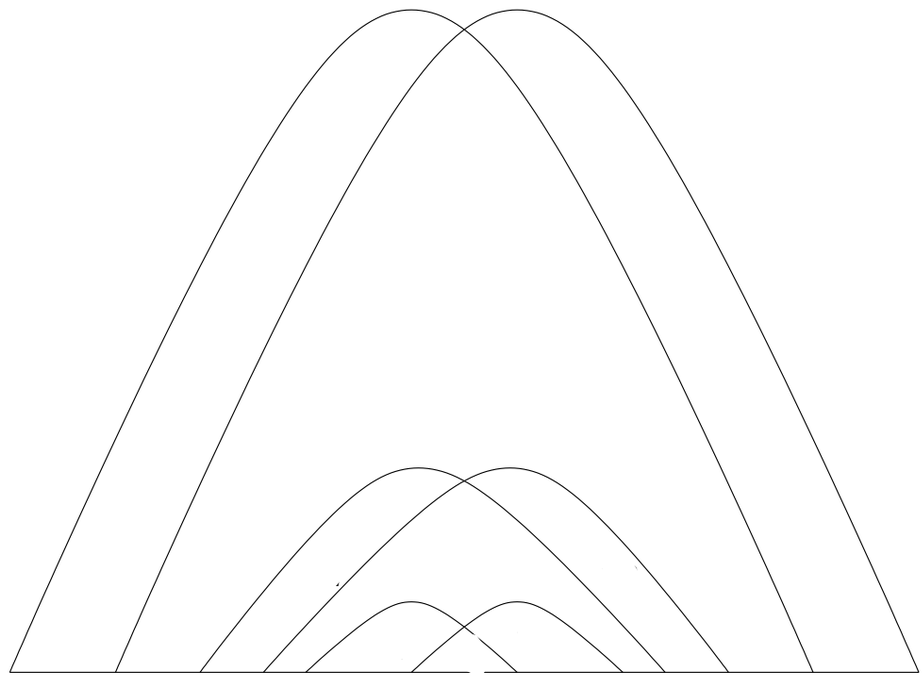
defined by formulas

$\aleph_0$-many

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

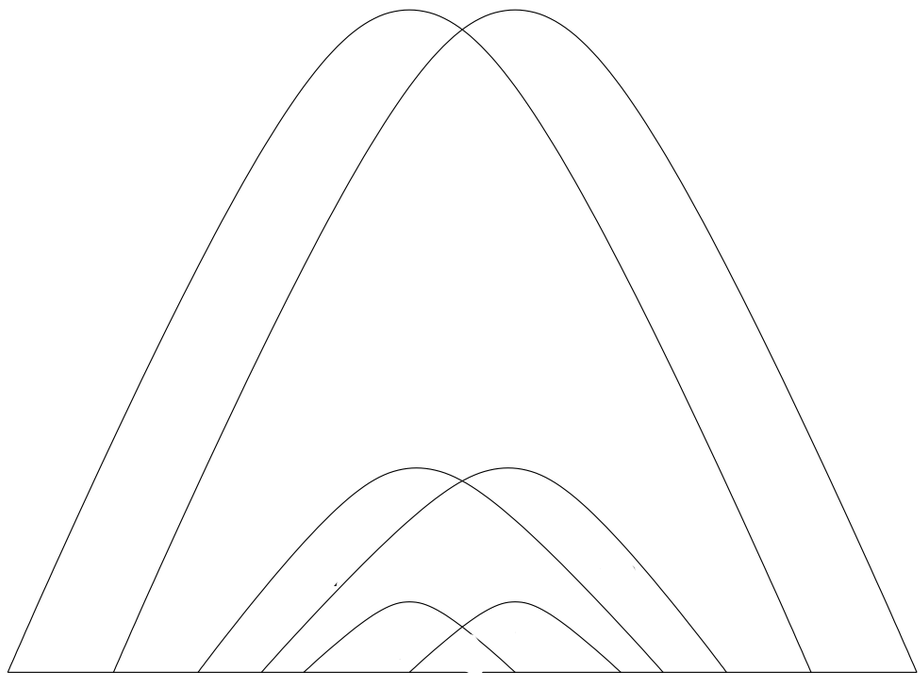
$\aleph_0$-many

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

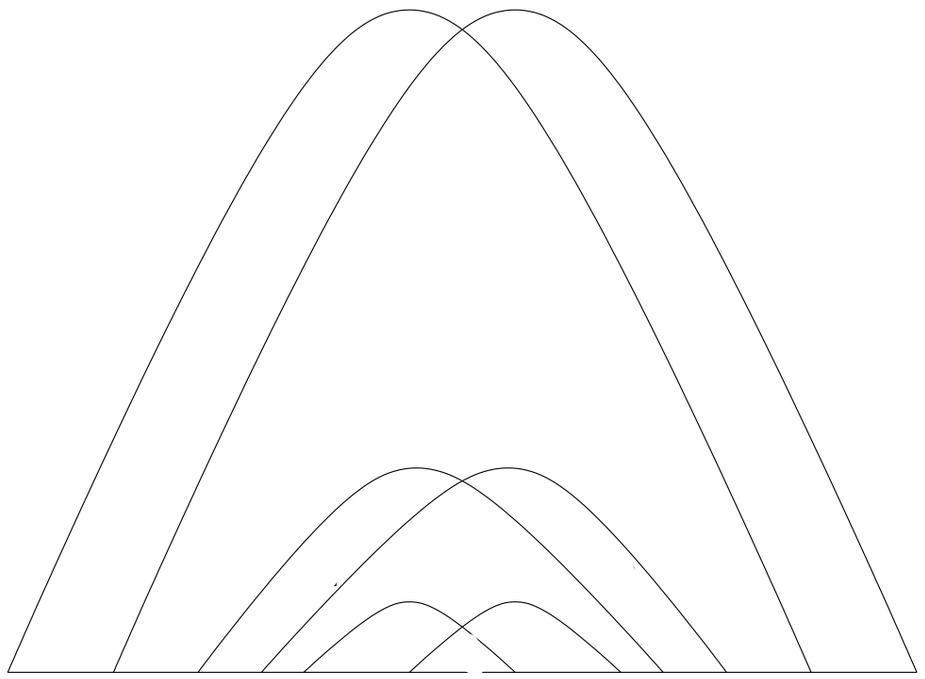
ω levels

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

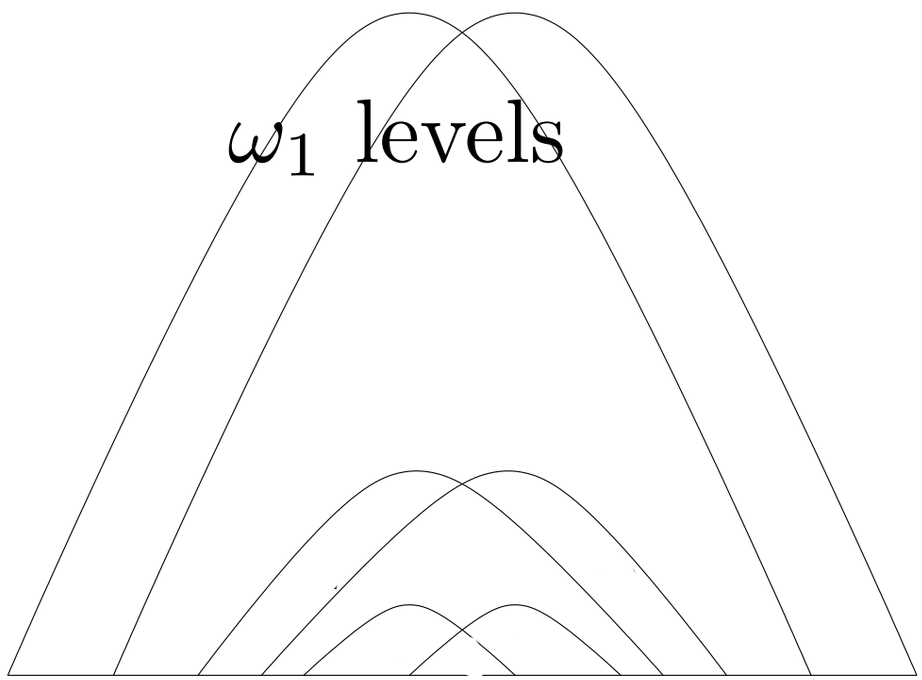
ω levels

The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

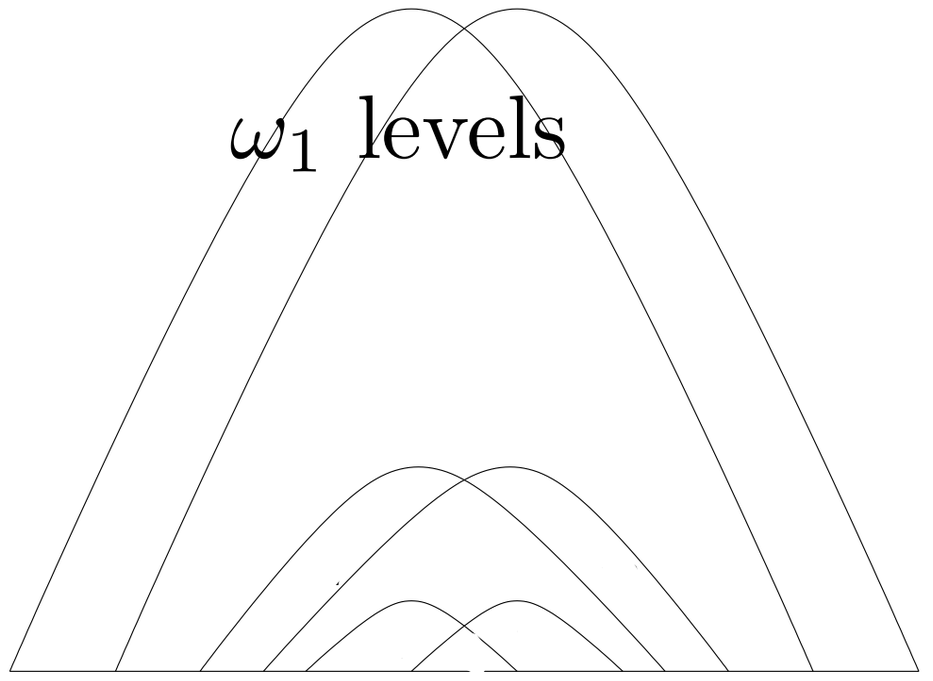
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

0

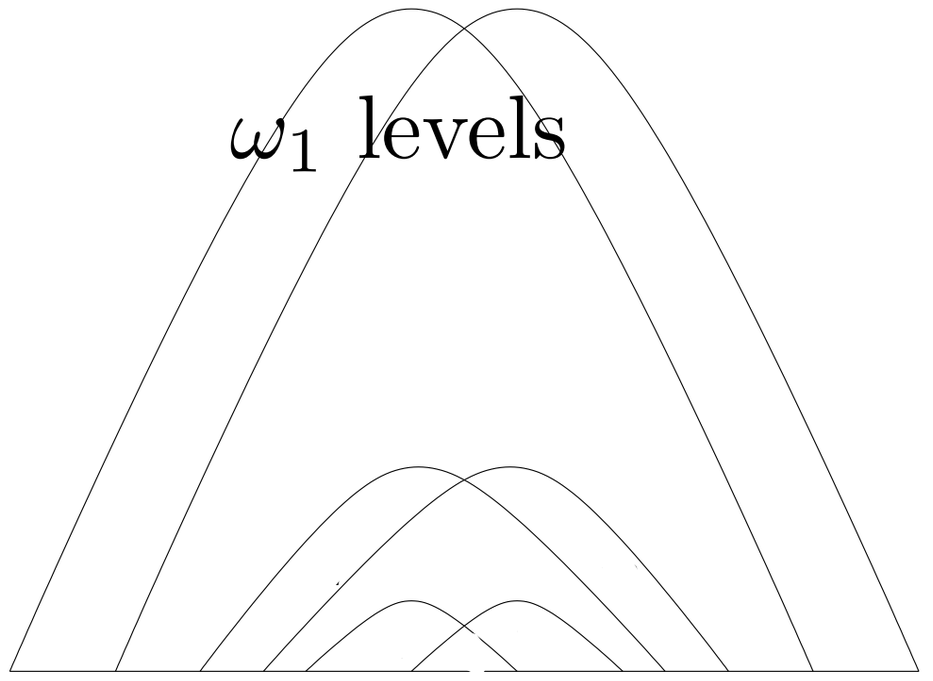
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

1

0

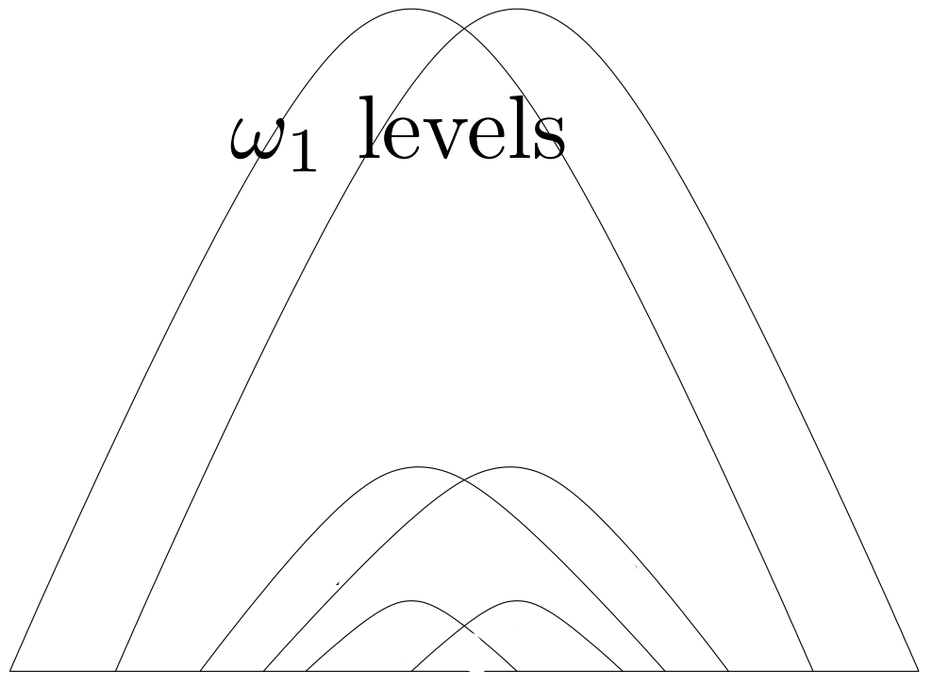
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

$\omega = \aleph_0$

1

0

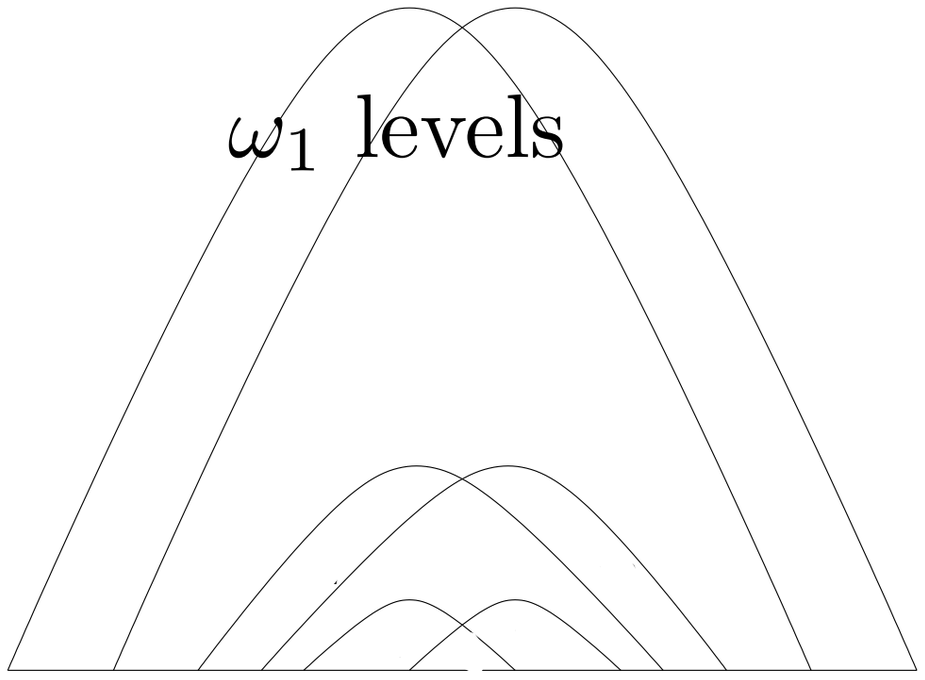
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

$\omega = \aleph_0$

1

0

$\omega_1 = \aleph_1$

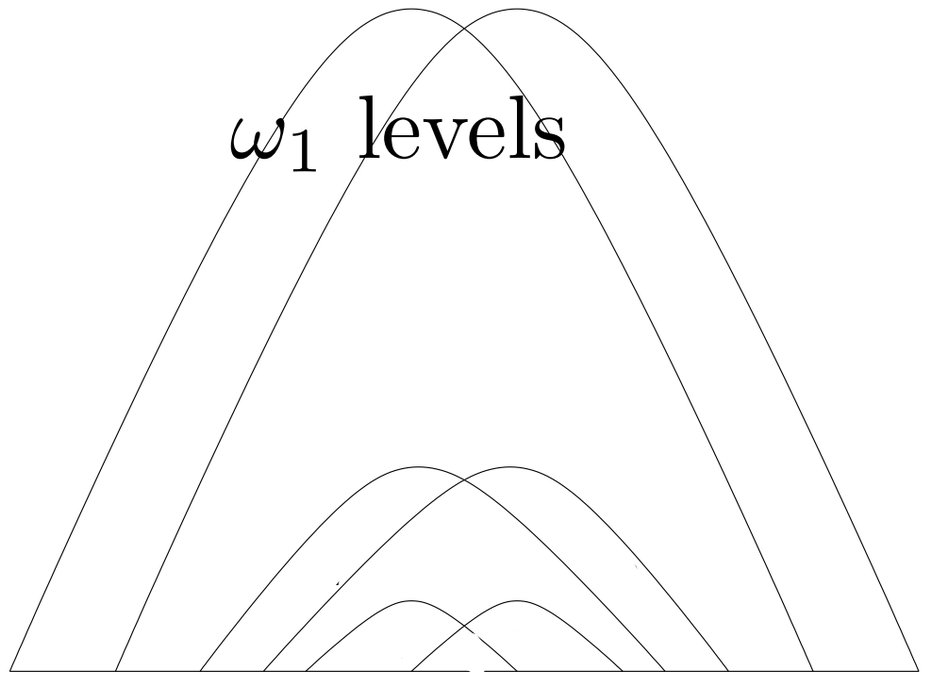
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

$\omega = \aleph_0$

1

0

$\omega_1 = \aleph_1 2^{\aleph_0}$

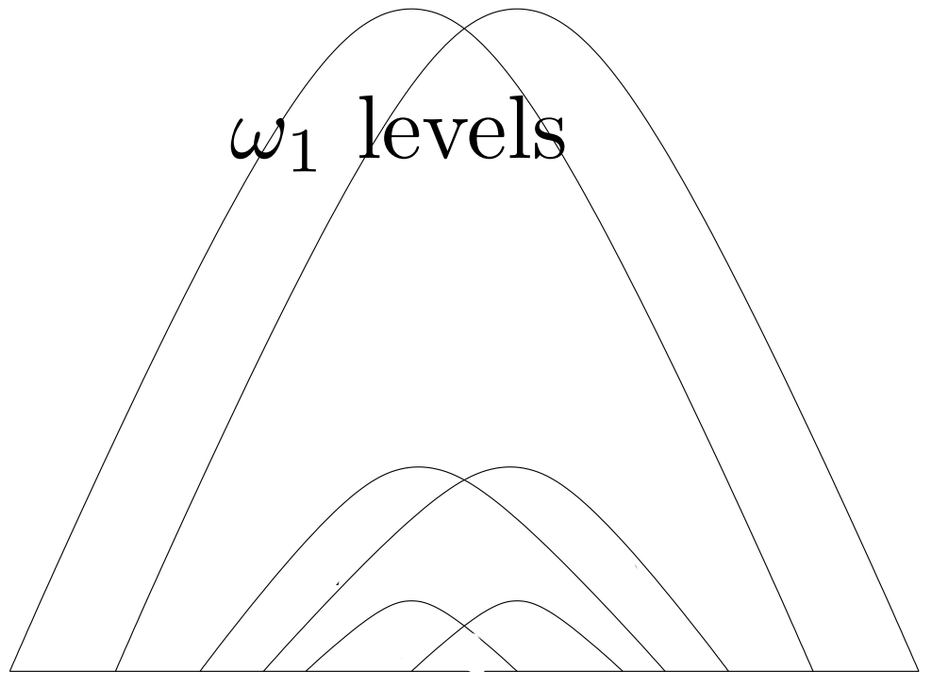
The Borel Hierarchy

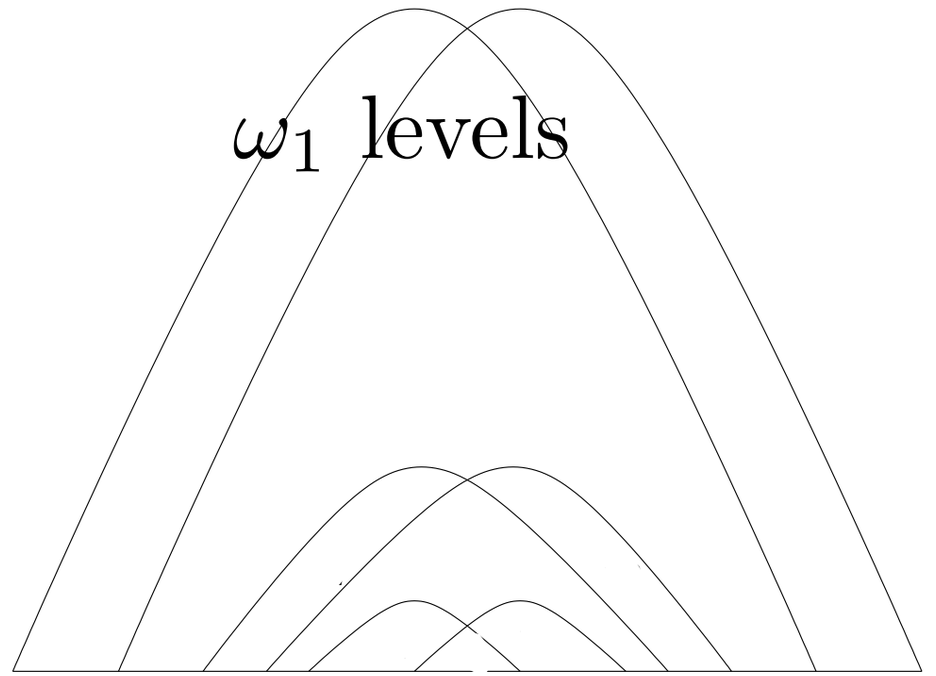
Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy		The Borel Hierarchy
	$2^{\aleph_0}$	
Subsets of $\mathbb{N}$		Subsets of $\mathbb{N}^\omega$
defined by formulas	$\omega_1 = \aleph_1$	most are not…
$\aleph_0$-many		$2^{\aleph_0}$-many
ω levels	$\omega = \aleph_0$	
	1	
	0	

The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

$\omega_1 \quad 2^{\aleph_0} \quad \aleph_1$

$\omega = \aleph_0$

1

0

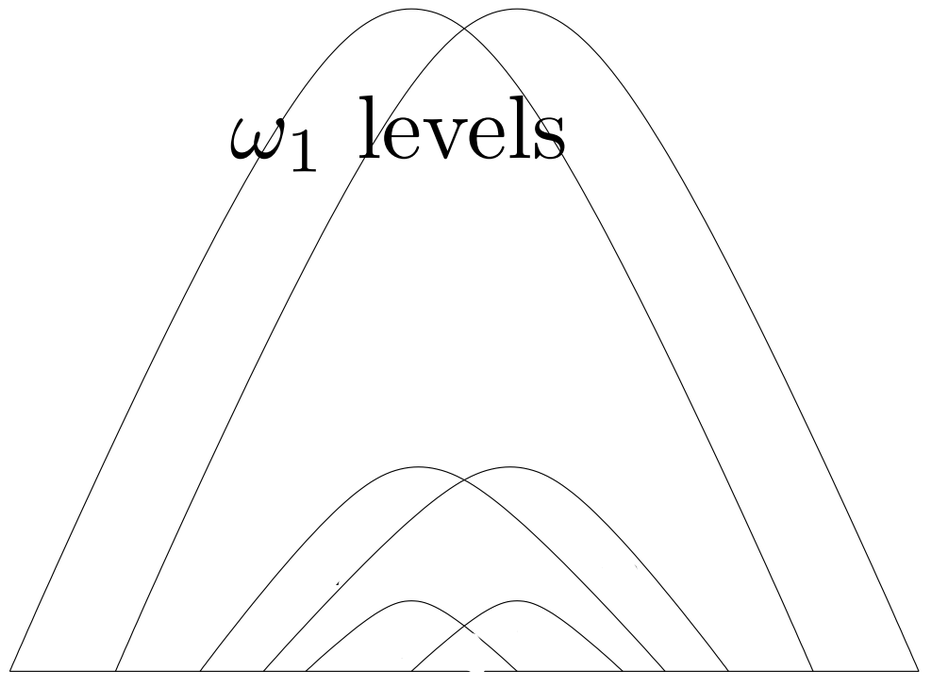
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

Subsets of $\mathbb{N}$

defined by formulas

$\aleph_0$-many

ω levels

$\omega = \aleph_0$

1

0

$\omega_1 = \aleph_1$

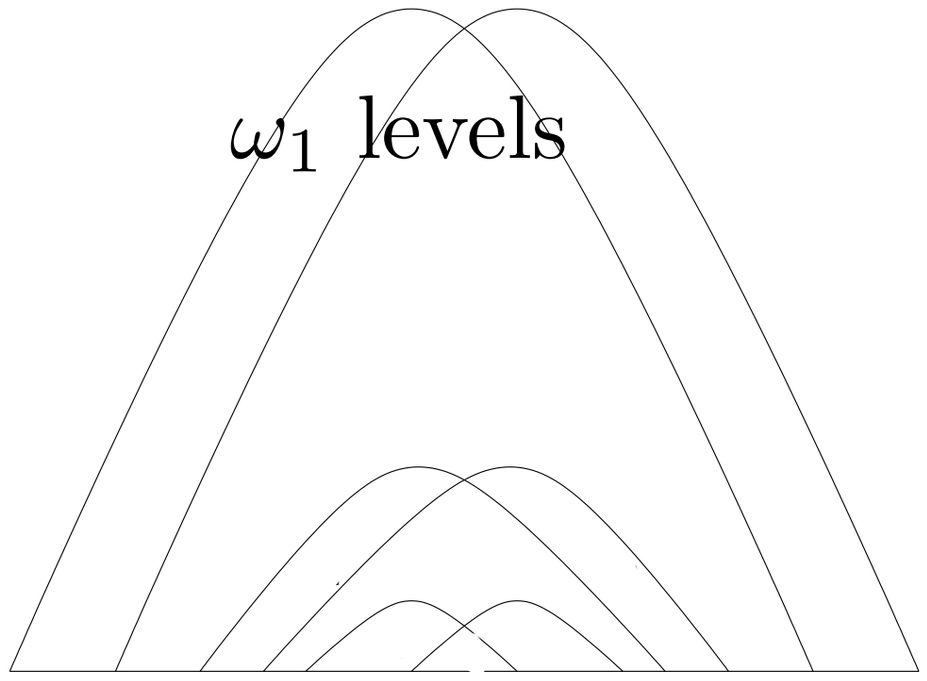
The Borel Hierarchy

Subsets of $\mathbb{N}^\omega$

most are not...

$2^{\aleph_0}$-many

ω_1 levels



The Arithmetical Hierarchy

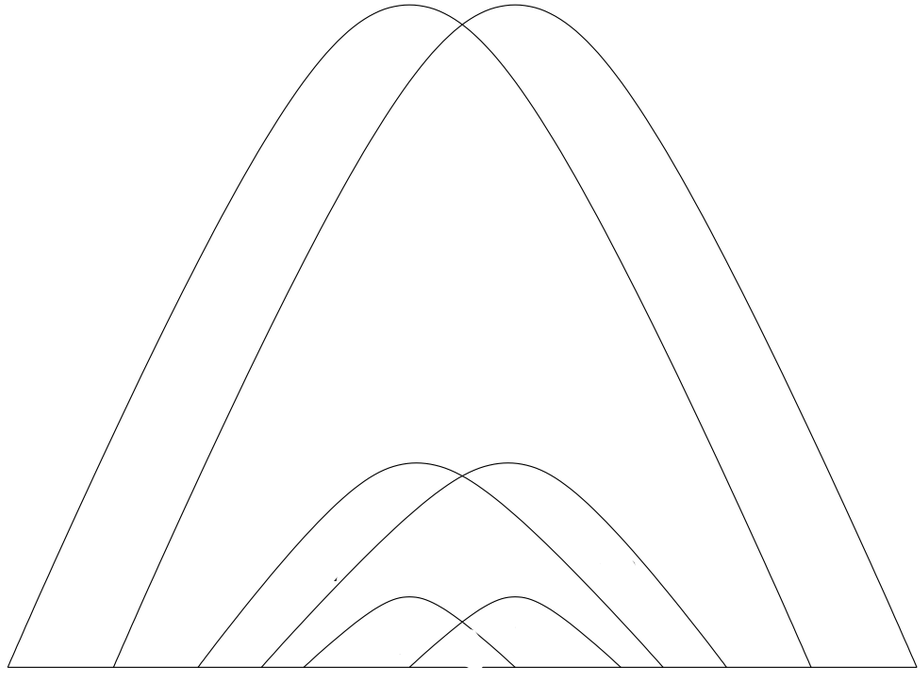
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

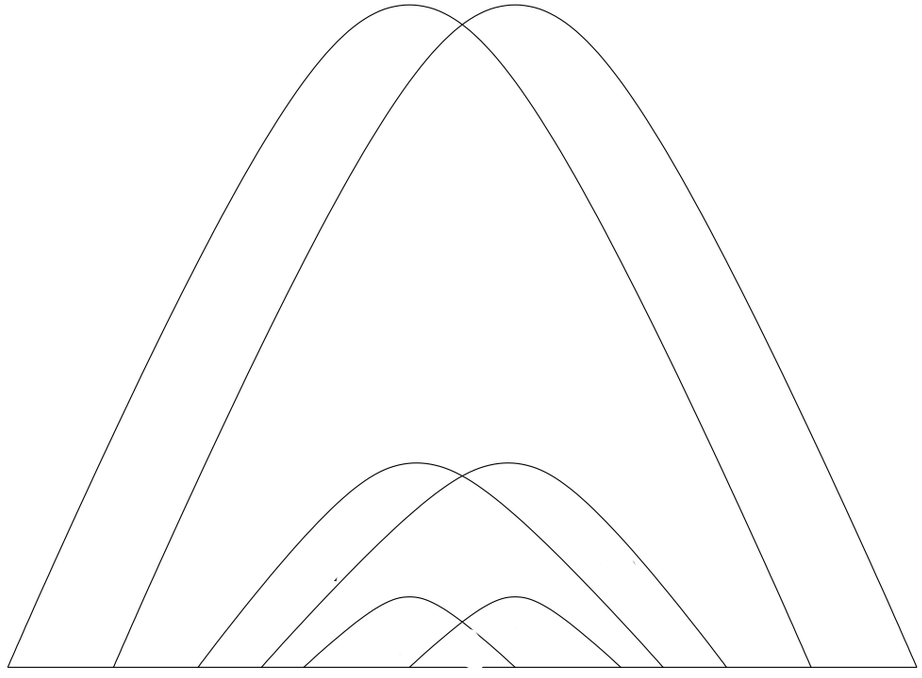
  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

The Borel Hierarchy



The Arithmetical Hierarchy

- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

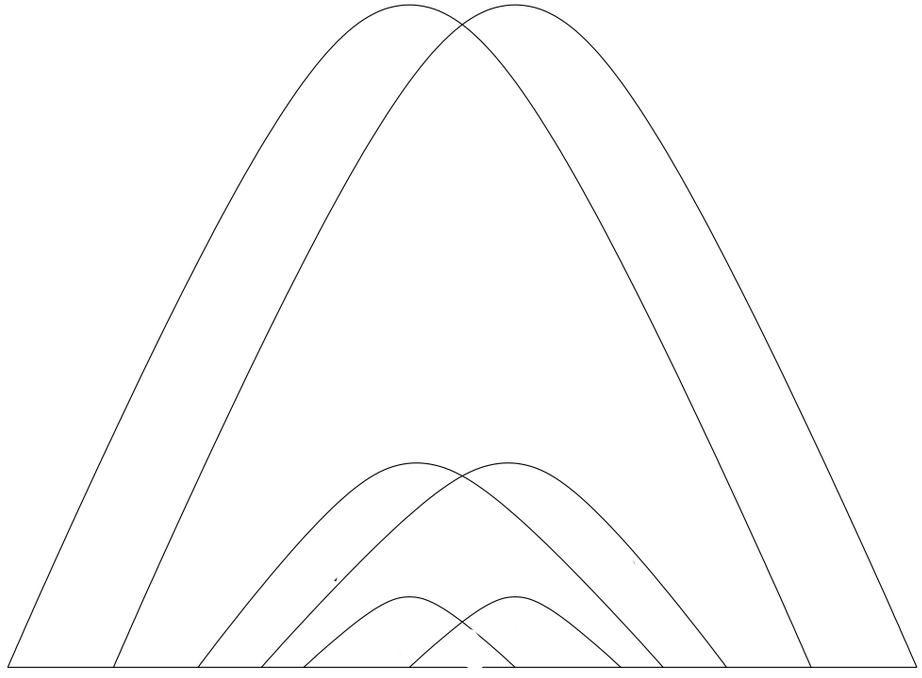
- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

The Borel Hierarchy

$W\mathbb{N}^\omega$



The Arithmetical Hierarchy

- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

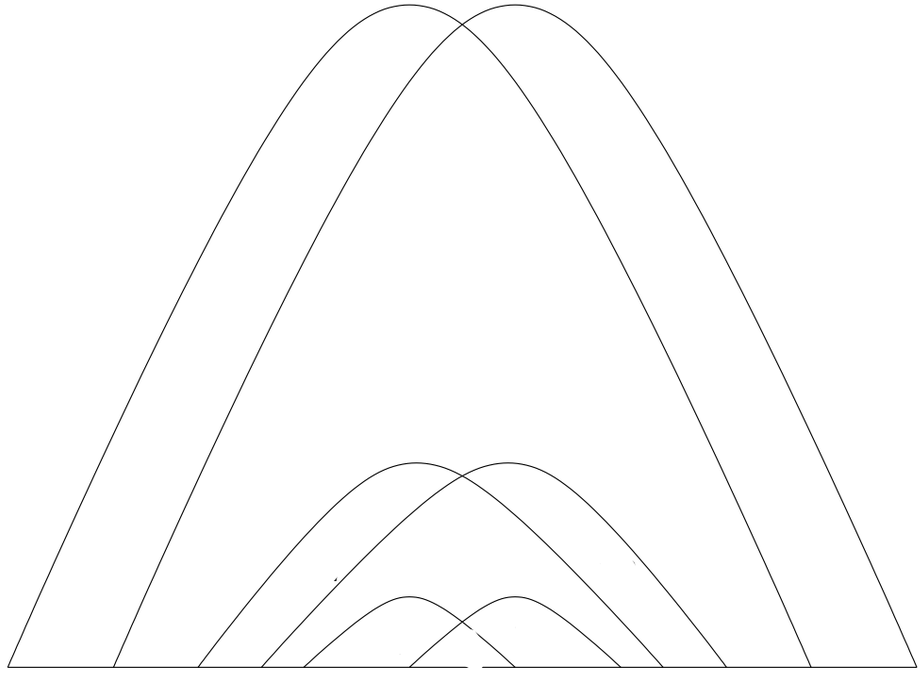
- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

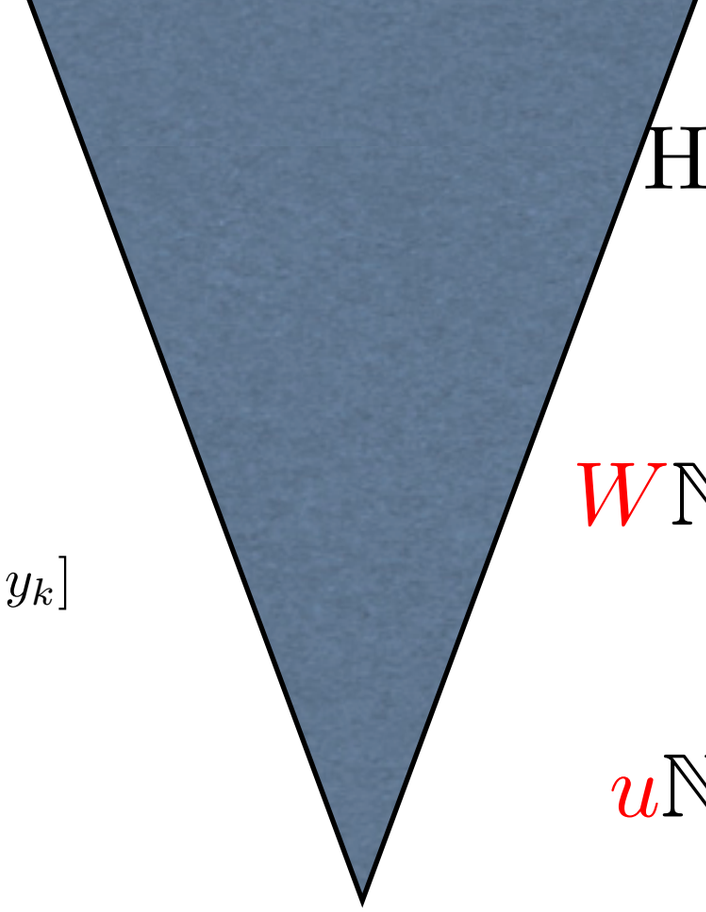
The Borel Hierarchy

$W\mathbb{N}^\omega$

$u\mathbb{N}^\omega$



The Arithmetical Hierarchy



Hierarchy

$W\mathbb{N}^\omega$

$u\mathbb{N}^\omega$

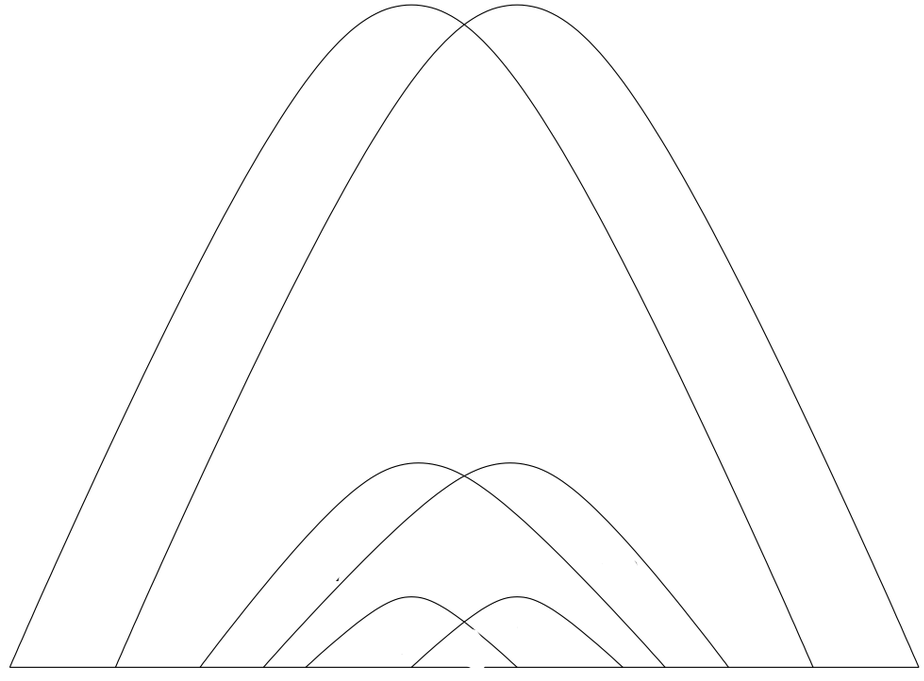
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

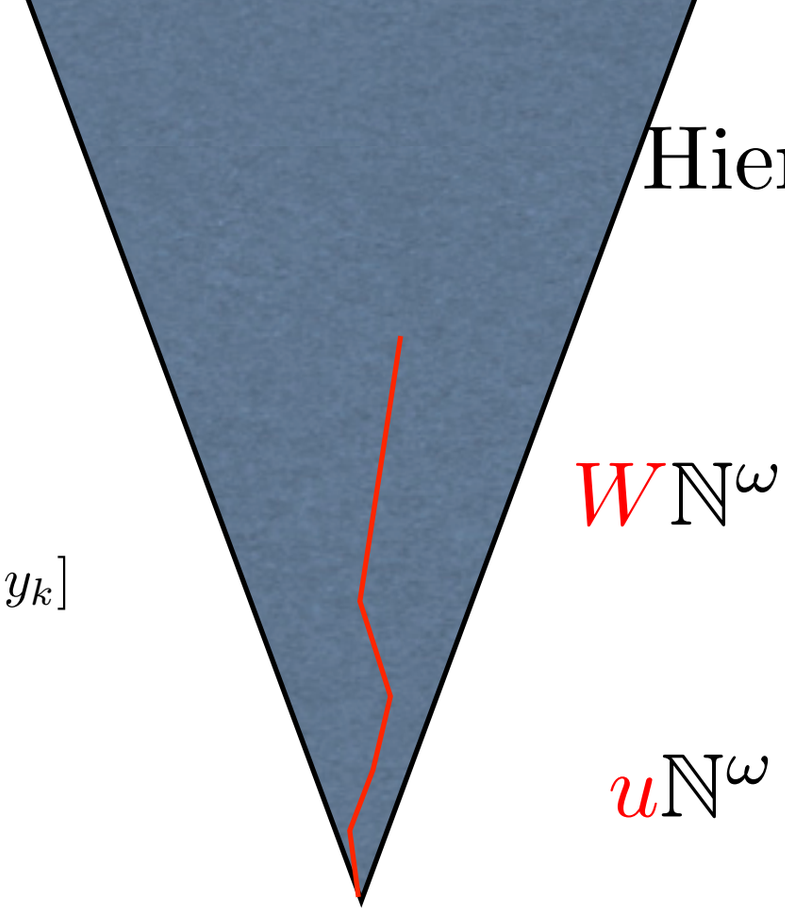
- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy



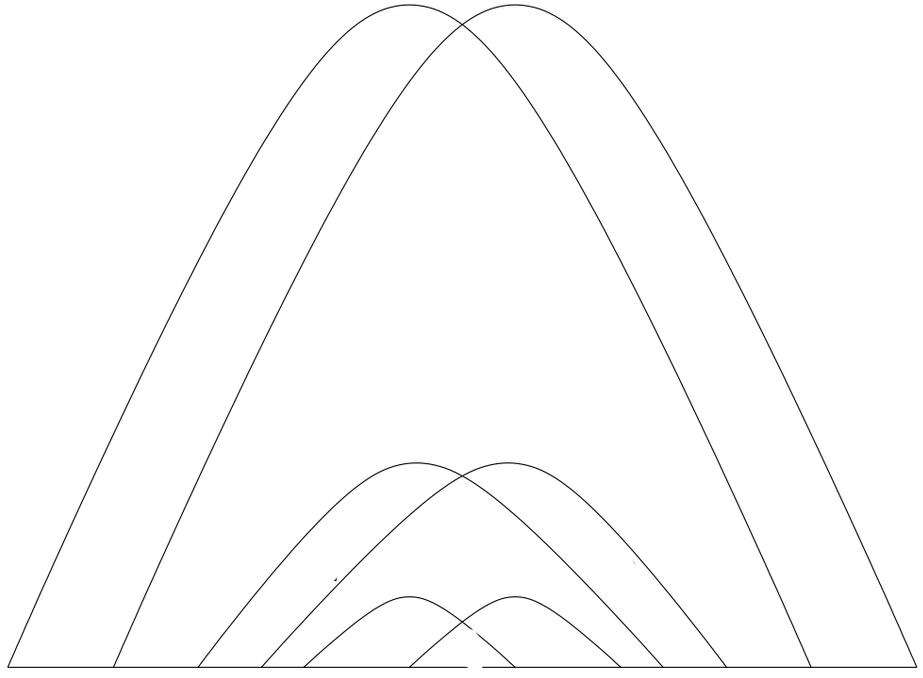
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

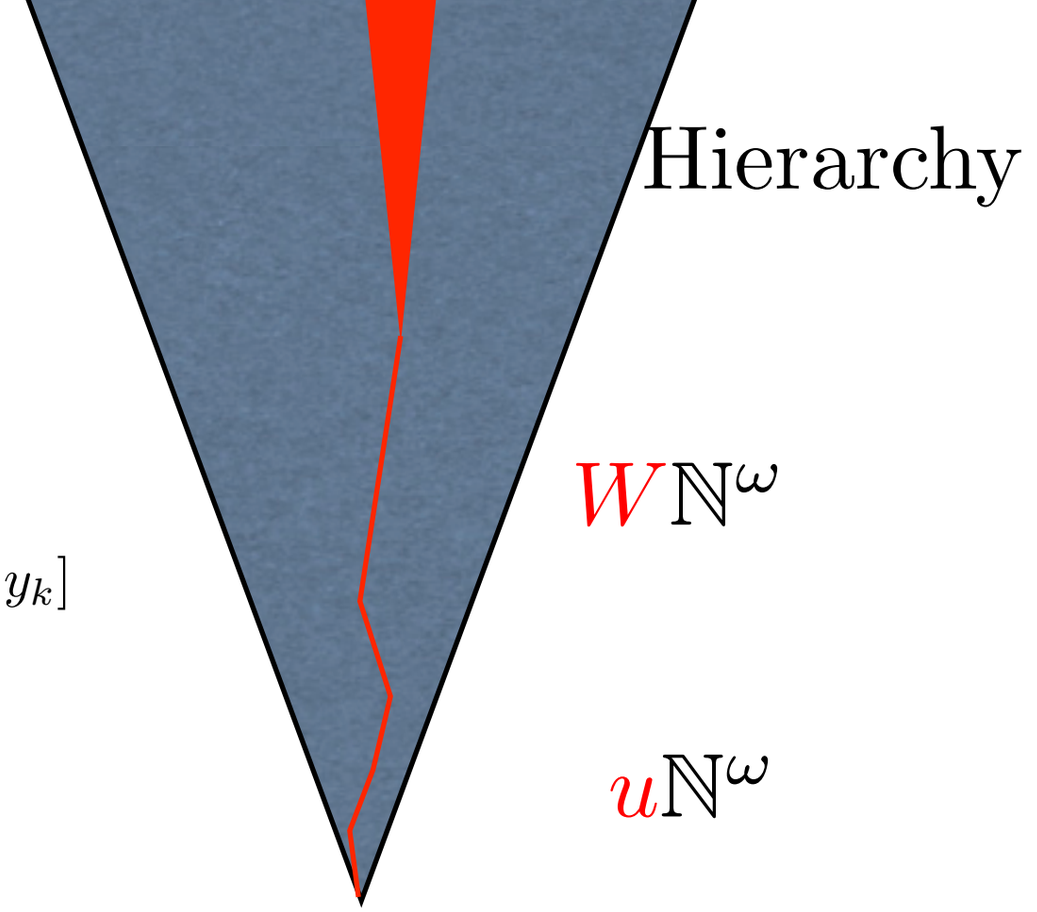
- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy



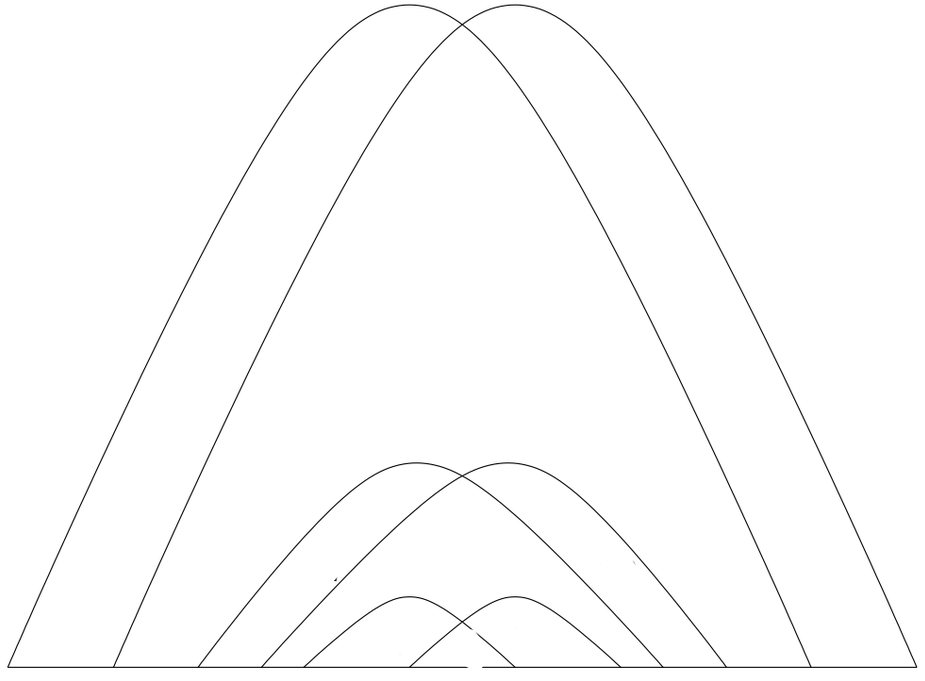
- $(y_1, \ldots, y_k) \in A \in \Sigma^0_1 \iff \begin{cases} A \text{ is r.e.} \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi^0_n \iff A^{\complement} \in \Sigma^0_n$

- $(y_1, \ldots, y_k) \in A \in \Sigma^0_{n+1} \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ B \in \pi^0_n \end{cases}$

- $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$



The Arithmetical Hierarchy

- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

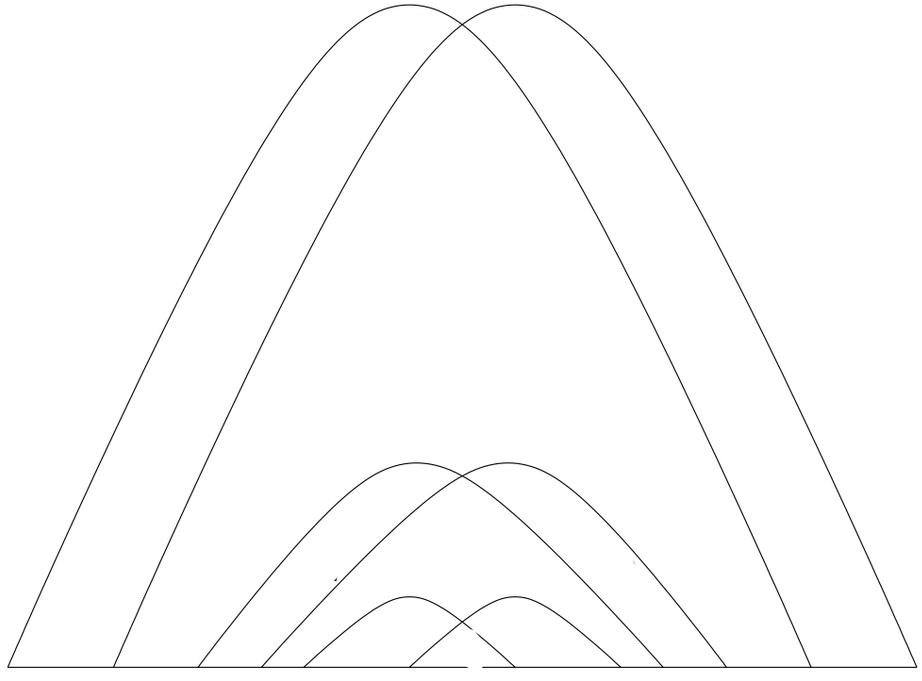
- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

The Borel Hierarchy

$W\mathbb{N}^\omega$

$u\mathbb{N}^\omega$



The Arithmetical Hierarchy

The Borel Hierarchy

- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

$W\mathbb{N}^\omega$

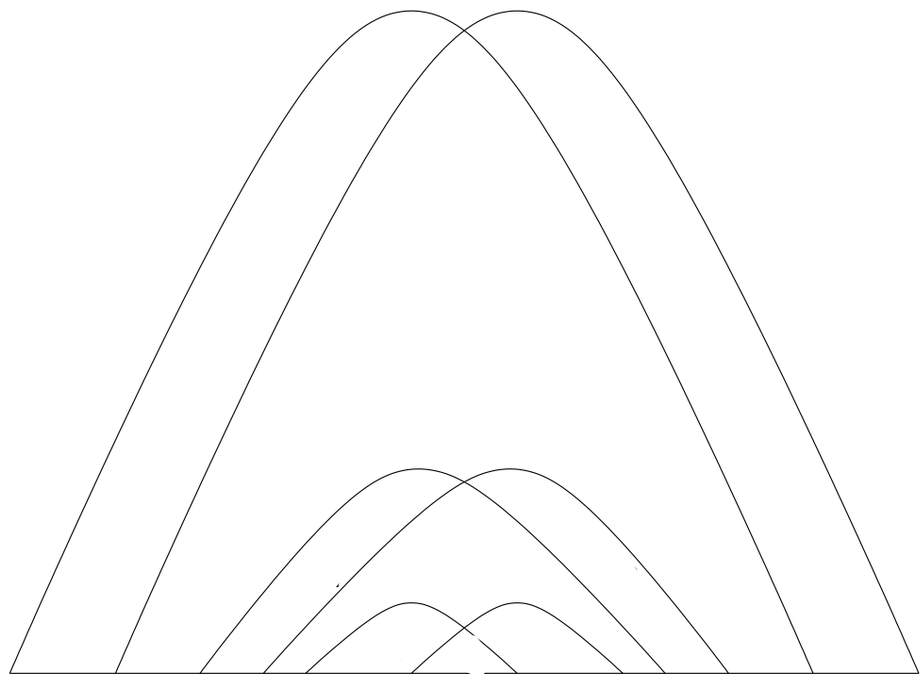
$u\mathbb{N}^\omega$

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

$A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

The Borel Hierarchy

- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

$W\mathbb{N}^\omega$

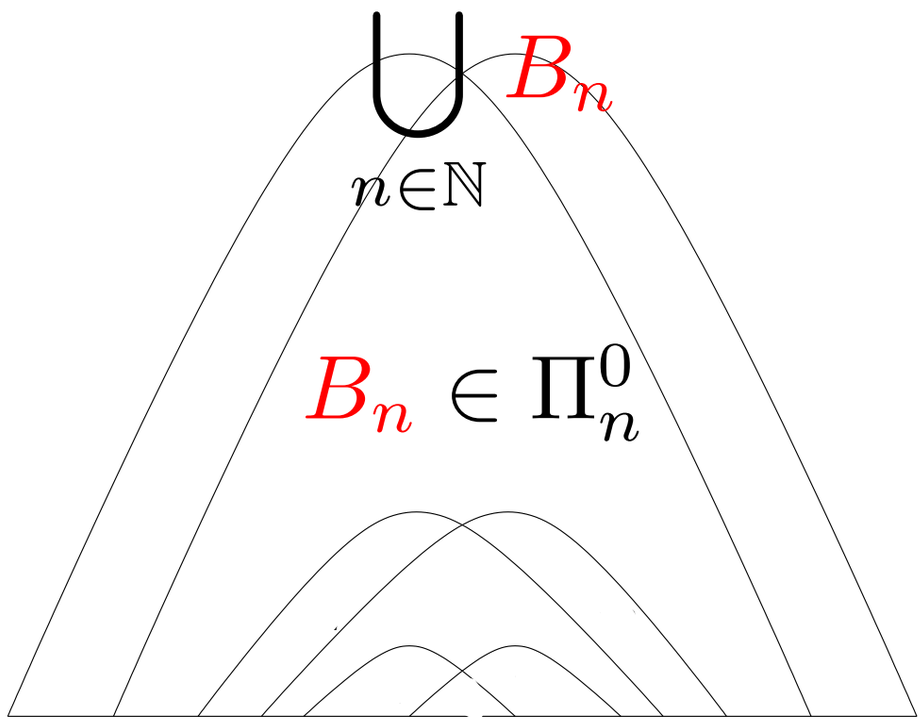
$u\mathbb{N}^\omega$

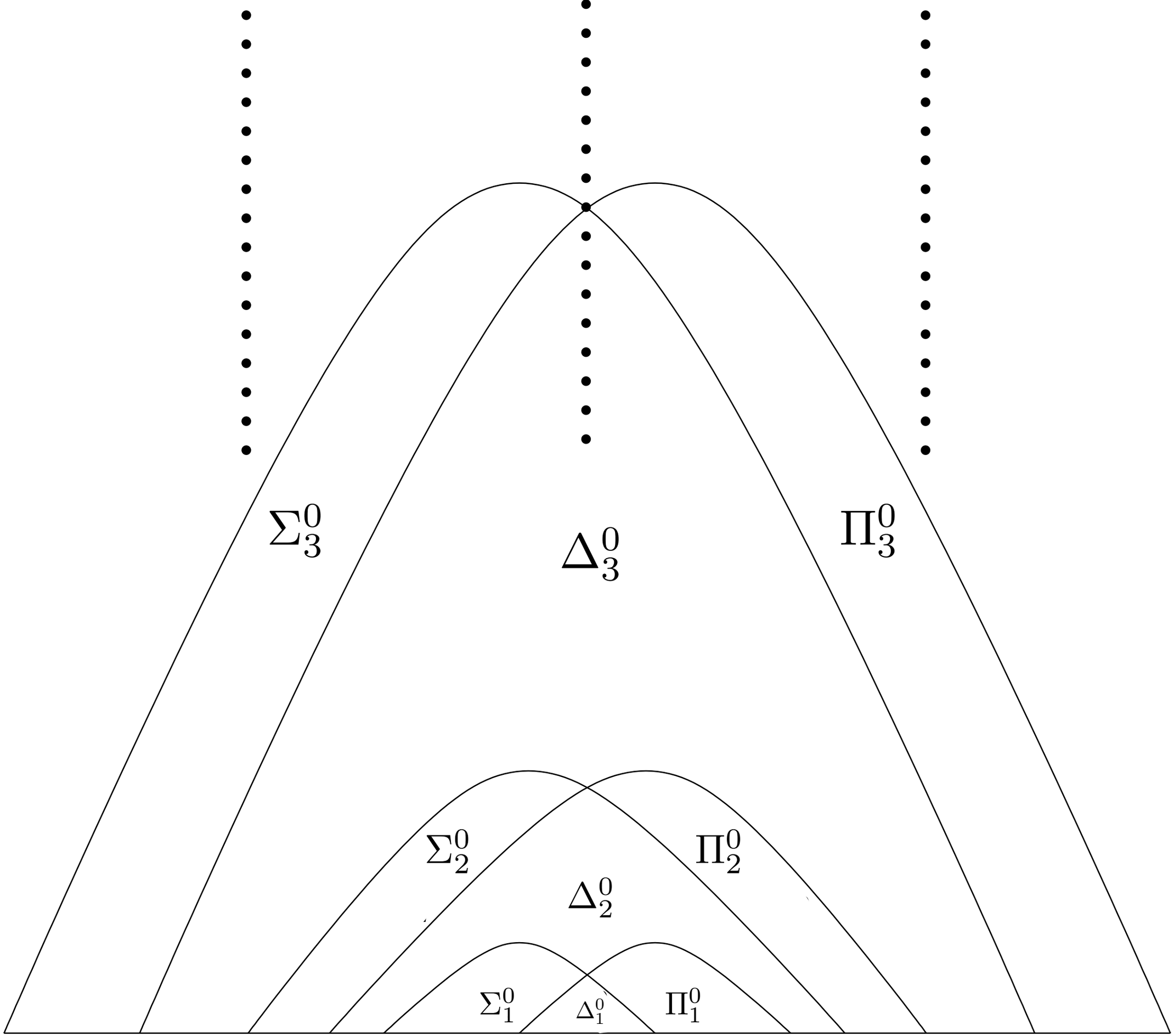
- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

$A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

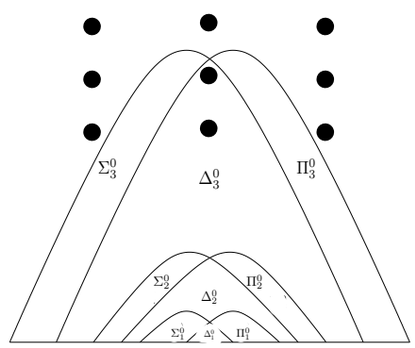



Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0

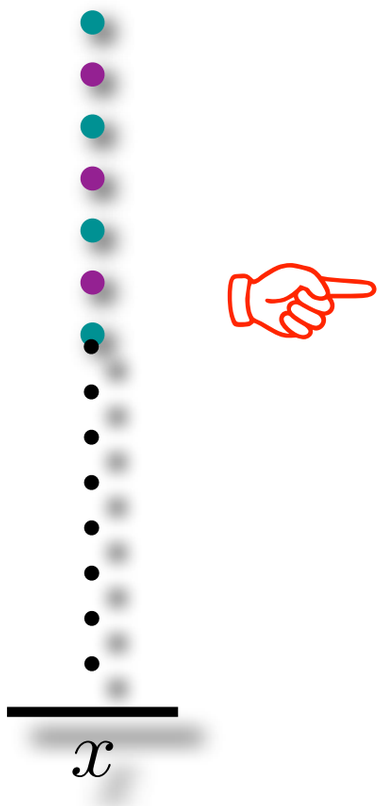


An example: evaluation game for 1st order logic

non determined game

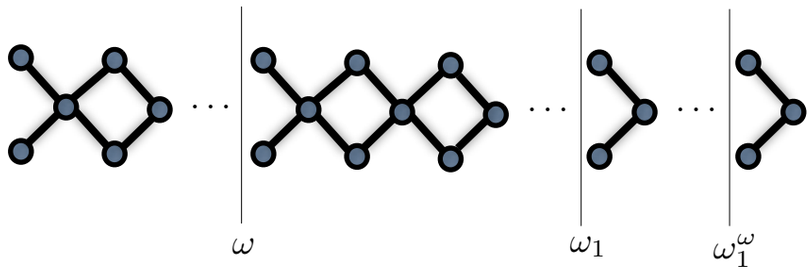


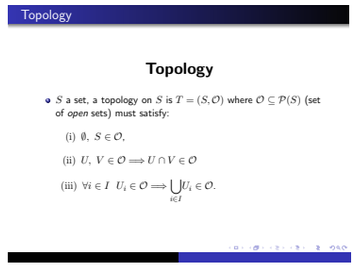
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

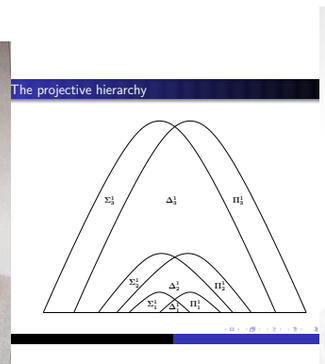
Games that construct mappings



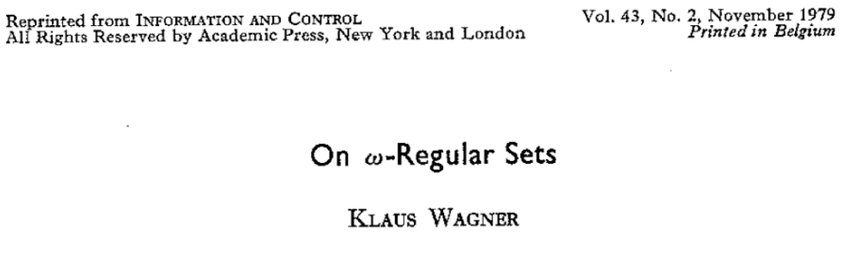










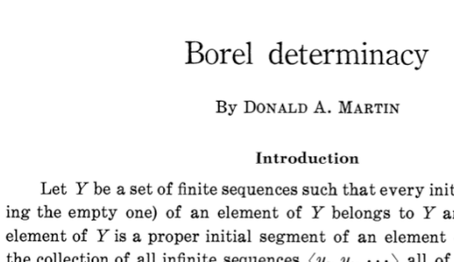


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

Topology

- S a set, a topology on S is $T = (S, \mathcal{O})$ where $\mathcal{O} \subseteq \mathcal{P}(S)$ (set of *open* sets) must satisfy:

(i) $\emptyset,\ S \in \mathcal{O},$

(ii) $U,\ V \in \mathcal{O} \Longrightarrow U \cap V \in \mathcal{O}$

(iii) $\forall i \in I \quad U_i \in \mathcal{O} \Longrightarrow \bigcup_{i \in I} U_i \in \mathcal{O}.$

The Borel hierarchy

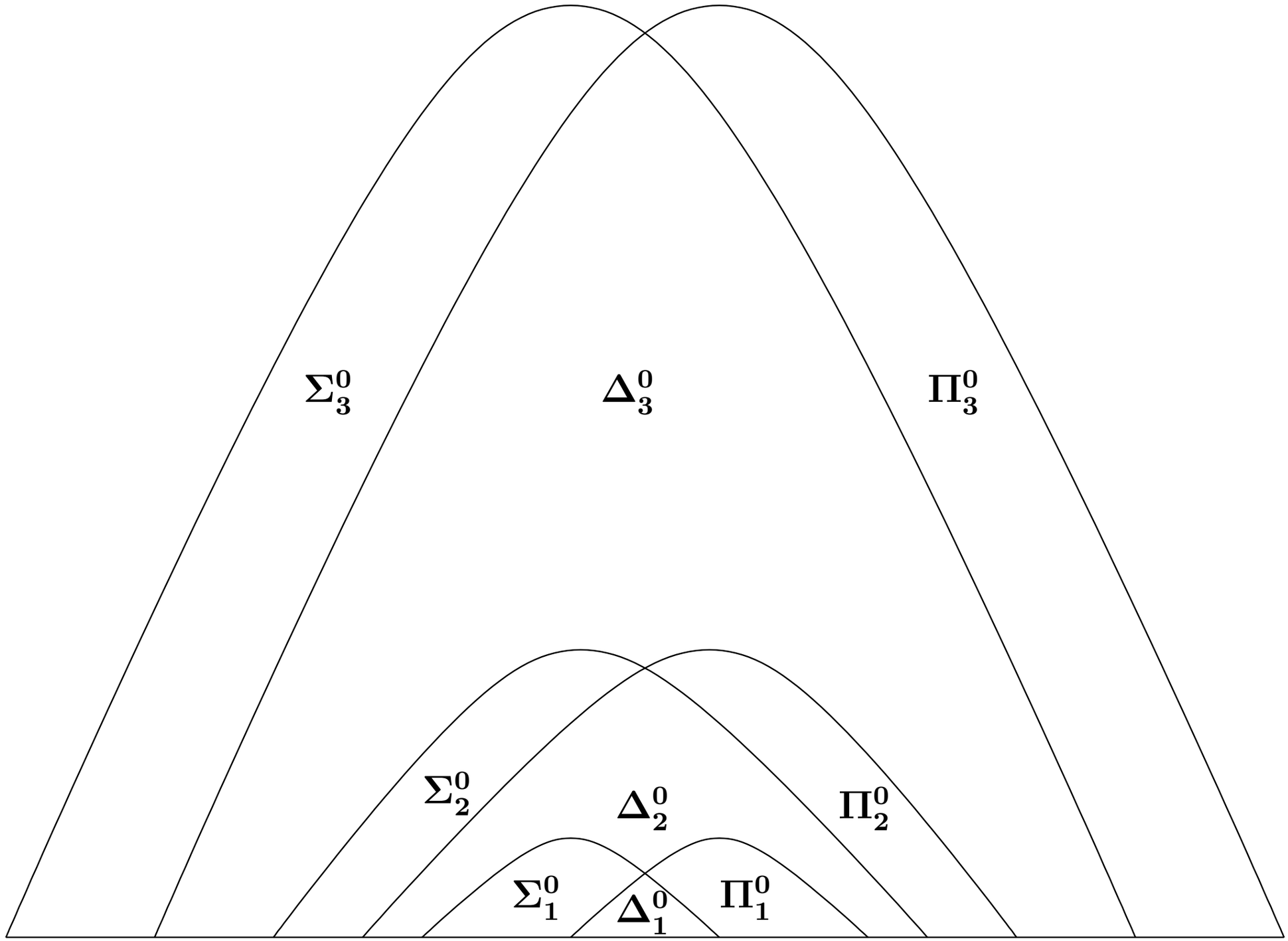
Definition

1. $\mathbf{\Sigma}^0_1 = \{open\ sets\}$
2. $\mathbf{\Pi}^0_\alpha = \{\Sigma \setminus A : \ A \in \mathbf{\Sigma}^0_\alpha\}$
3. $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$
4. $\mathbf{\Sigma}^0_\alpha = \{A = \bigcup_{n\in\mathbb{N}} A_n : \ A_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta\}$

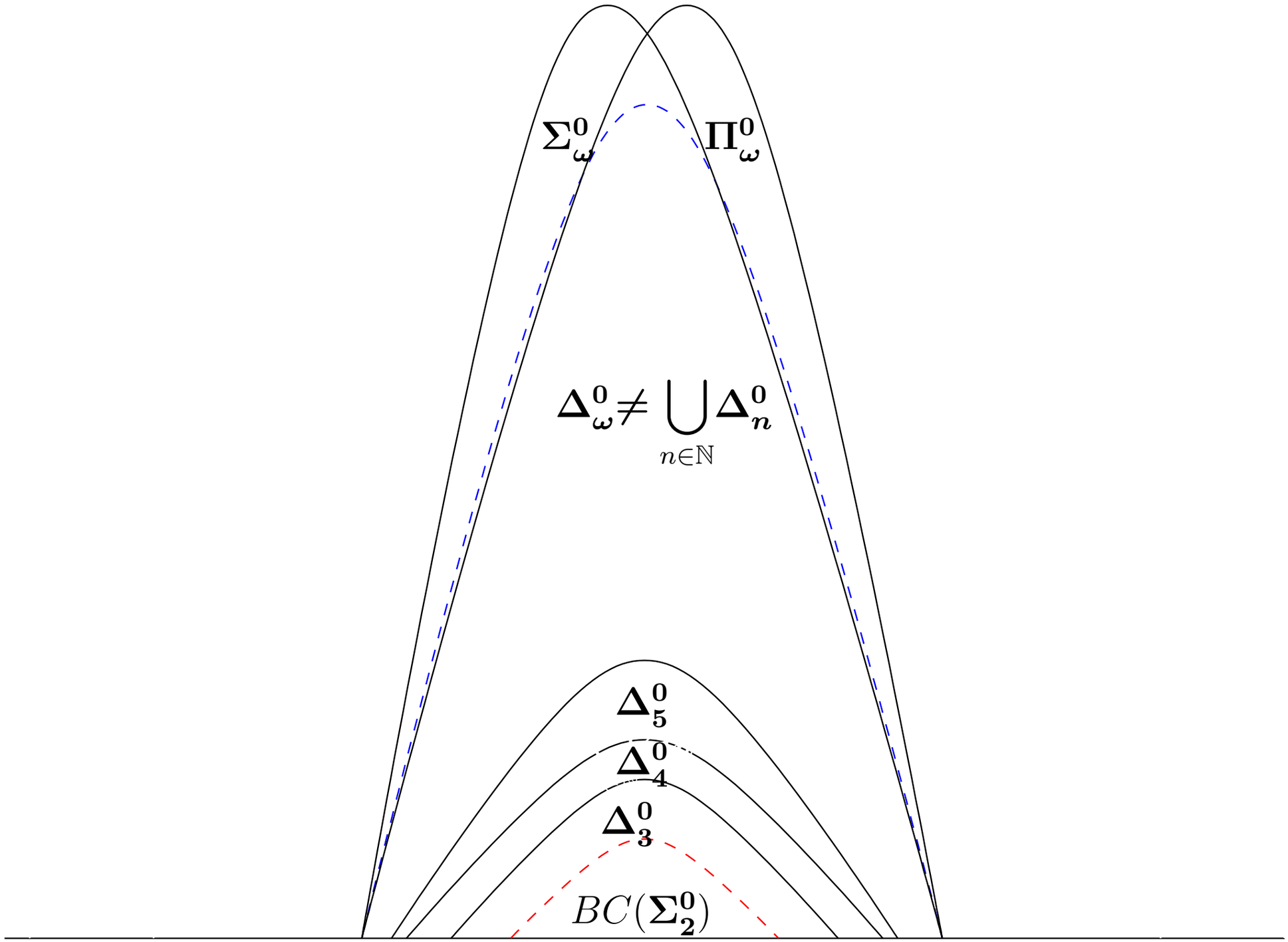
$$\mathcal{B} = \bigcup_{\alpha\in On} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha\in On} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$$

In particular, $\mathbf{\Pi}^0_1$ is the class of all closed sets, $\mathbf{\Sigma}^0_2$ the class of all countable union of closed sets, etc...

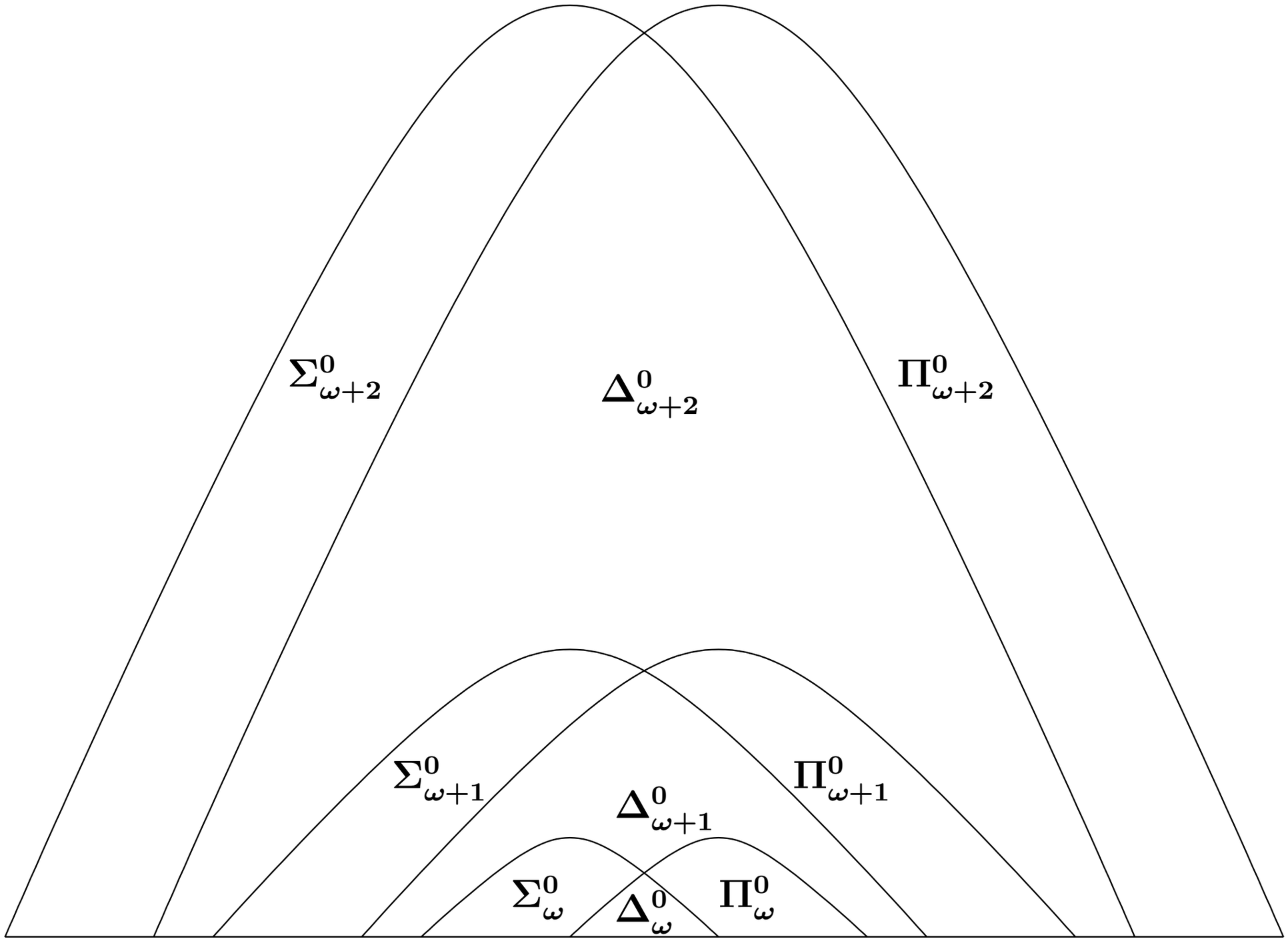
The Borel hierarchy



The Borel hierarchy



The Borel hierarchy



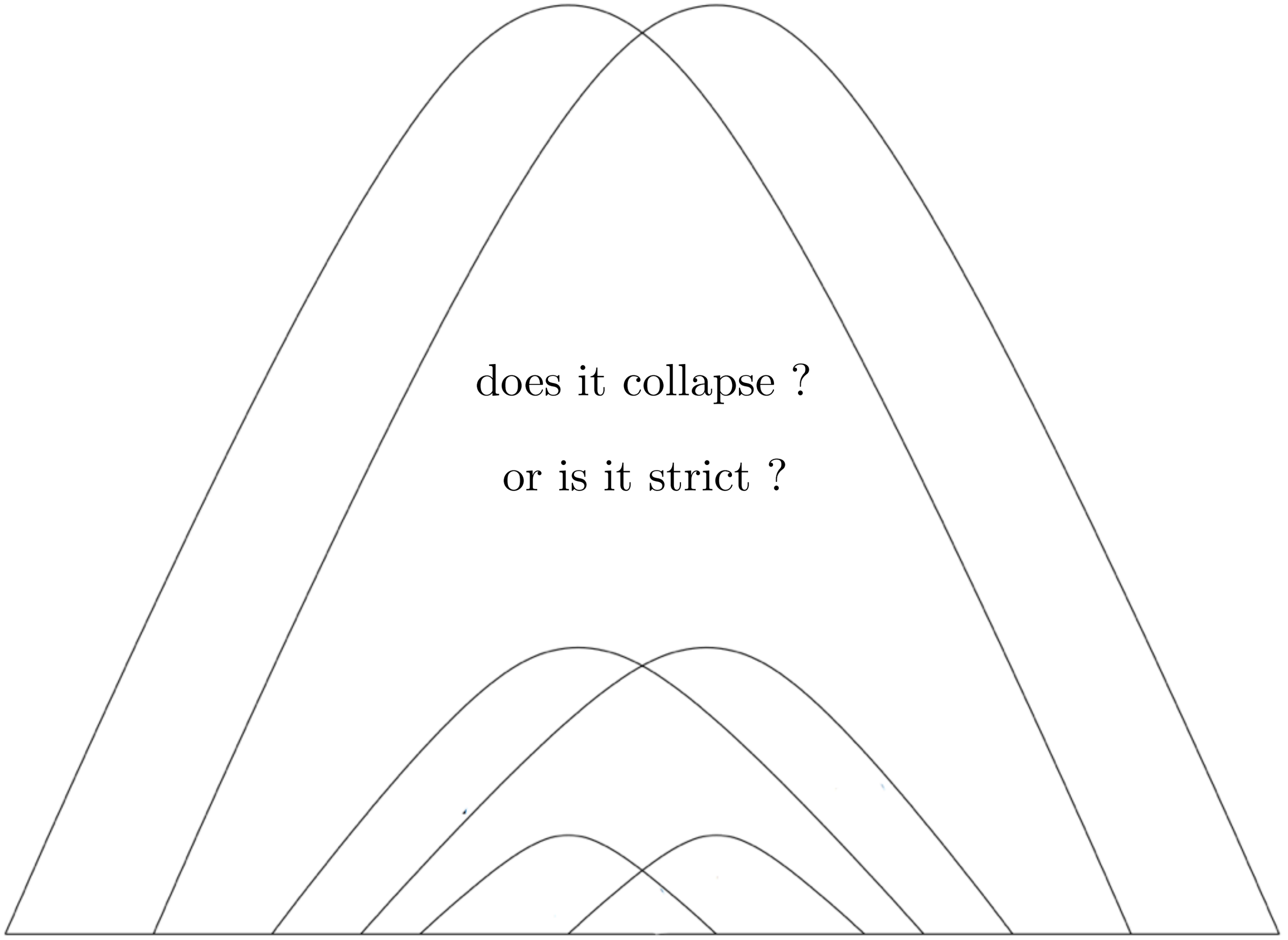
Borel Sets

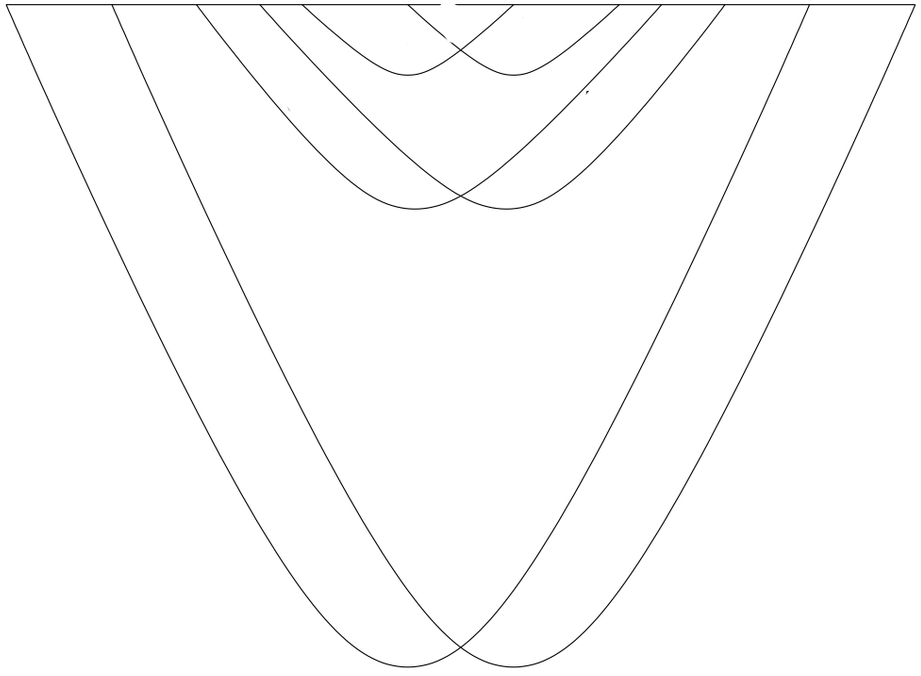
- $\mathbf{\Sigma}^0_{\omega_1} = \mathbf{\Pi}^0_{\omega_1} = \mathbf{\Delta}^0_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\alpha$
- $\mathbf{\Sigma}^0 \leftrightsquigarrow \bigcup_{n\in\mathbb{N}} \leftrightsquigarrow \exists^{\mathbb{N}}$
- $\mathbf{\Pi}^0 \leftrightsquigarrow \bigcap_{n\in\mathbb{N}} \leftrightsquigarrow \forall^{\mathbb{N}}$
- $\mathbf{\Delta}^0 \leftrightsquigarrow \bigcup_{n\in\mathbb{N}} \cap \bigcap_{n\in\mathbb{N}} \leftrightsquigarrow \exists^{\mathbb{N}} \wedge \forall^{\mathbb{N}}$
- $\mathbf{\Sigma}_\alpha \leftrightsquigarrow \alpha \times \left(\bigcup_{n\in\mathbb{N}} + \complement \right) \leftrightsquigarrow \underbrace{\exists^{\mathbb{N}}\forall^{\mathbb{N}}\exists^{\mathbb{N}}\forall^{\mathbb{N}}\ldots\exists^{\mathbb{N}}\forall^{\mathbb{N}}}_{\alpha\times}$

UNIL | Université de Lausanne

does it collapse ?

or is it strict ?





The Arithmetical Hierarchy

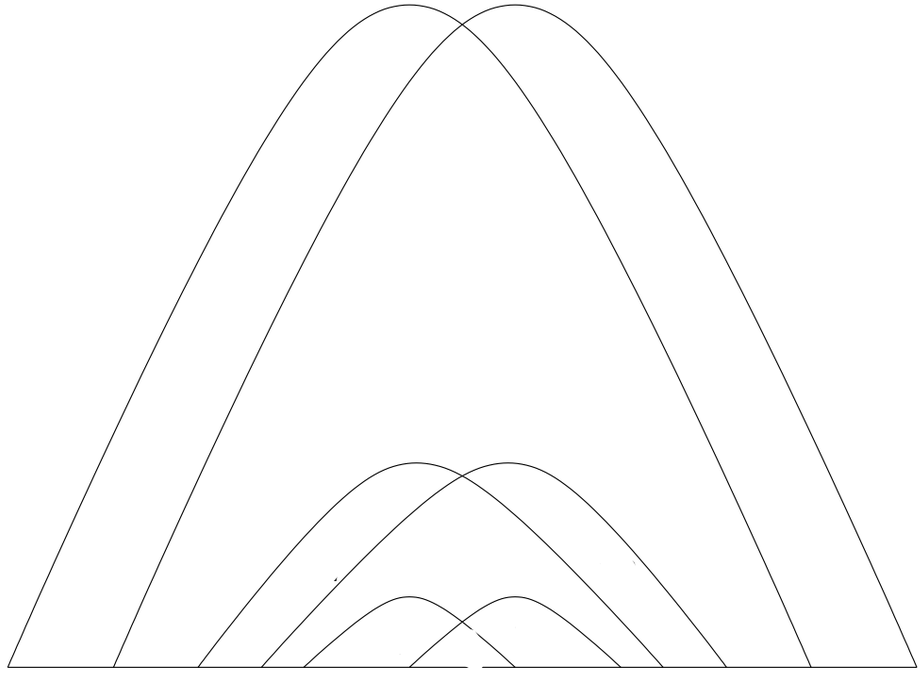
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

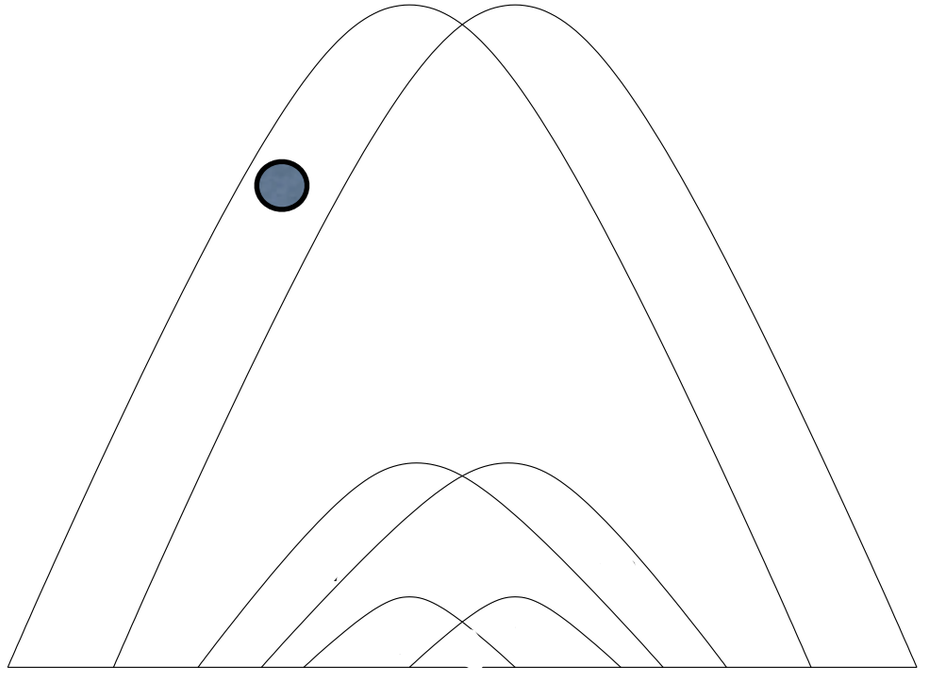
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

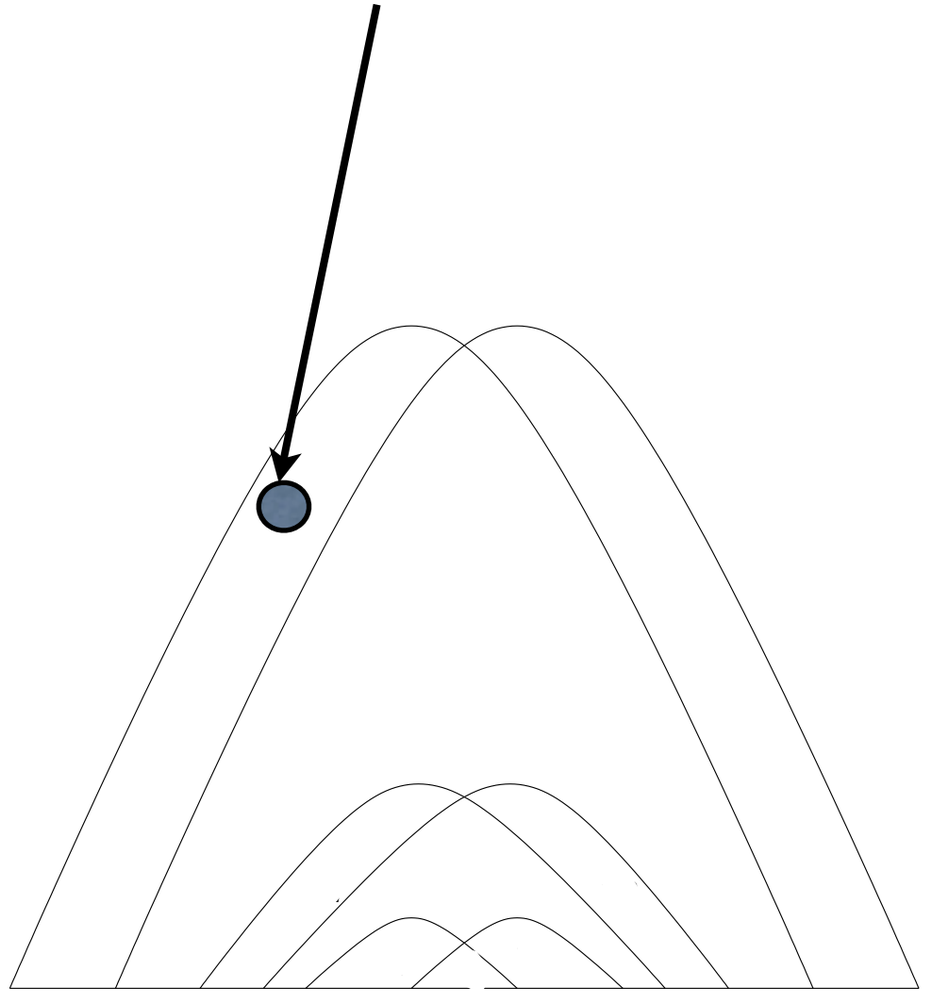
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

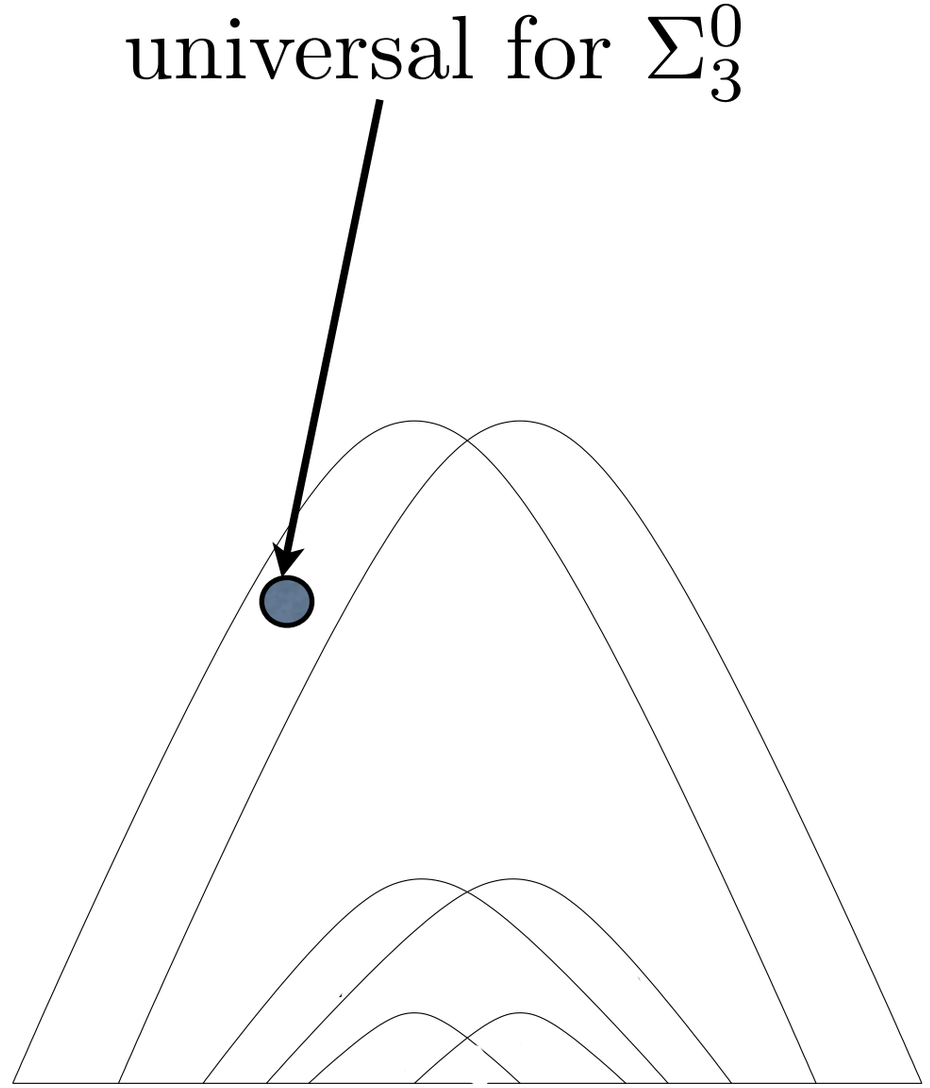
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



The Arithmetical Hierarchy

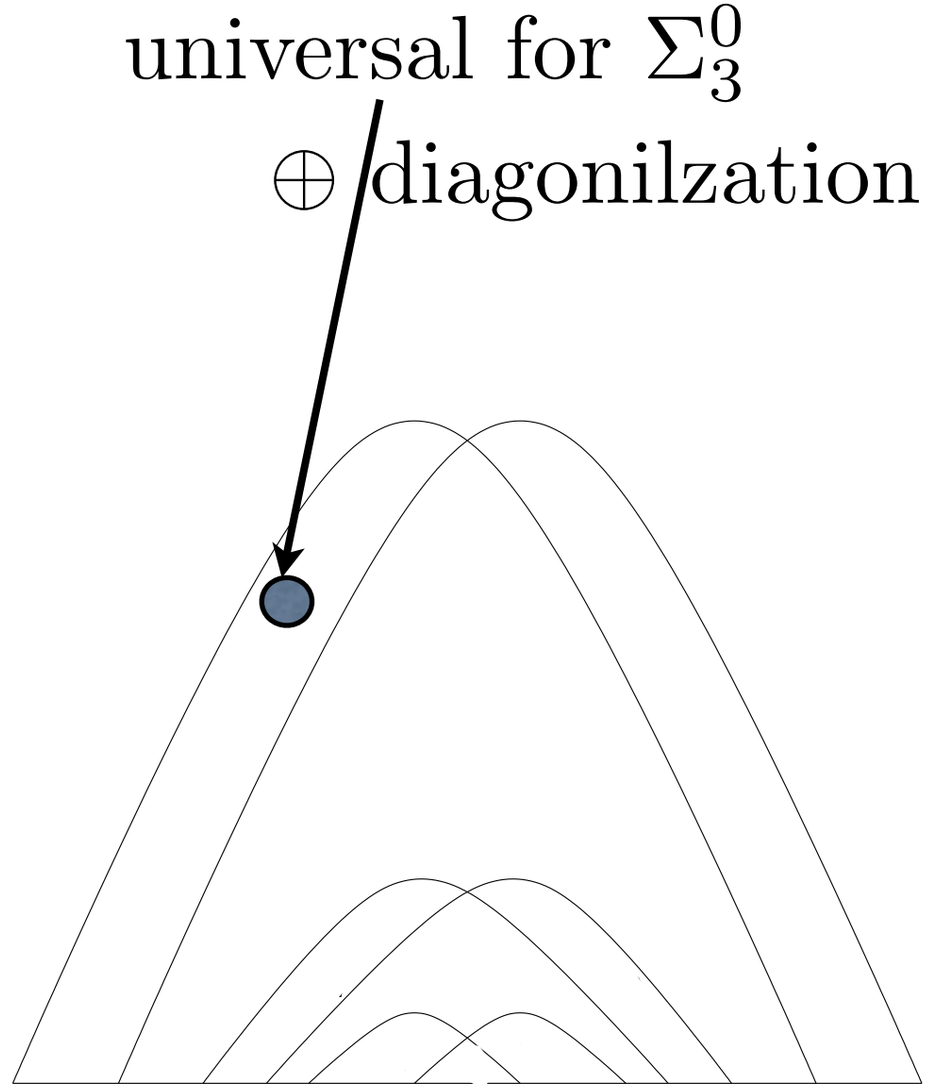
- $(y_1, \ldots, y_k) \in A \in \Sigma_1^0 \iff \begin{cases} A \text{ is r.e.} \\ \\ \exists x \ \phi[x, y_1, \ldots, y_k] \end{cases}$

  where ϕ is free of unbounded quantifiers

- $A \in \Pi_n^0 \iff A^{\complement} \in \Sigma_n^0$

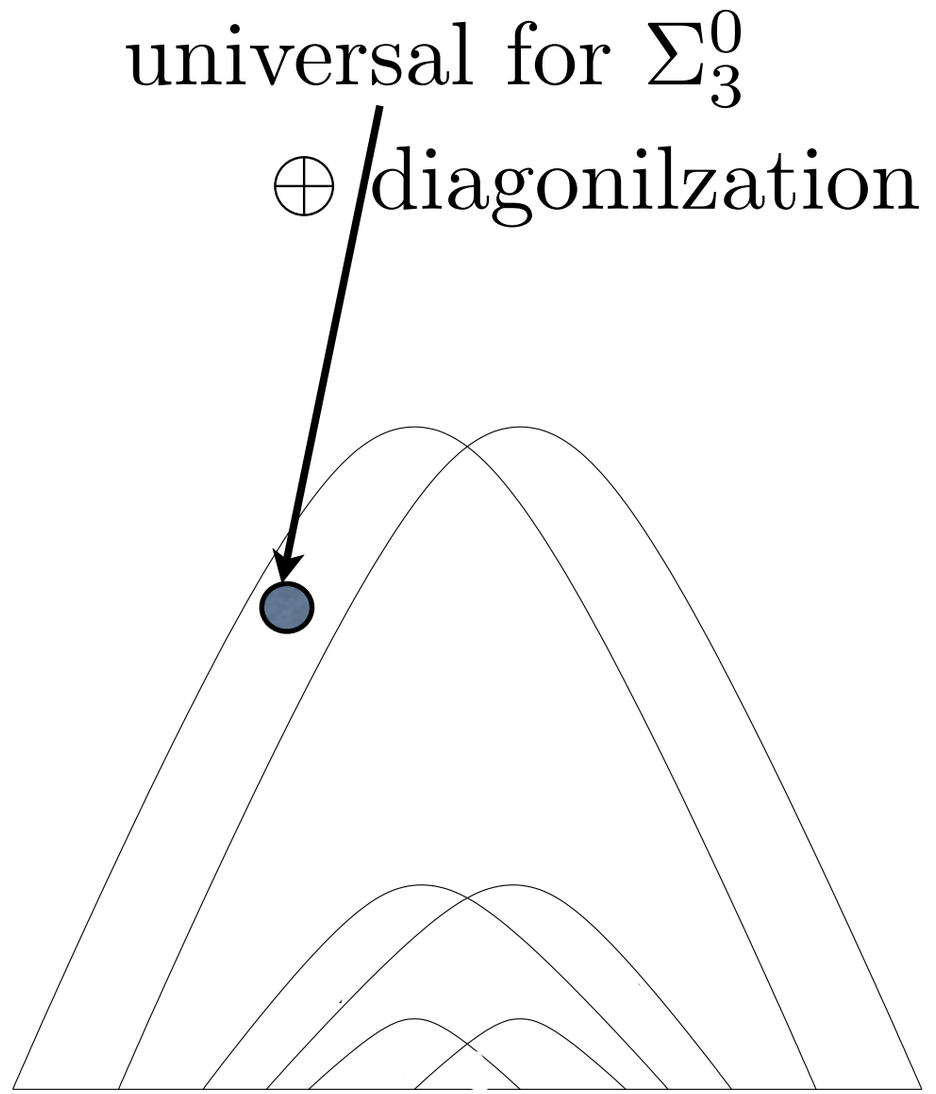
- $(y_1, \ldots, y_k) \in A \in \Sigma_{n+1}^0 \iff \begin{cases} \exists x \ (x, y_1, \ldots, y_k) \in B \\ \\ B \in \pi_n^0 \end{cases}$

- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$



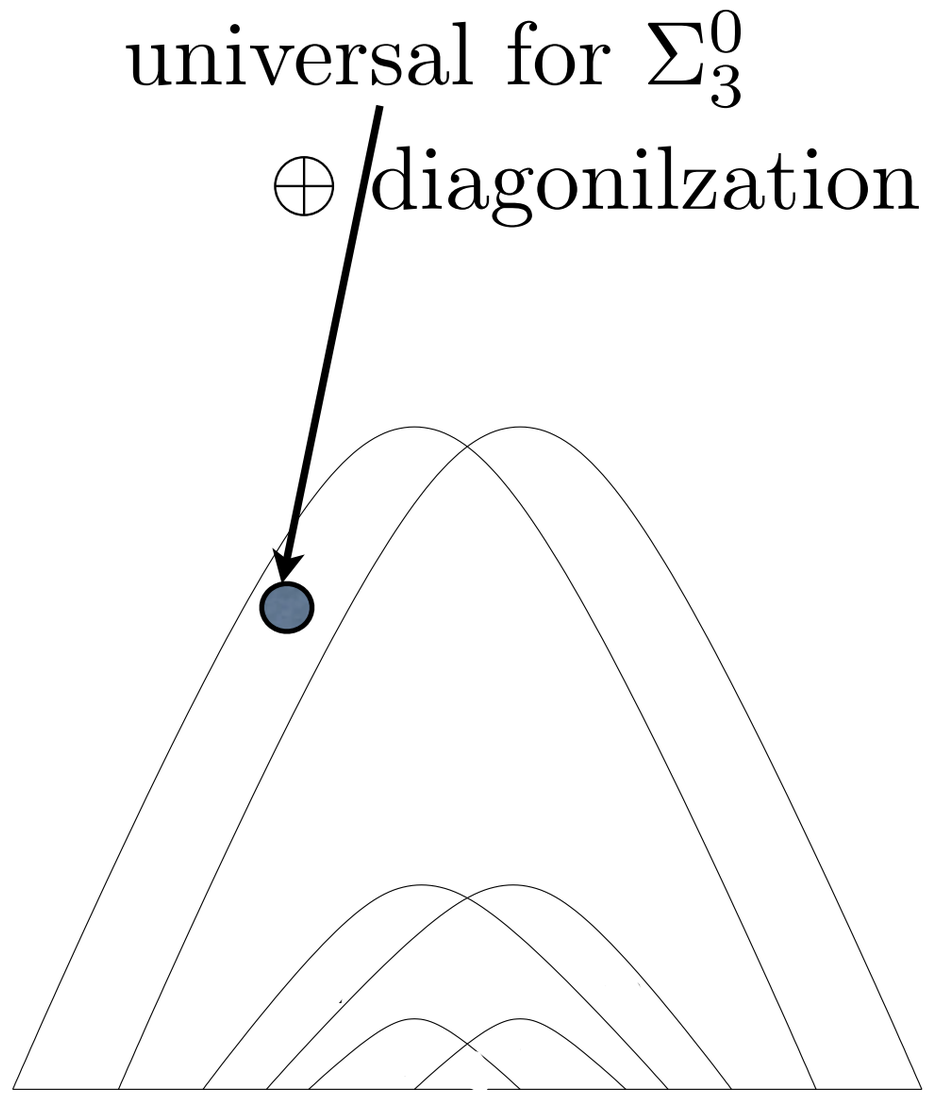
The Borel Hierarchy

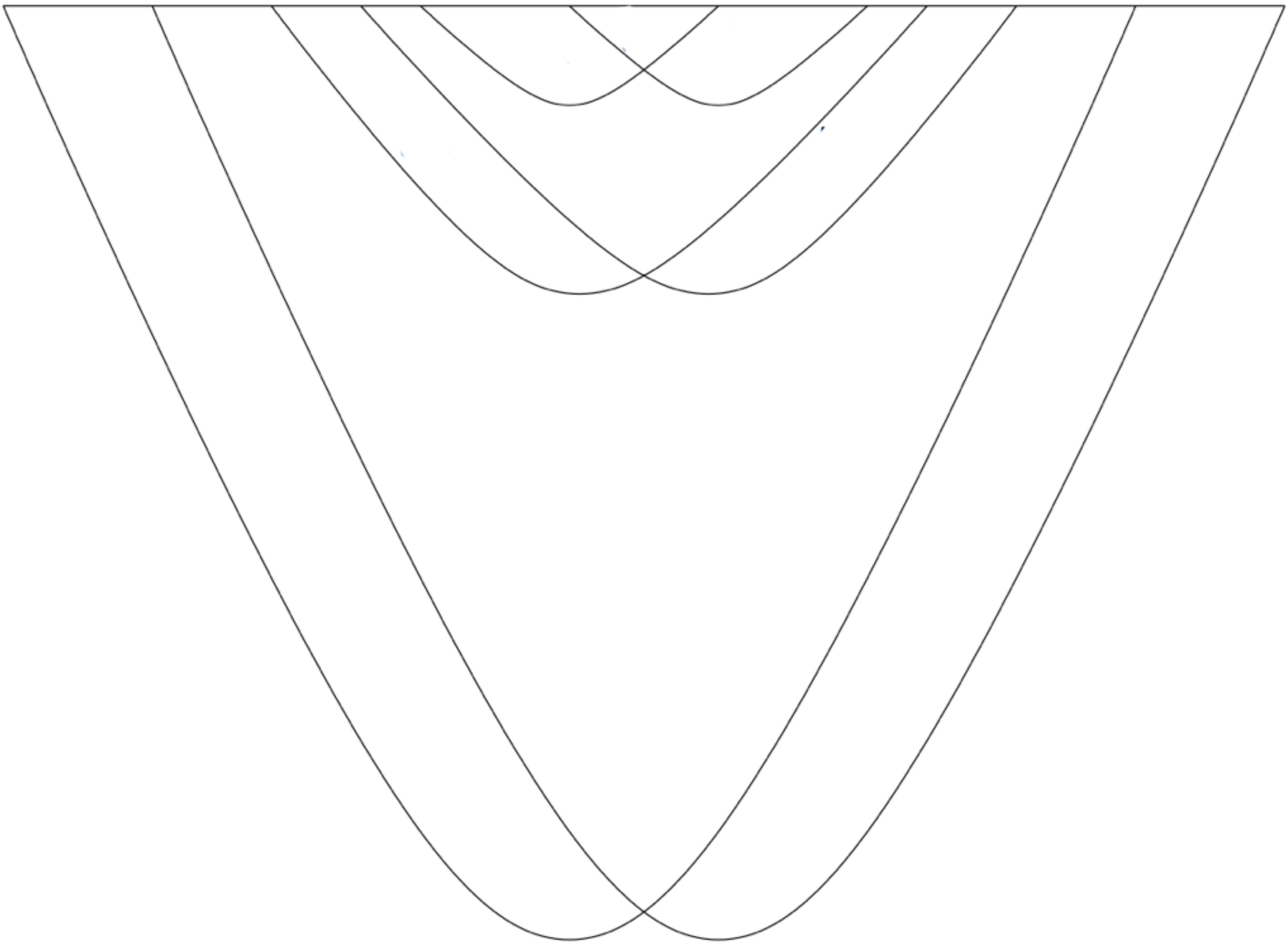
Same ideas work !



The Borel Hierarchy

Other proof with games !!!





Topology

- S a set, a topology on S is $T = (S, \mathcal{O})$ where $\mathcal{O} \subseteq \mathcal{P}(S)$ (set of *open* sets) must satisfy:

(i) $\emptyset,\ S \in \mathcal{O},$

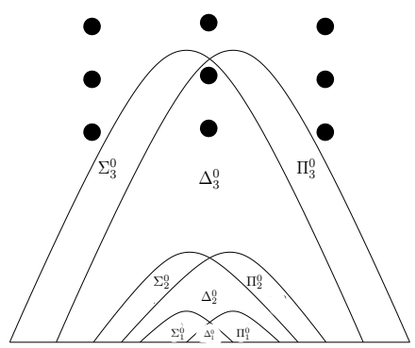
(ii) $U,\ V \in \mathcal{O} \Longrightarrow U \cap V \in \mathcal{O}$

(iii) $\forall i \in I \quad U_i \in \mathcal{O} \Longrightarrow \bigcup_{i \in I} U_i \in \mathcal{O}.$

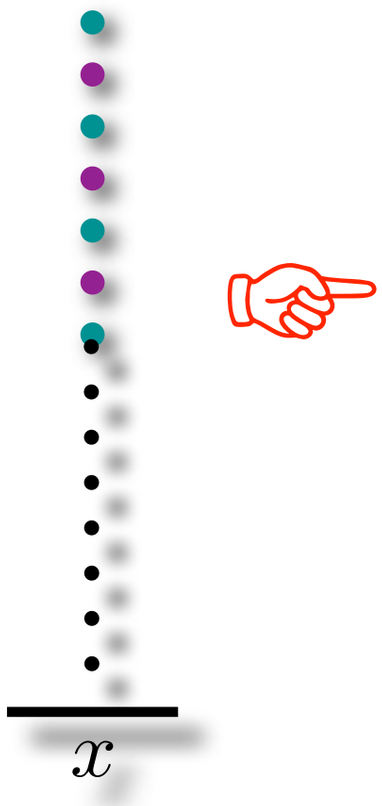


An example: evaluation game for 1st order logic

non determined game

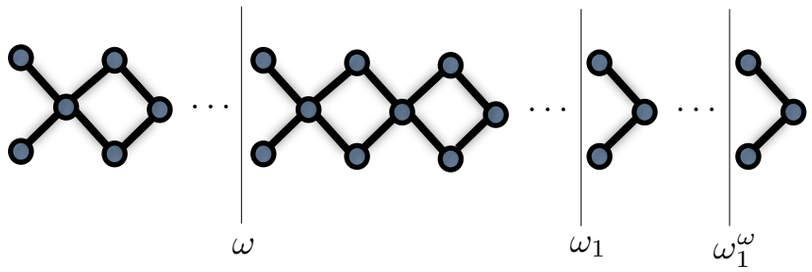


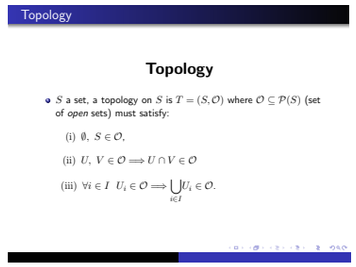
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

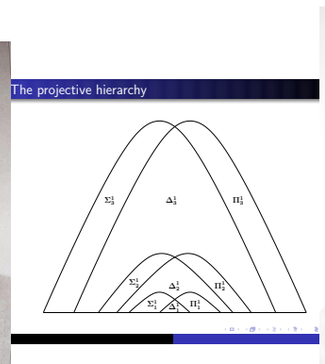
Games that construct mappings



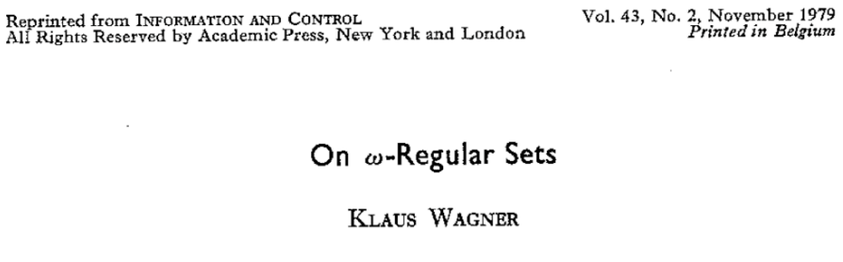










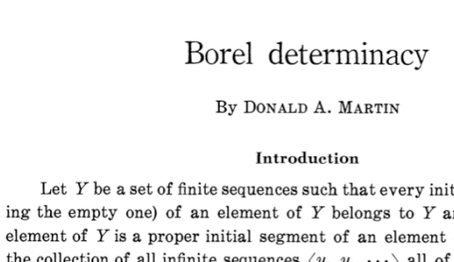


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

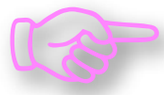
Annals of Mathematics, **102** (1975), 363-371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots\rangle$ all of whose finite initial segments belong to Y. For each $A \subseteq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0\rangle \in Y$, II picks y_1 with $\langle y_0, y_1\rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2\rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega\rangle \in A$. ($\omega$ = the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements of Y of even length such that always $\langle y_0, \cdots, y_{2n-1}, s(\langle y_0, \cdots, y_{2n-1}\rangle)\rangle \in Y$. A play $\langle y_0, y_1, \cdots\rangle$ of $\mathcal{G}(A, Y)$ is a *play according to* the strategy s if $(\forall n)(y_{2n} = s(\langle y_0, \cdots, y_{2n-1}\rangle))$. The notions of a *strategy for* II and a *play according to* a strategy for II are similarly defined. s is a winning strategy for I (II) if every play according to s is a win for I (II). $\mathcal{G}(A, Y)$ is *determined* if one of the players has a winning strategy.



Determinacy

UNIL | Université de Lausanne

Determinacy

(1953) D. Gale & M. Stewart Σ^0_1 determinacy

UNIL | Université de Lausanne

Determinacy

(1953) D. Gale & M. Stewart Σ^0_1 determinacy

(1955) M. Wolfe Σ^0_2 determinacy

Determinacy

- (1953) D. Gale & M. Stewart Σ^0_1 determinacy
- (1955) M. Wolfe Σ^0_2 determinacy
- (1964) M. Davis Σ^0_3 determinacy

Determinacy

(1953) D. Gale & M. Stewart Σ^0_1 determinacy

(1955) M. Wolfe Σ^0_2 determinacy

(1964) M. Davis Σ^0_3 determinacy

(1972) J. B. Paris Σ^0_4 determinacy

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

UNIL | Université de Lausanne

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

UNIL | Université de Lausanne

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85–101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326–357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245–266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287–291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661–667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841–847.

UNIL | Université de Lausanne

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85–101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326–357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245–266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287–291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661–667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841–847.

UNIL | Université de Lausanne

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85–101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326–357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245–266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287–291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661–667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841–847.

UNIL | Université de Lausanne

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85–101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326–357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245–266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287–291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661–667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841–847.

UNIL | Université de Lausanne

(1953) D. Gale & M. Stewart Σ^0_1 determinacy

(1955) M. Wolfe Σ^0_2 determinacy

(1964) M. Davis Σ^0_3 determinacy

(1972) J. B. Paris Σ^0_4 determinacy

(1953) D. Gale & M. Stewart Σ^0_1 determinacy

(1955) M. Wolfe Σ^0_2 determinacy

(1964) M. Davis Σ^0_3 determinacy

(1972) J. B. Paris Σ^0_4 determinacy

(1975) D. A. Martin Borel determinacy

References

[1] Andreas Blass, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] Morton Davis, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85–101.

[3] Harvey Friedman, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326–357.

[4] D. Gale and F. M. Stewart, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245–266.

[5] Donald A. Martin, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287–291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. Paris, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661–667.

[8] P. Wolfe, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841–847.

UNIL | Université de Lausanne

Annals of Mathematics, **102** (1975), 363-371

Borel determinacy

By Donald A. Martin

Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots \rangle$ all of whose finite initial segments belong to Y. For each $A \subseteq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0 \rangle \in Y$, II picks y_1 with $\langle y_0, y_1 \rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2 \rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega \rangle \in A$. ($\omega$ = the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements of Y of even length such that always $\langle y_0, \cdots, y_{2n-1}, s(\langle y_0, \cdots, y_{2n-1} \rangle) \rangle \in Y$.

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved

THEOREM. *All Borel games are determined.*

THE ROCKEFELLER UNIVERSITY

REFERENCES

[1] ANDREAS BLASS, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.
[2] MORTON DAVIS, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.
[3] HARVEY FRIEDMAN, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.
[4] D. GALE and F. M. STEWART, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.
[5] DONALD A. MARTIN, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.
[6] ———, *Borel and projective games*, Springer-Verlag, to appear.
[7] J. B. PARIS, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.
[8] P. WOLFE, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

(Received April 20, 1975)

UNIL | Université de Lausanne

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved



THEOREM. *All Borel games are determined.*

THE ROCKEFELLER UNIVERSITY

REFERENCES

[1] ANDREAS BLASS, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.

[2] MORTON DAVIS, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.

[3] HARVEY FRIEDMAN, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.

[4] D. GALE and F. M. STEWART, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.

[5] DONALD A. MARTIN, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.

[6] ———, *Borel and projective games*, Springer-Verlag, to appear.

[7] J. B. PARIS, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.

[8] P. WOLFE, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

(Received April 20, 1975)

UNIL | Université de Lausanne

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved

THEOREM. *All Borel games are determined.*

THE ROCKEFELLER UNIVERSITY

REFERENCES

[1] ANDREAS BLASS, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.
[2] MORTON DAVIS, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.
[3] HARVEY FRIEDMAN, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.
[4] D. GALE and F. M. STEWART, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.
[5] DONALD A. MARTIN, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.
[6] ———, *Borel and projective games*, Springer-Verlag, to appear.
[7] J. B. PARIS, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.
[8] P. WOLFE, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

(Received April 20, 1975)

UNIL | Université de Lausanne

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved

THEOREM. *All Borel games are determined.*

THE ROCKEFELLER UNIVERSITY

REFERENCES

[1] ANDREAS BLASS, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.
[2] MORTON DAVIS, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.
[3] HARVEY FRIEDMAN, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.
[4] D. GALE and F. M. STEWART, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.
[5] DONALD A. MARTIN, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.
[6] ———, *Borel and projective games*, Springer-Verlag, to appear.
[7] J. B. PARIS, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.
[8] P. WOLFE, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

(Received April 20, 1975)

UNIL | Université de Lausanne

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved

THEOREM. *All Borel games are determined.*

THE ROCKEFELLER UNIVERSITY

REFERENCES

[1] ANDREAS BLASS, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.
[2] MORTON DAVIS, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.
[3] HARVEY FRIEDMAN, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.
[4] D. GALE and F. M. STEWART, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.
[5] DONALD A. MARTIN, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.
[6] ———, *Borel and projective games*, Springer-Verlag, to appear.
[7] J. B. PARIS, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.
[8] P. WOLFE, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

(Received April 20, 1975)

UNIL | Université de Lausanne

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved

THEOREM. *All Borel games are determined.*

THE ROCKEFELLER UNIVERSITY

REFERENCES

[1] ANDREAS BLASS, Equivalence of two strong forms of determinacy, Proc. A.M.S., to appear.
[2] MORTON DAVIS, Infinite games with perfect information, Advances in game theory, Annals of Math. Studies No. 52, Princeton University Press (1964), 85-101.
[3] HARVEY FRIEDMAN, Higher set theory and mathematical practice, Annals of Math. Logic **2** (1971), 326-357.
[4] D. GALE and F. M. STEWART, Infinite games with perfect information, Contributions to the theory of games, Annals of Math. Studies No. 28, Princeton University Press (1953), 245-266.
[5] DONALD A. MARTIN, Measurable cardinals and analytic games, Fund. Math. **66** (1970), 287-291.
[6] ———, *Borel and projective games*, Springer-Verlag, to appear.
[7] J. B. PARIS, $ZF \vdash \Sigma^0_4$ determinateness, Jour. Symbolic Logic **37** (1972), 661-667.
[8] P. WOLFE, The strict determinateness of certain infinite games, Pac. Jour. of Math. **5** (1955), 841-847.

(Received April 20, 1975)

UNIL | Université de Lausanne

UNIL | Université de Lausanne

(1960-70's) "Borel determinacy is too good to be true"

(1960-70's) "Borel determinacy is too good to be true"

(1960-70's) Tony Martin proves consequences of Borel determinacy determinacy, in order to disprove Borel determinacy

(1960-70's) "Borel determinacy is too good to be true"

(1960-70's) Tony Martin proves consequences of Borel determinacy determinacy, in order to disprove Borel determinacy

(1971) H. Friedman shows that Σ^0_α determinacy requires α iterations of the power set axiom

(1960-70's) "Borel determinacy is too good to be true"

(1960-70's) Tony Martin proves consequences of Borel determinacy determinacy, in order to disprove Borel determinacy

(1971) H. Friedman shows that Σ^0_α determinacy requires α iterations of the power set axiom

any proof should involve these large objects

UNIL | Université de Lausanne

when moves are integers, (quasi)strategies are trees whose nodes are integers ⇒ sets of integers

when moves are sets of integers, (quasi)strategies are trees whose nodes are sets of integers ⇒ sets of sets of integers

when moves are.....

games with auxiliary moves.....

when moves are integers, (quasi)strategies are trees whose nodes are integers ⇒ sets of integers

when moves are sets of integers, (quasi)strategies are trees whose nodes are sets of integers ⇒ sets of sets of integers

when moves are.....

games with auxiliary moves.....



games with auxiliary moves.....



games with auxiliary moves.....



games with auxiliary moves.....

UNIL | Université de Lausanne

$$\mathcal{G}_*(A)$$

$$\mathcal{G}_*(A)$$

Alice

Bob

$$\mathcal{G}_*(A)$$

Alice

Bob



$$\mathcal{G}_*(A)$$

Alice

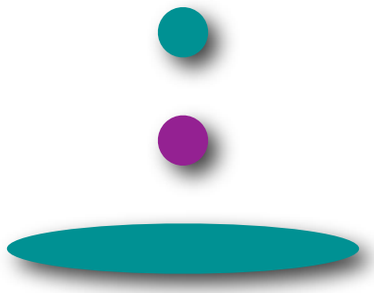
Bob



$$\mathcal{G}_*(A)$$

Alice

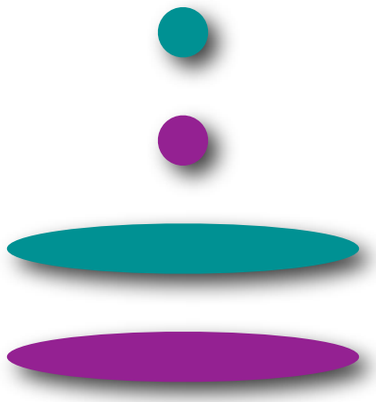
Bob



$$\mathcal{G}_*(A)$$

Alice

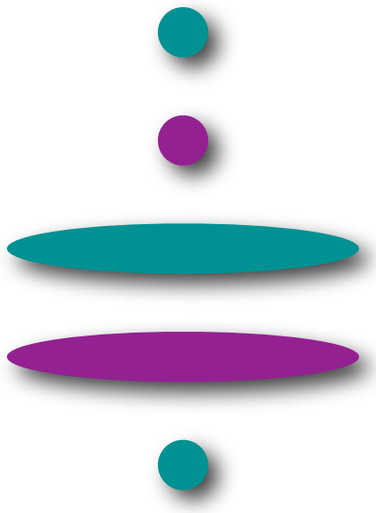
Bob



$$\mathcal{G}_*(A)$$

Alice

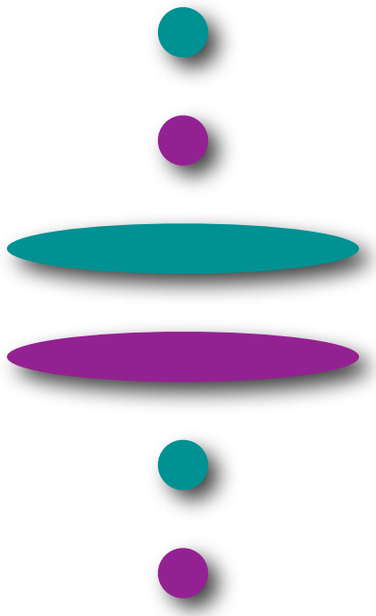
Bob



$$\mathcal{G}_*(A)$$

Alice

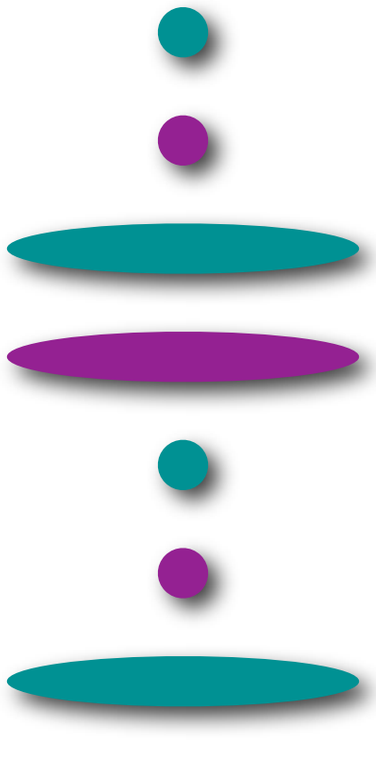
Bob



$\mathcal{G}_*(A)$

Alice

Bob

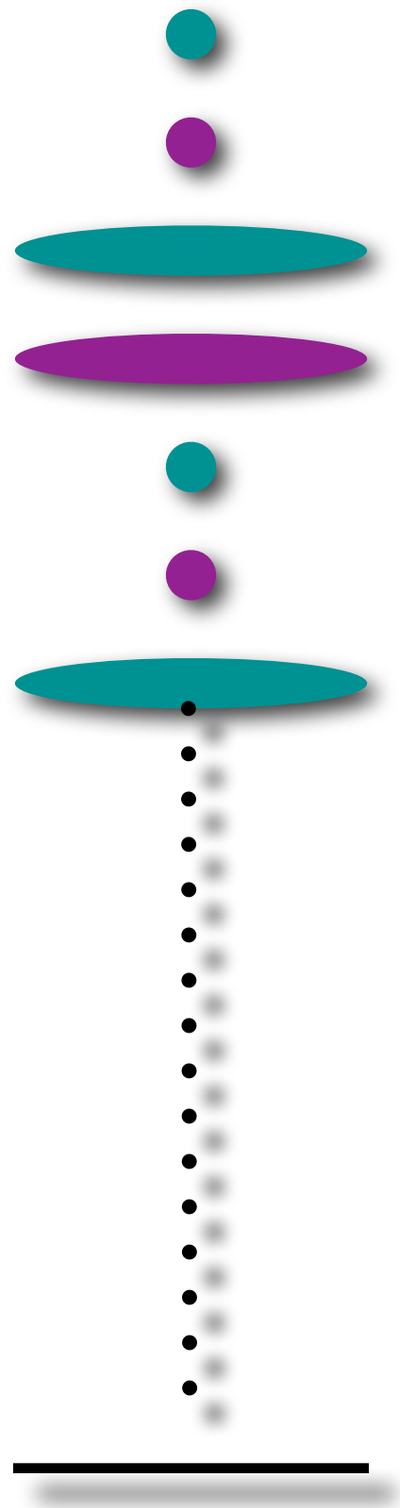


UNIL | Université de Lausanne

$\mathcal{G}_*(A)$

Alice

Bob

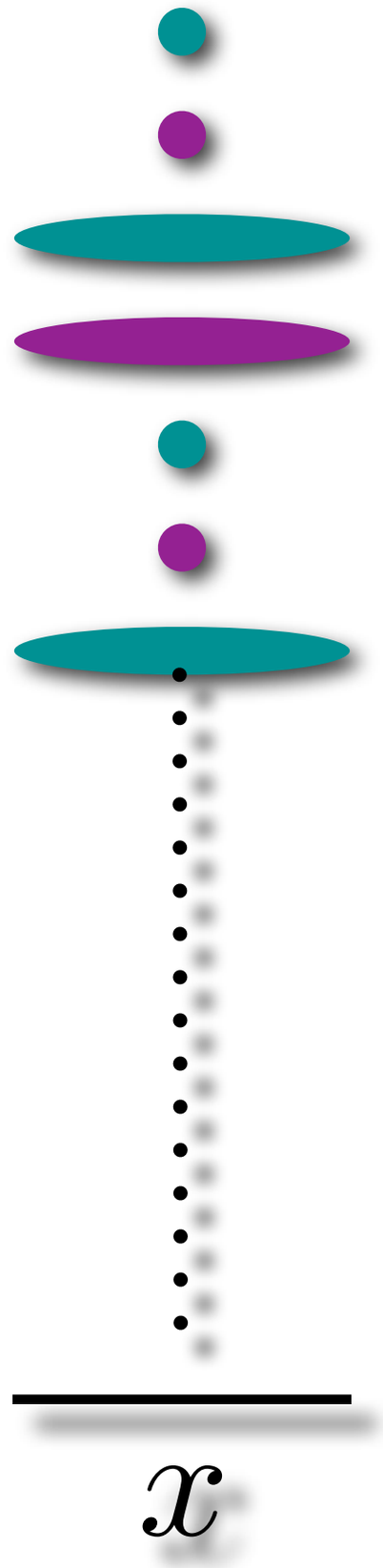


UNIL | Université de Lausanne

$\mathcal{G}_*(A)$

Alice

Bob



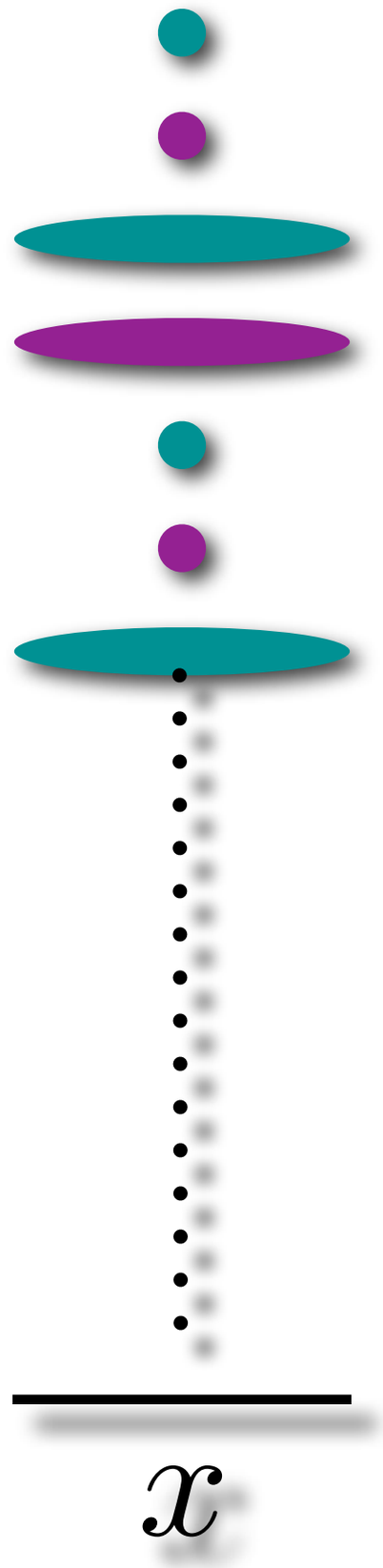
x

UNIL | Université de Lausanne

$\mathcal{G}_*(A)$

Alice

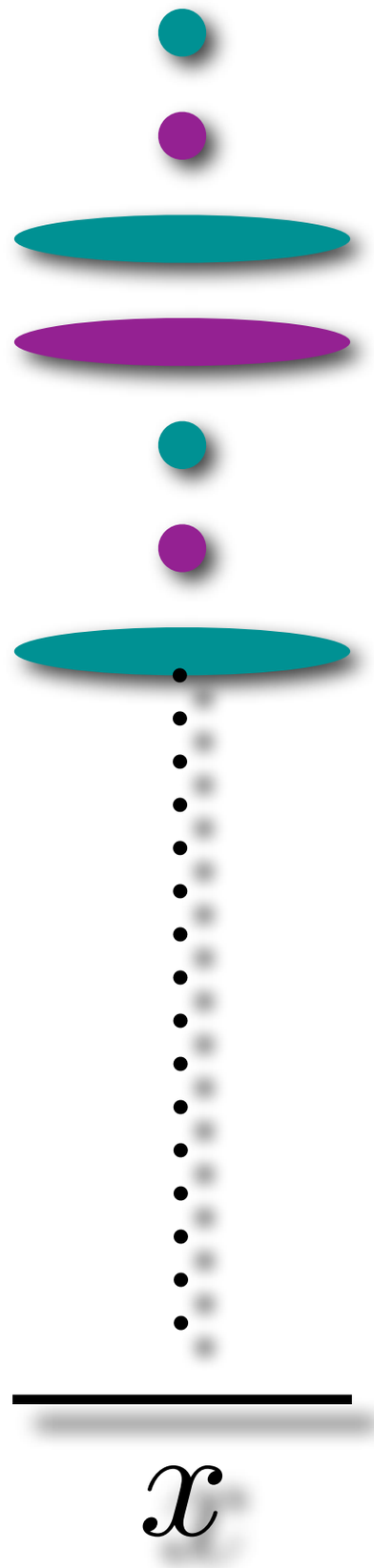
Bob



x

UNIL | Université de Lausanne

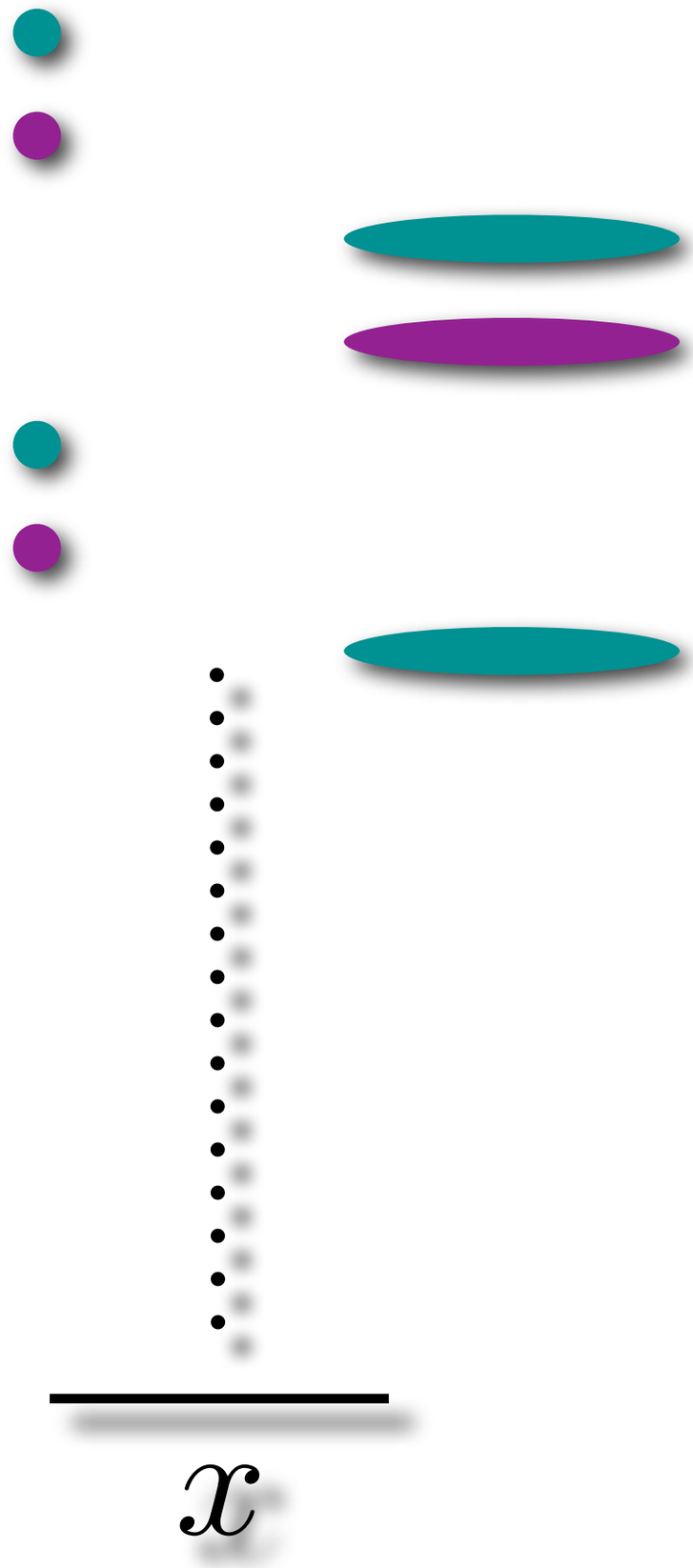
$$\mathcal{G}_*(A)$$



$$x$$

UNIL | Université de Lausanne

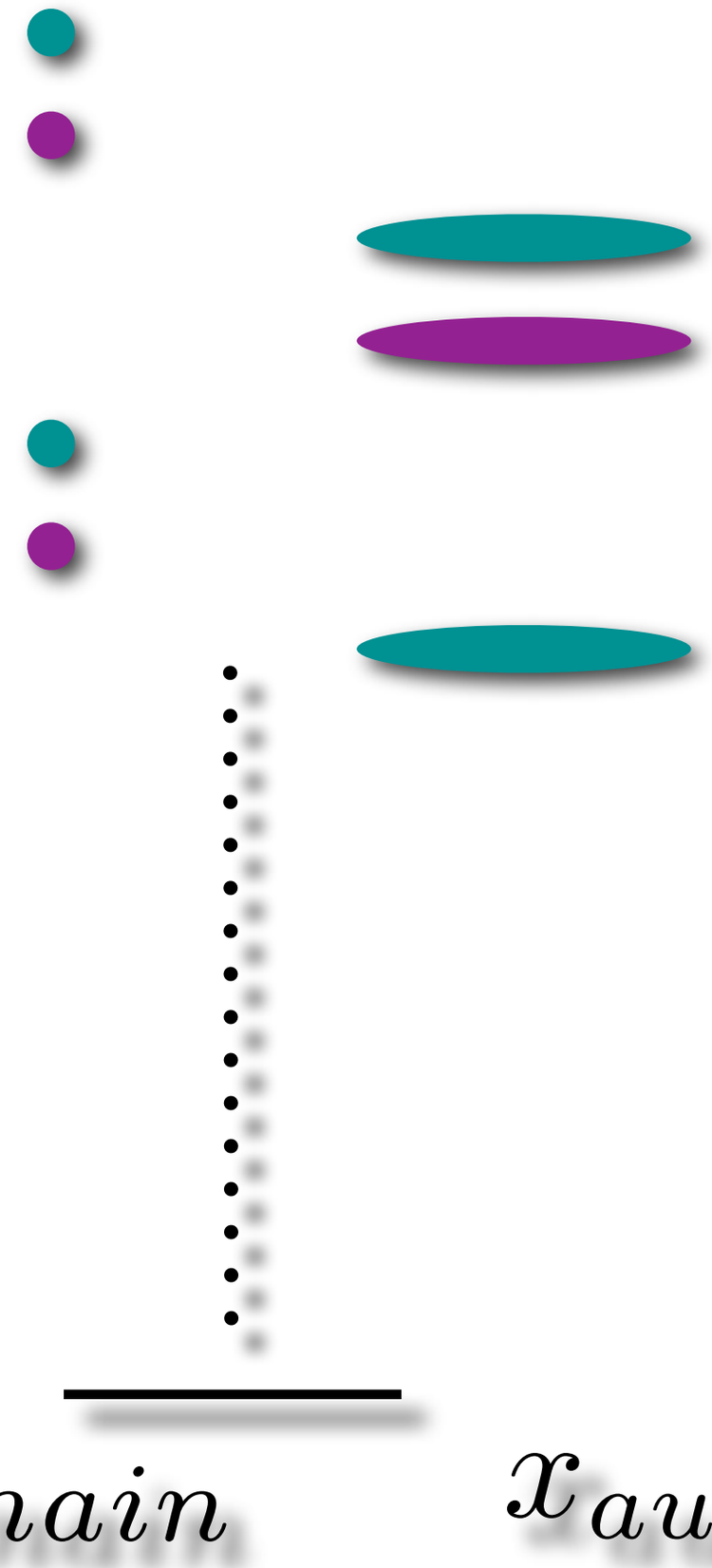
$$\mathcal{G}_*(A)$$



$$x$$

UNIL | Université de Lausanne

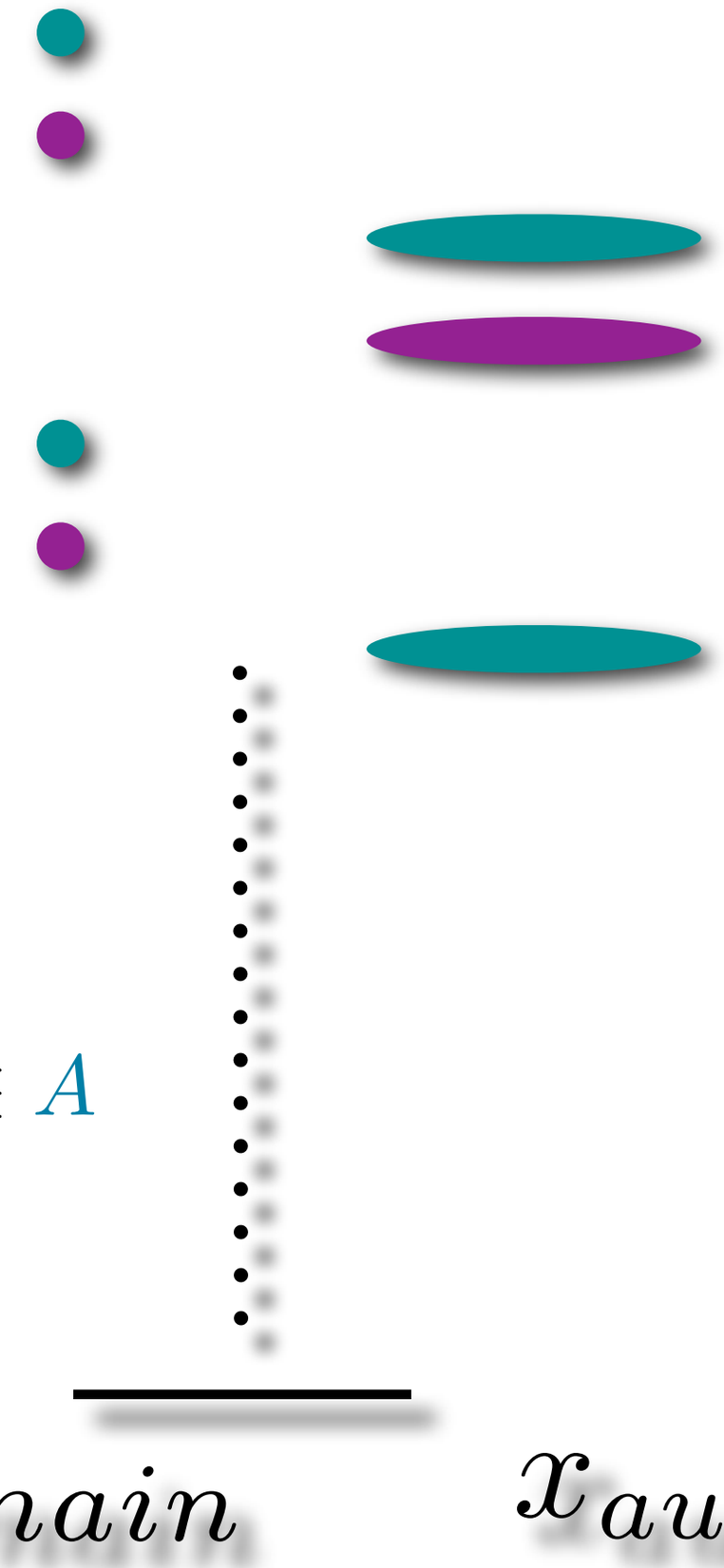
$\mathcal{G}_*(A)$



x_{main} x_{aux}

UNIL | Université de Lausanne

$\mathcal{G}_*(A)$

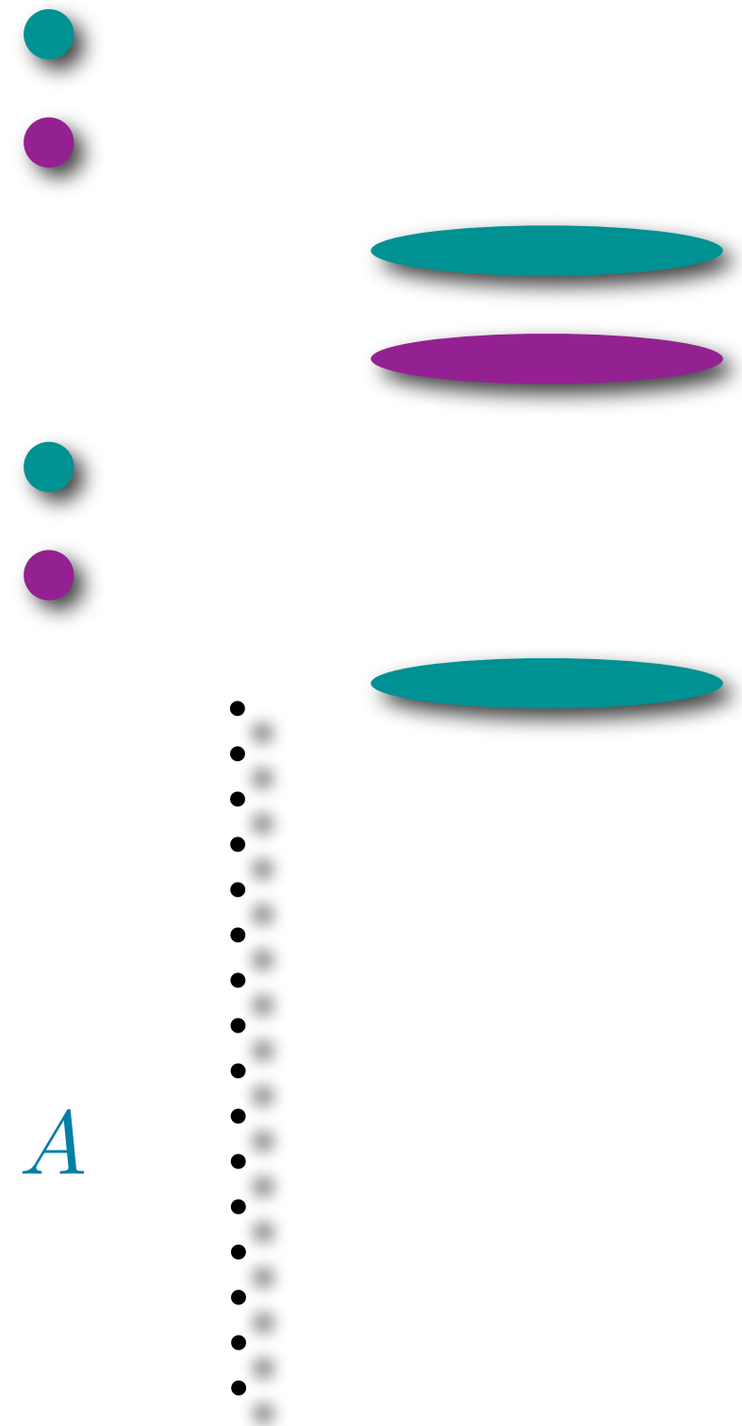


Alice wins $\iff x_{main} \in A$

x_{main} x_{aux}

UNIL | Université de Lausanne

$\mathcal{G}(\hat{A})$

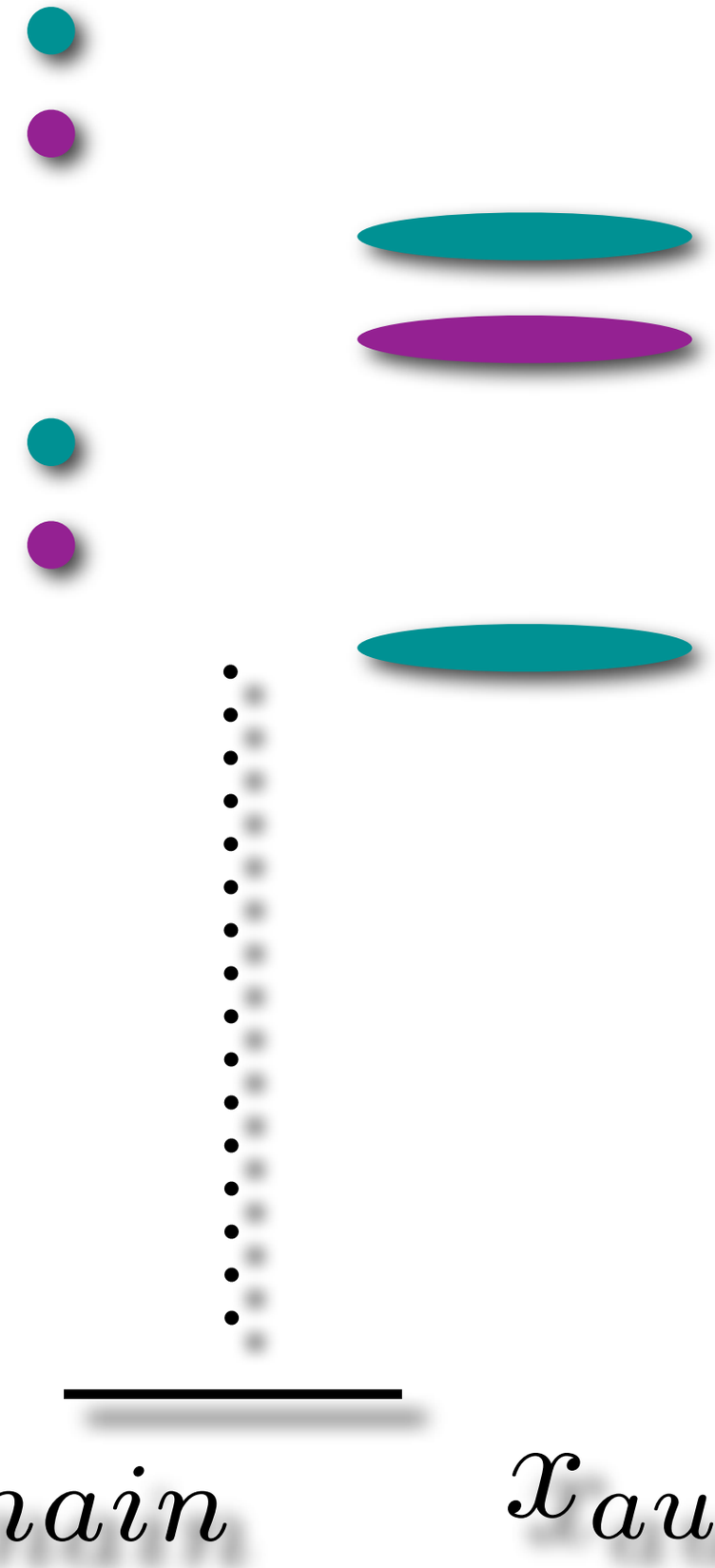


Alice wins $\iff x_{main} \in A$

x_{main} x_{aux}

UNIL | Université de Lausanne

$\mathcal{G}(\hat{A})$



Alice wins

$\Longleftrightarrow$

$x \in \hat{A}$

$\Longleftrightarrow$

$x_{main} \in A$

x_{main} x_{aux}

UNIL | Université de Lausanne

Main Ideas of Borel Determinacy

From Moschovakis *Descriptive Set Theory*

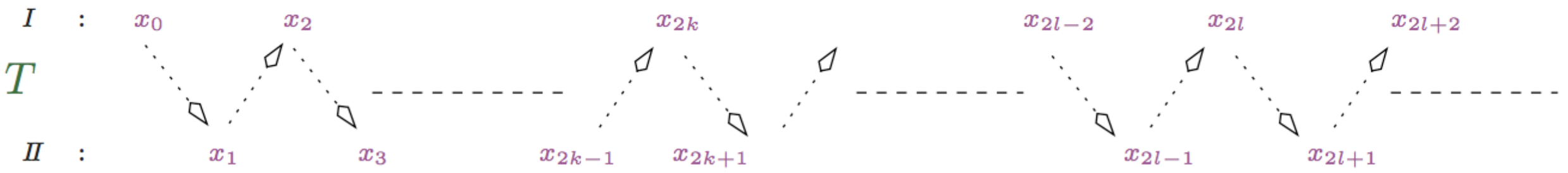
- T a n.e.p. tree, and $A \subseteq [T]$
- $A \in \mathbf{\Sigma}^0_3$, prove $\mathbb{G}(A, T)$ is determined assuming $Det(\mathbf{\Sigma}^0_2)$
- $x \in A \iff \underbrace{\underbrace{\exists i \ \underbrace{\forall j \ \underbrace{x \notin [T_{i,j}]}_{\mathbf{\Sigma}^0_1}}_{\mathbf{\Pi}^0_2}}_{\mathbf{\Sigma}^0_3}}$
- $\ulcorner \ , \ \urcorner$ be a bijection $\mathbb{N} \times \mathbb{N} \longmapsto \mathbb{N}$, and $\llcorner \ \lrcorner$ its inverse

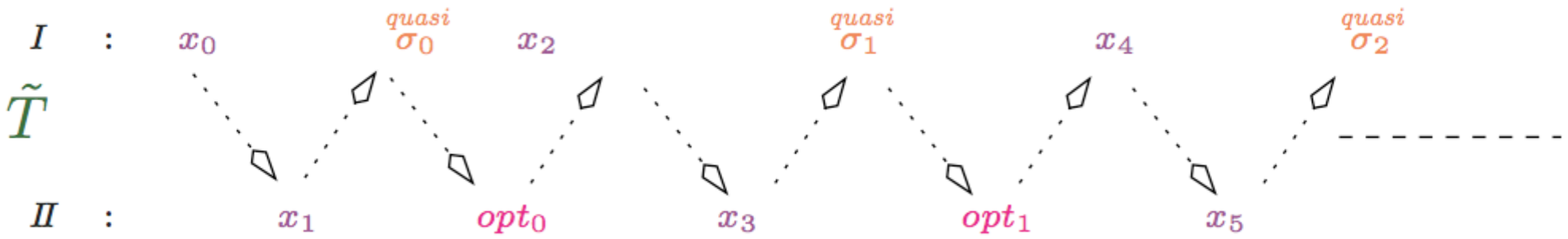
$$\ulcorner i, j \urcorner = n, \quad \llcorner n \lrcorner = (i, j)$$

- Design $\tilde{T}$, and $\tilde{A} \subseteq [\tilde{T}]$ s.t.
  - $\tilde{A} \in \mathbf{\Sigma}_2^0$
  - $\tilde{x} \in \tilde{A}$ yields $x \in A$
  - $\tilde{\sigma}$ w.s. for I in $\mathbb{G}(\tilde{A}, \tilde{T})$ yields σ w.s. for I in $\mathbb{G}(A, T)$
  - $\tilde{\sigma}$ w.s. for II in $\mathbb{G}(\tilde{A}, \tilde{T})$ yields σ w.s. for II in $\mathbb{G}(A, T)$
  - $x \in [\sigma]$ induces $\tilde{x} \in [\tilde{\sigma}]$
- $|\tilde{T}| = 2^{|T|}$

  Moves in $\tilde{T}$ must have the cardinality of sets of moves in T
- $\tilde{x} \in \tilde{A} \in \mathbf{\Sigma}_2^0$, and $x \in A \in \mathbf{\Sigma}_3^0$

  function $x \longmapsto \tilde{x}$ must involve **priority** arguments





UNIL | Université de Lausanne

- ◦ $\overset{quasi}{\sigma_0}$ a I-imposed subgame on $x_0x_1\Sigma^* \cap T$
  - ◦ $\overset{quasi}{\sigma_1}$ a I-imposed subgame on $x_0x_1x_2x_3\Sigma^* \cap T$
  - ◦ $\overset{quasi}{\sigma_2}$ a I-imposed subgame on $x_0x_1x_2x_3x_4x_5\Sigma^* \cap T$

- opt_n is either
  - ◦ ***yes***,
  - ◦ $(\boldsymbol{no}, u_n)$

  with u_n of **odd length** extending $x_0x_1x_2 \dots x_{2n}x_{2n+1}$

Designing $\tilde{T}$ by auxiliary moves

1. If $opt_0 =$ ***yes***

   - $I\!I$ signifies he accepts the restriction from T to $\overset{quasi}{\sigma_0}$
   - $I\!I$ promises x remains inside $T_{\lfloor 0 \rfloor}$

     $\tilde{T}$ is designed so to prevent x from exiting $T_{\lfloor 0 \rfloor}$

   - Notice that $x \in T_{\lfloor 0 \rfloor}$

2. If $opt_0 = (\boldsymbol{no}, u_0)$

   - $I\!I$ signifies he refuses I's offer, and plays $u_0 \in {}^{quasi}\sigma_0 \setminus T_{\lfloor 0 \rfloor}$
   - From now on, both players are restricted to play according to u_0

     for example: if $x_0x_1 = 00$, and $u_0 = 001101$, then

$$x_0x_1x_2x_3x_4x_5 = 001101$$

$\tilde{T}$ is designed so to prevent x from exiting $u_0\Sigma^*$

- Notice that $x \notin T_{\lfloor 0 \rfloor}$

UNIL | Université de Lausanne

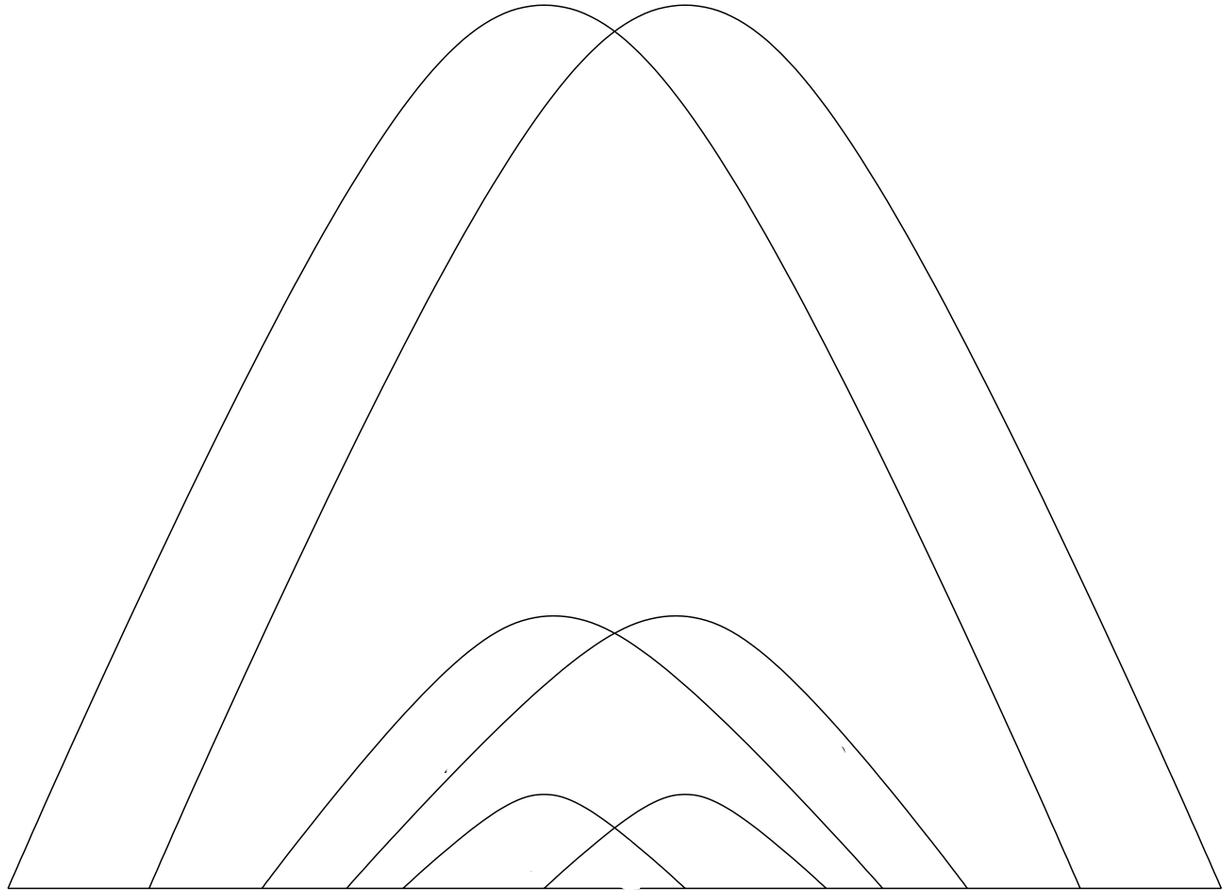
Defining $\tilde{A}$

- $\tilde{x} \in \tilde{A} \iff x \in A$
-

$$\begin{aligned}
\tilde{x} \in \tilde{A} &\iff x \in A \\
&\iff \exists i\ \forall j\ x \notin [T_{i,j}] \\
&\iff \exists i\ \forall j\ opt_{\ulcorner i,j \urcorner} = \textbf{\textit{no}} \\
&\iff \exists i\ \forall j\ \underbrace{\underbrace{\underbrace{\tilde{x}_{4\ulcorner i,j \urcorner+3} = \textbf{\textit{no}}}_{\mathbf{\Delta}_1^0}}_{\mathbf{\Pi}_1^0}}_{\mathbf{\Sigma}_2^0}
\end{aligned}$$

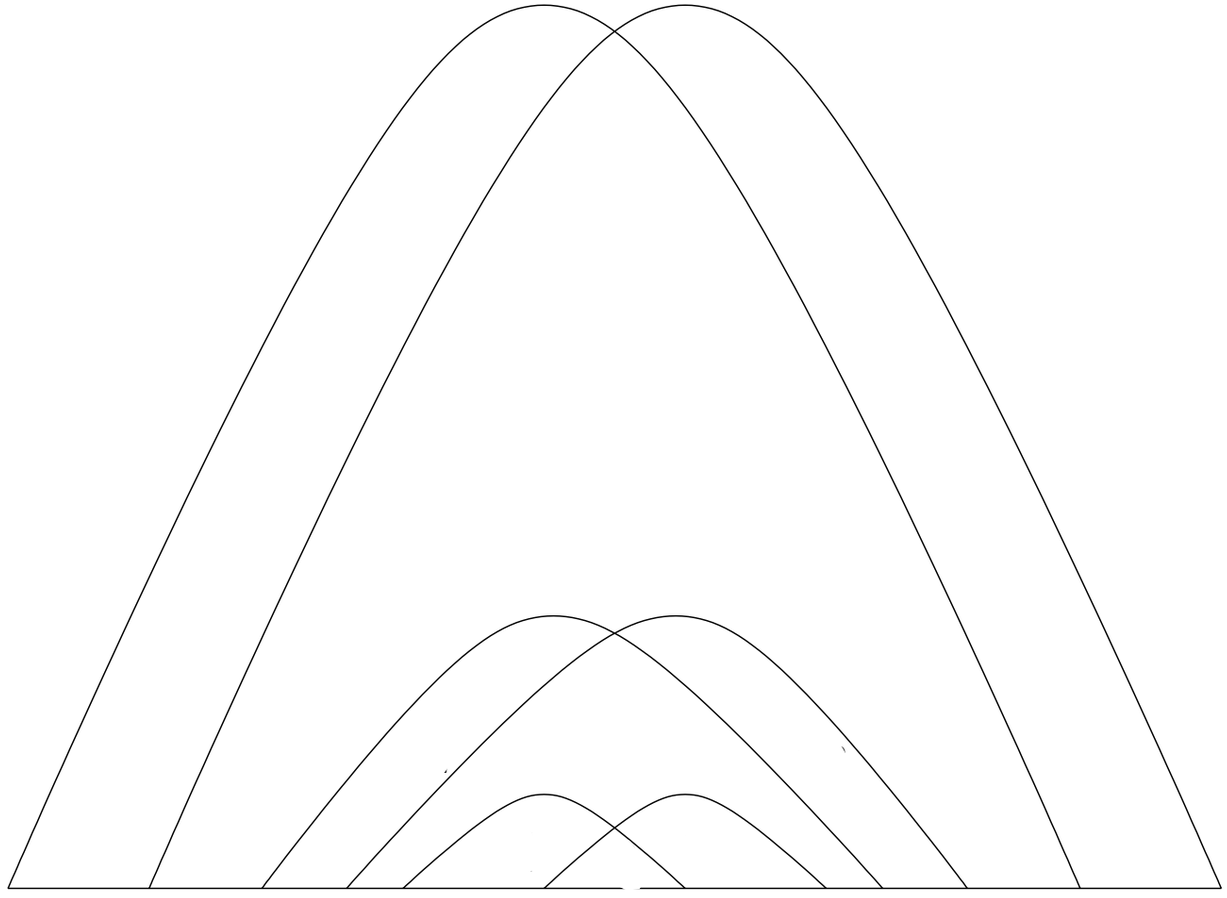
-

$\mathbb{G}(\tilde{A}, \tilde{T})$ is determined



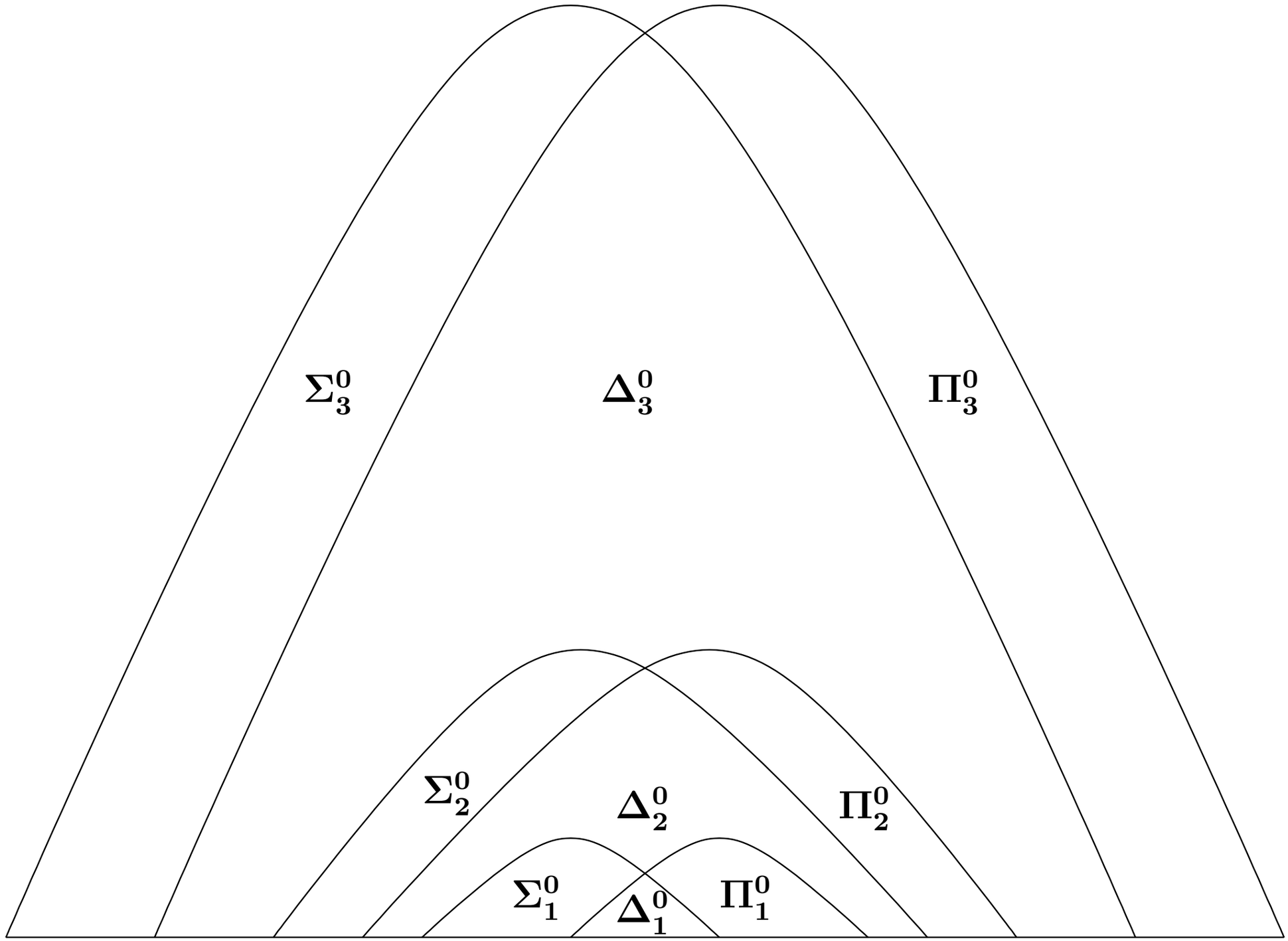
UNIL | Université de Lausanne

Why should we care ?

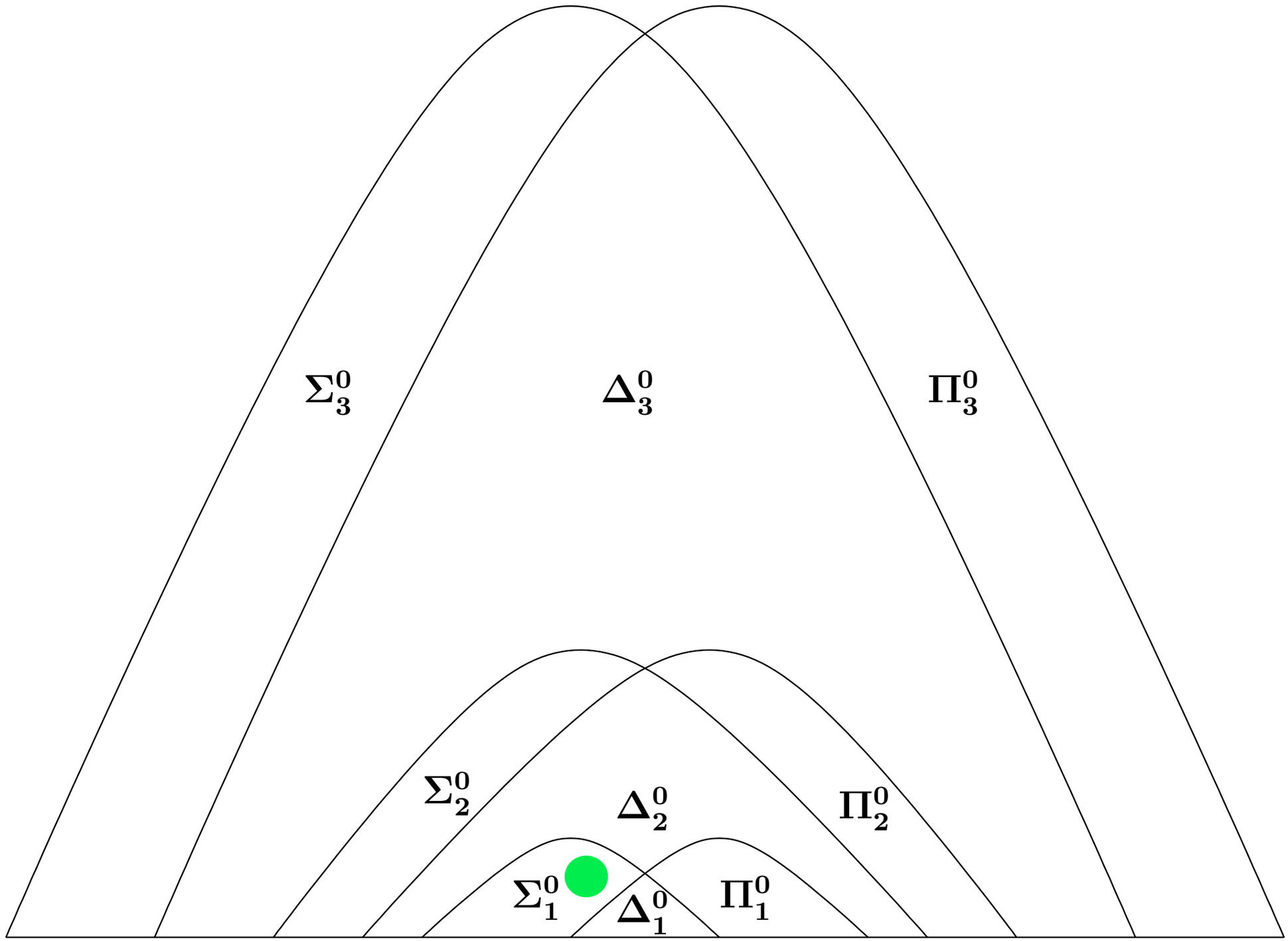


UNIL | Université de Lausanne

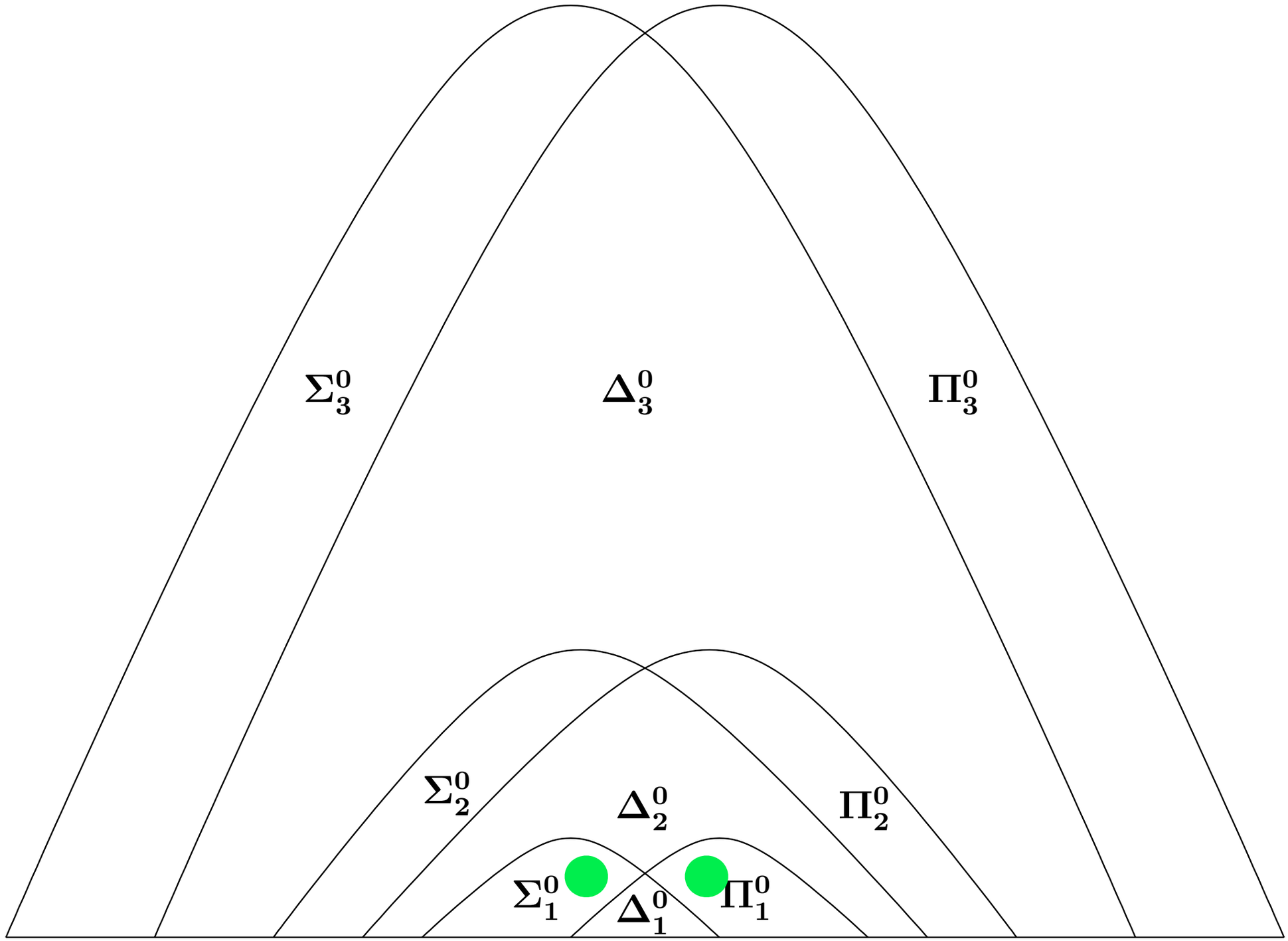
The Borel hierarchy



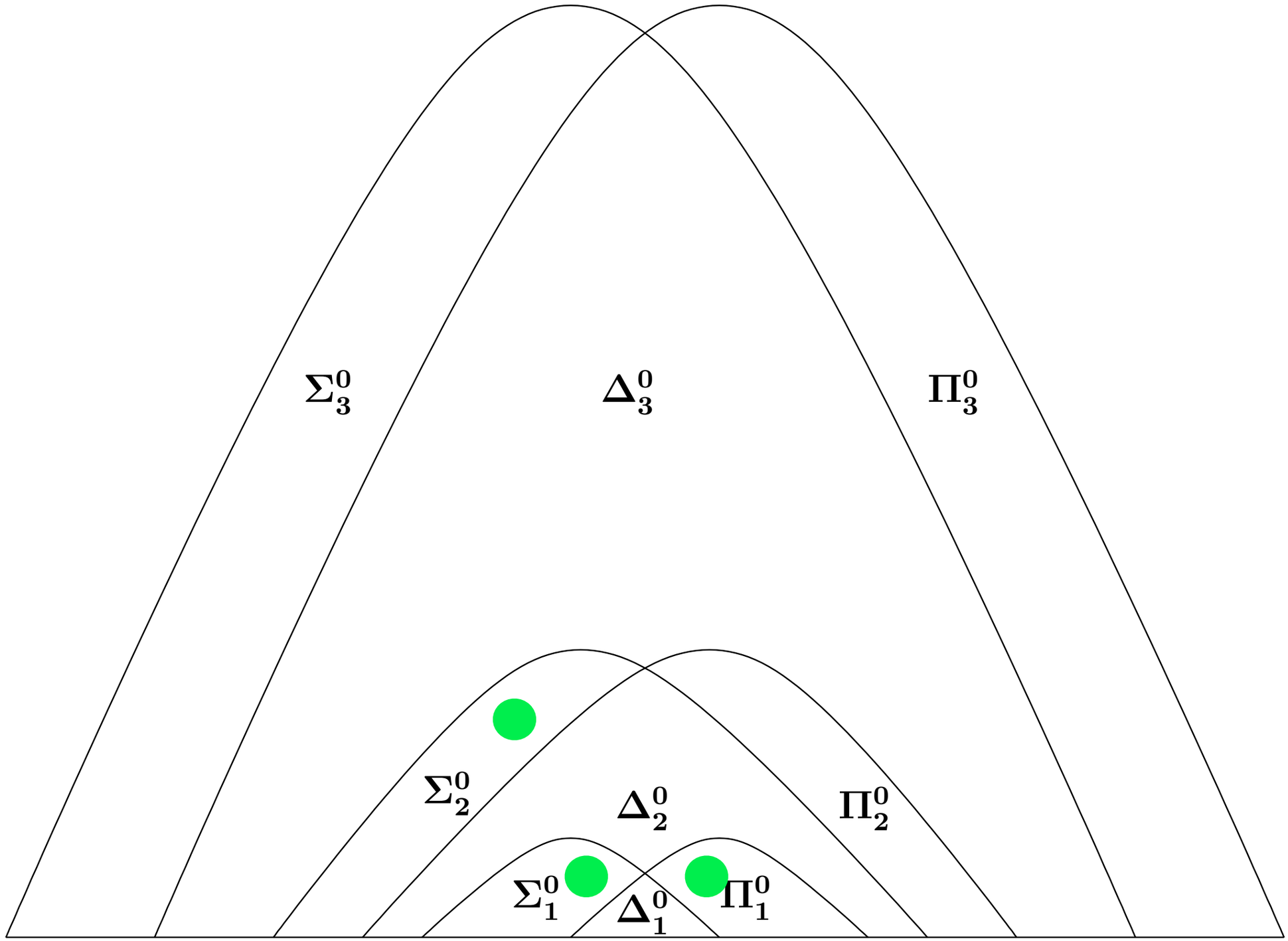
The Borel hierarchy



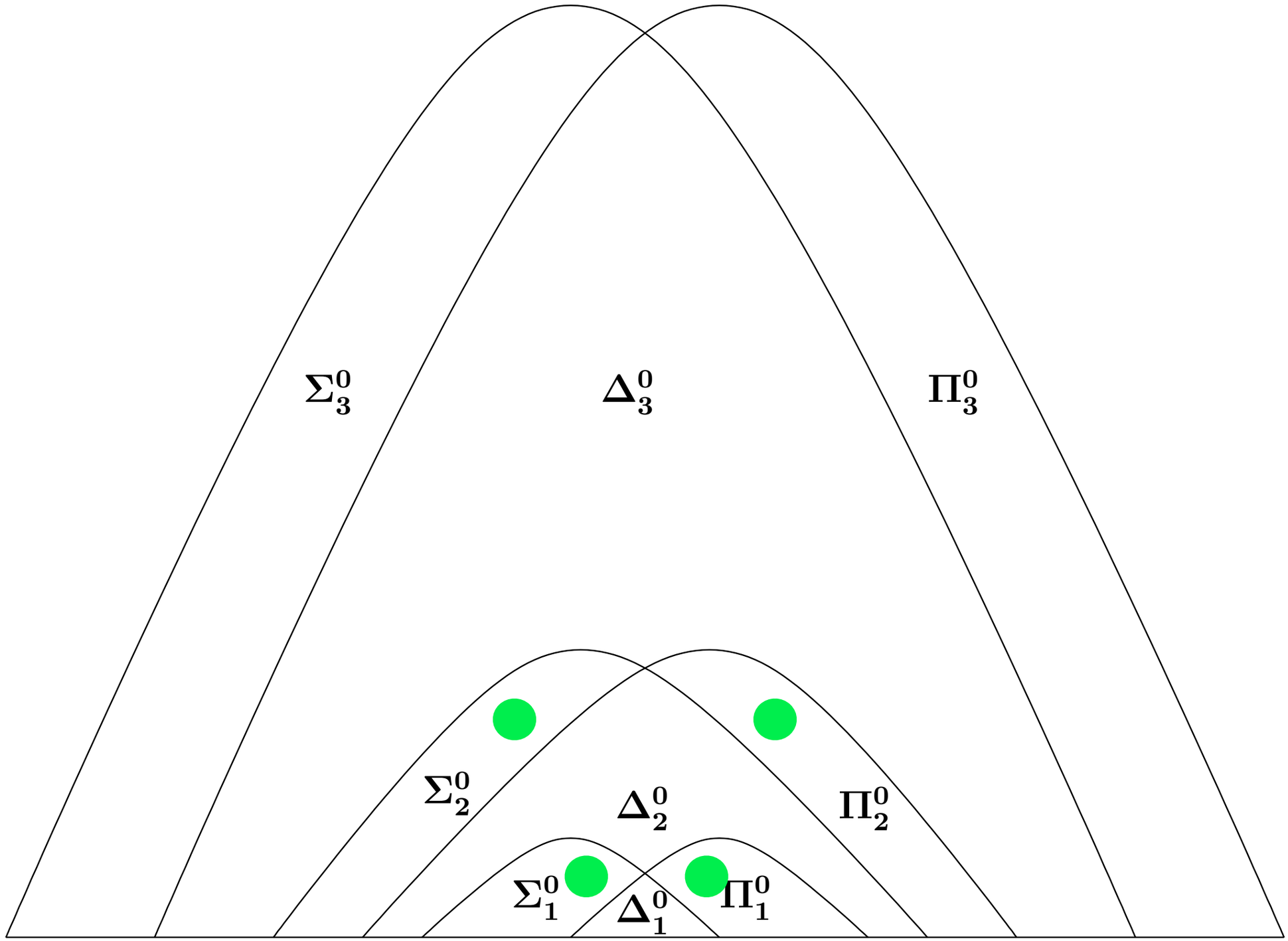
The Borel hierarchy



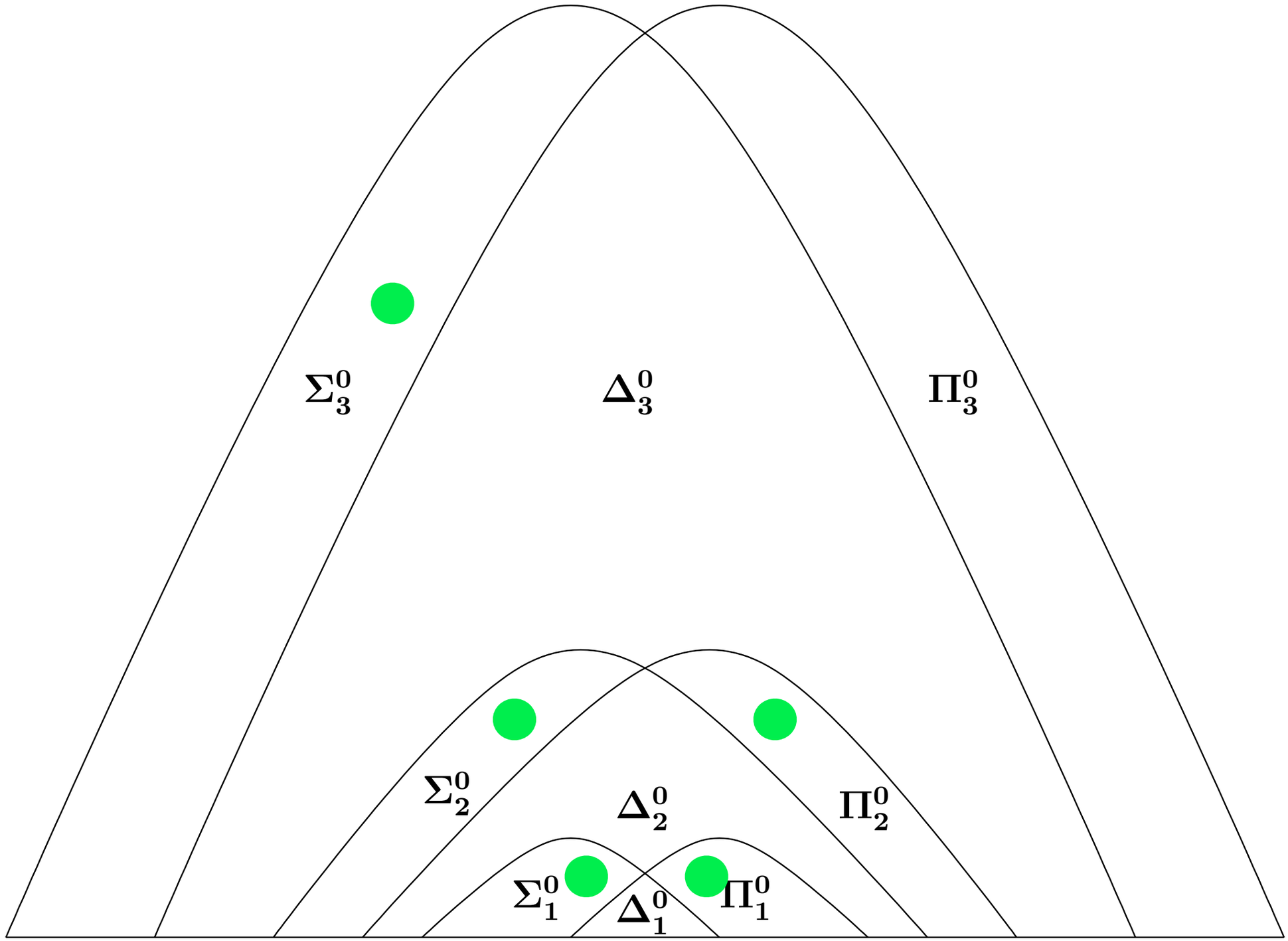
The Borel hierarchy



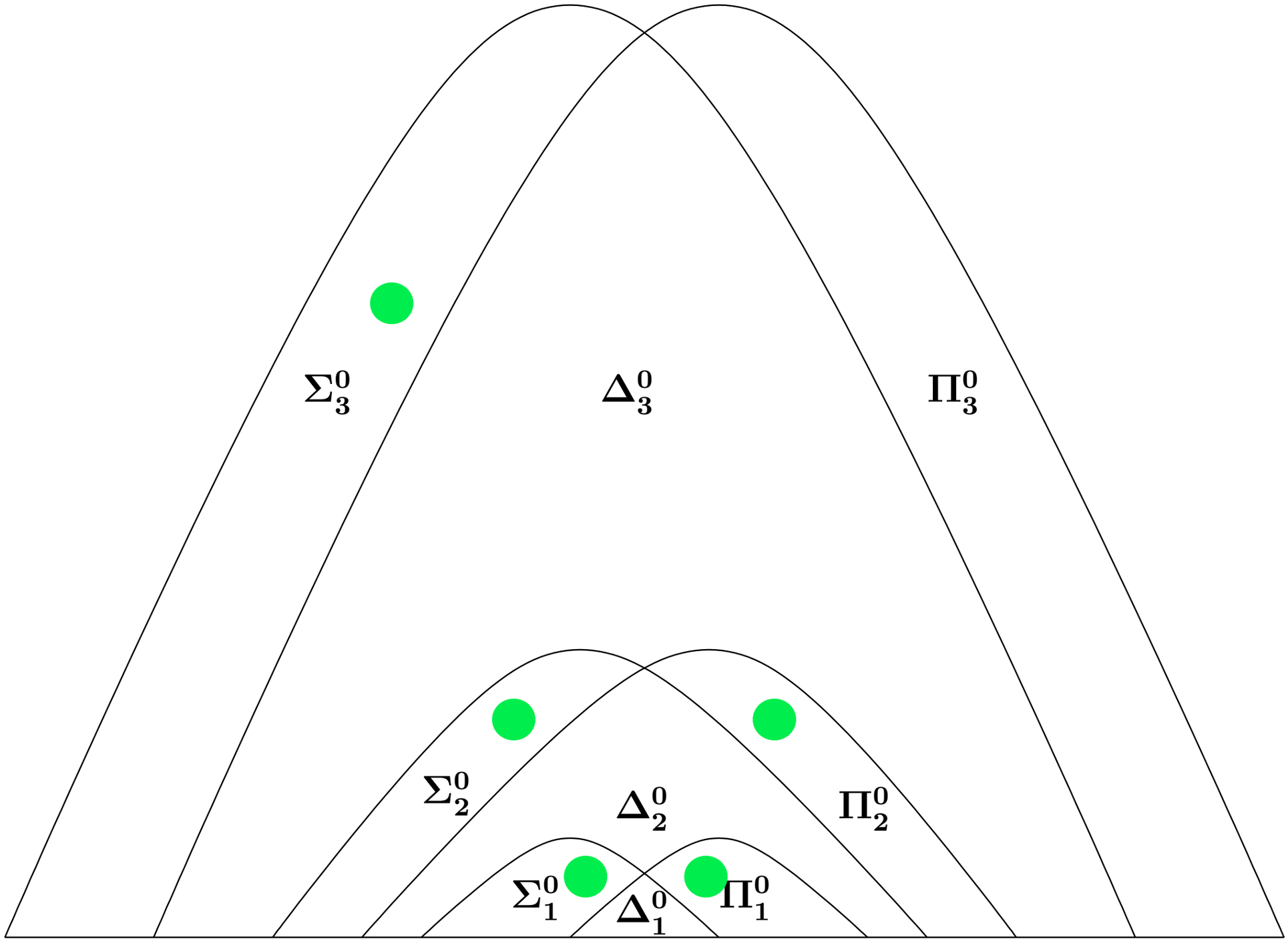
The Borel hierarchy

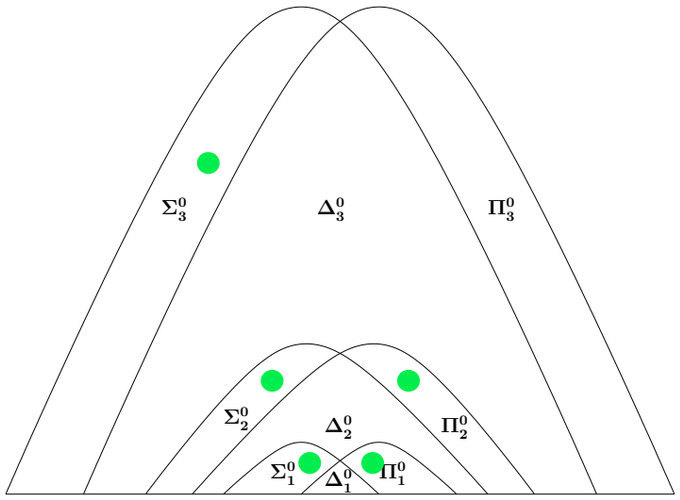


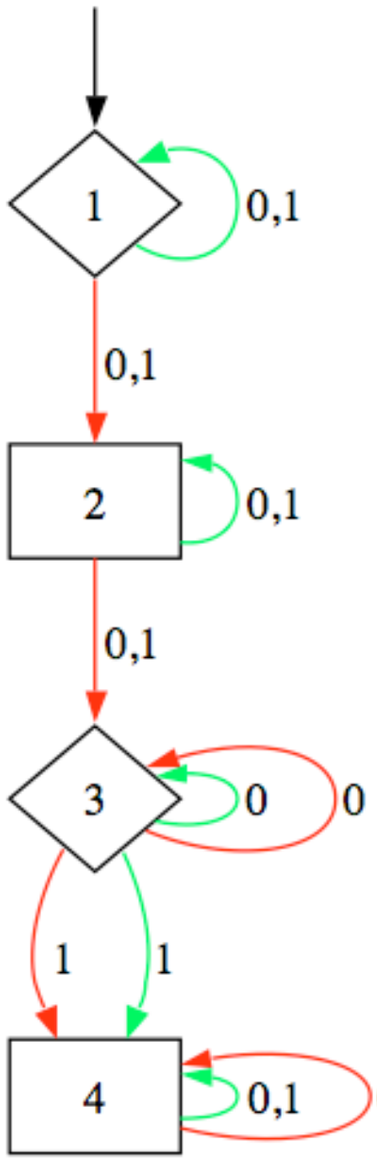
The Borel hierarchy



The Borel hierarchy



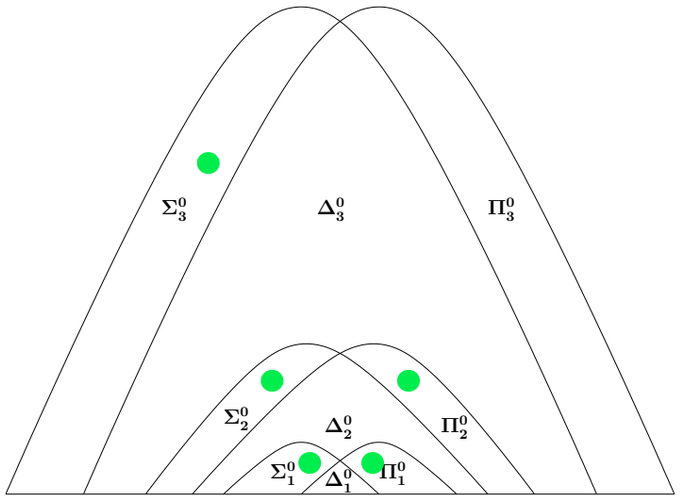


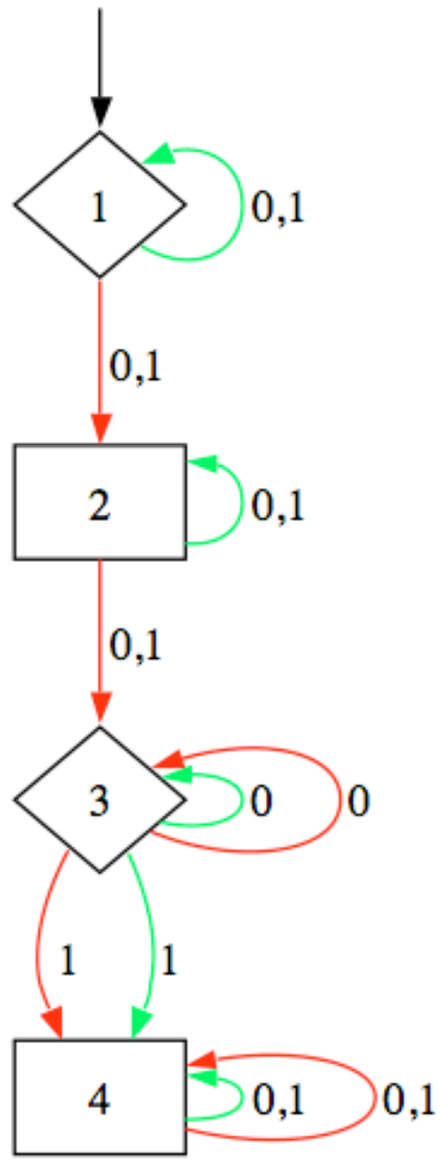


T accepted

$$\Longleftrightarrow$$

$$\exists k\ \forall l\ \exists m \quad \underbrace{\rightarrow\rightarrow \ldots \rightarrow\rightarrow}_{k} \underbrace{\rightarrow\rightarrow \ldots \rightarrow\rightarrow}_{l} \underbrace{\overset{0}{\rightarrow}\overset{0}{\rightarrow} \ldots \overset{0}{\rightarrow}}_{m} \overset{1}{\rightarrow} \quad \in T$$



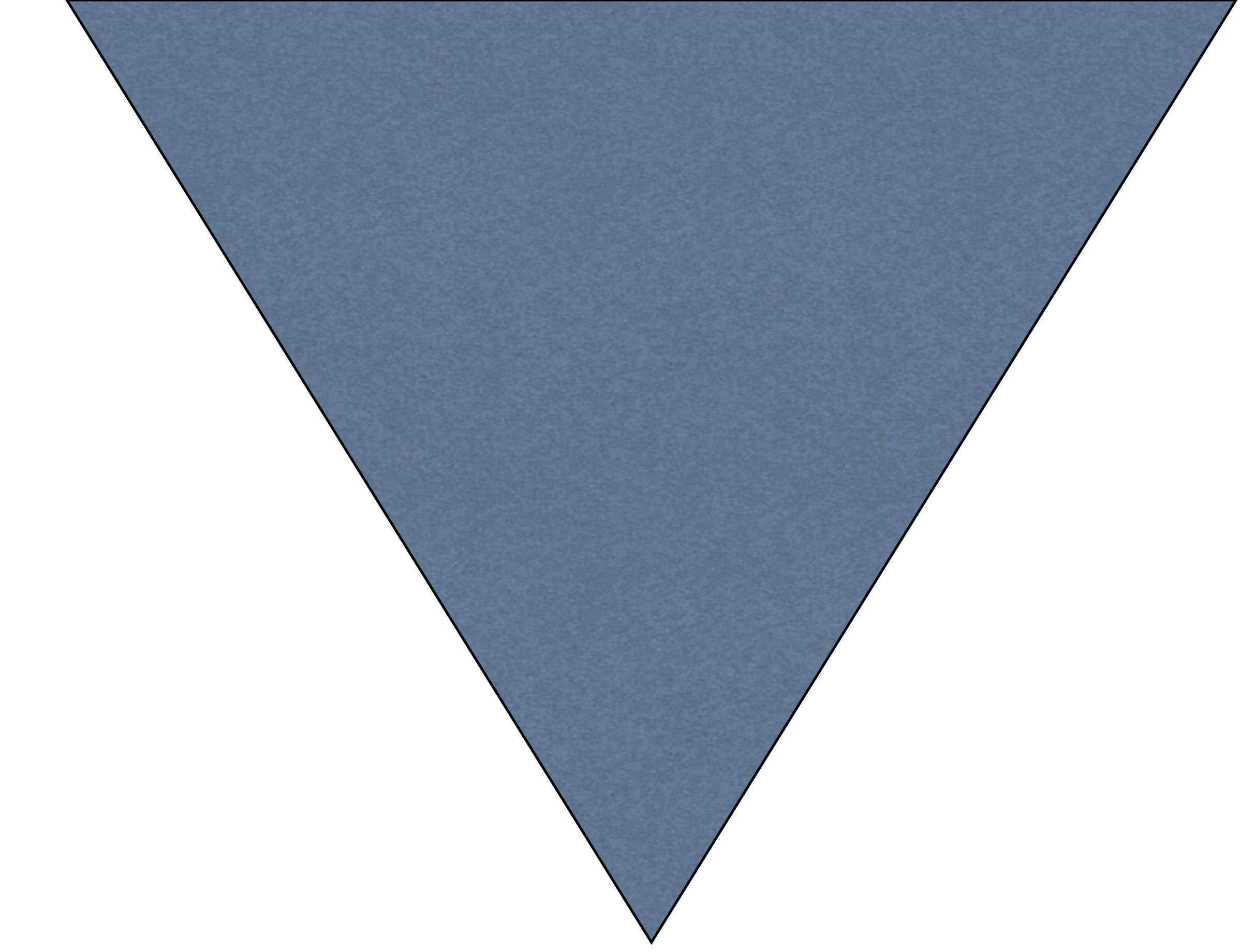


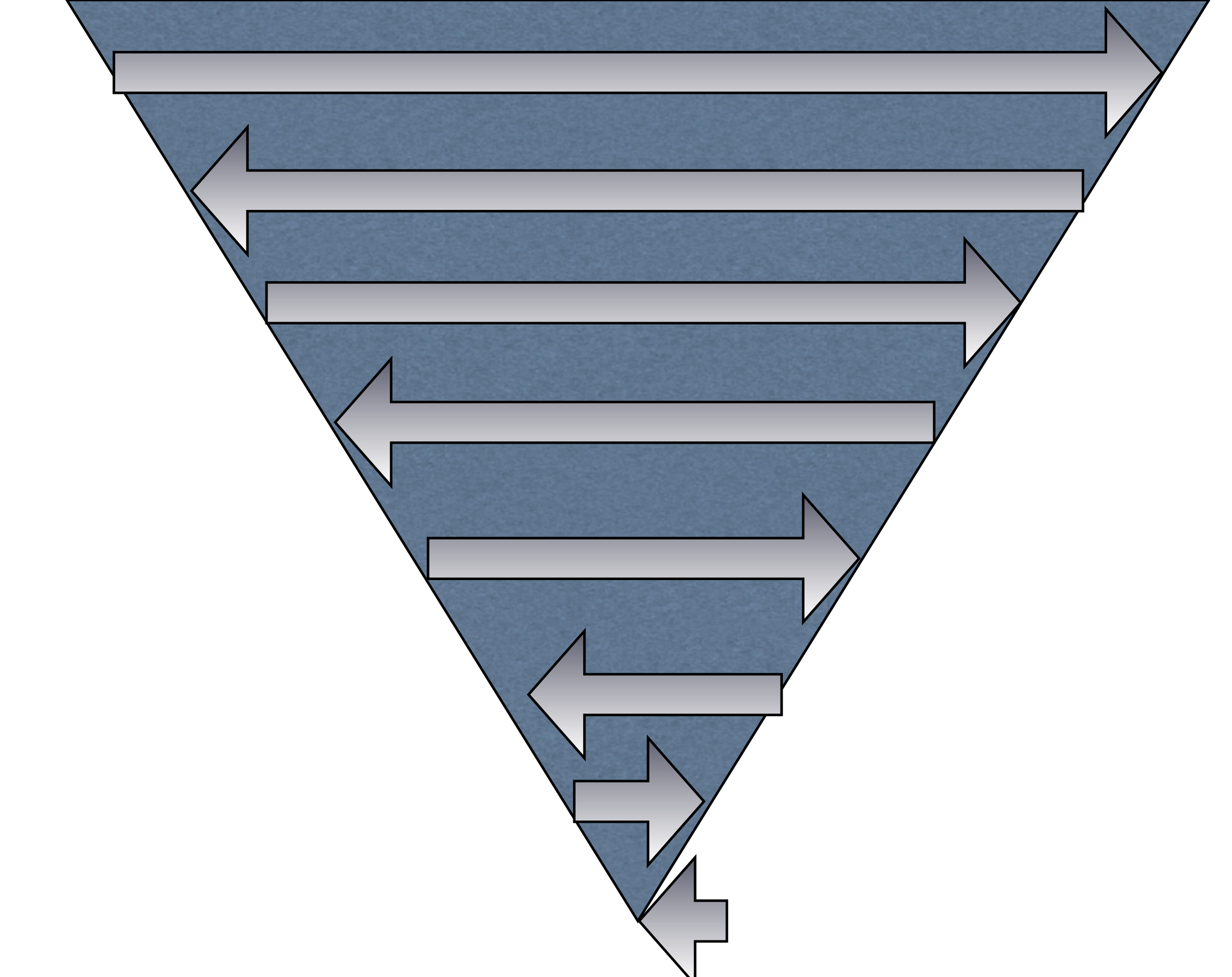
$$\in \boldsymbol{\Sigma}^0_3$$

T accepted

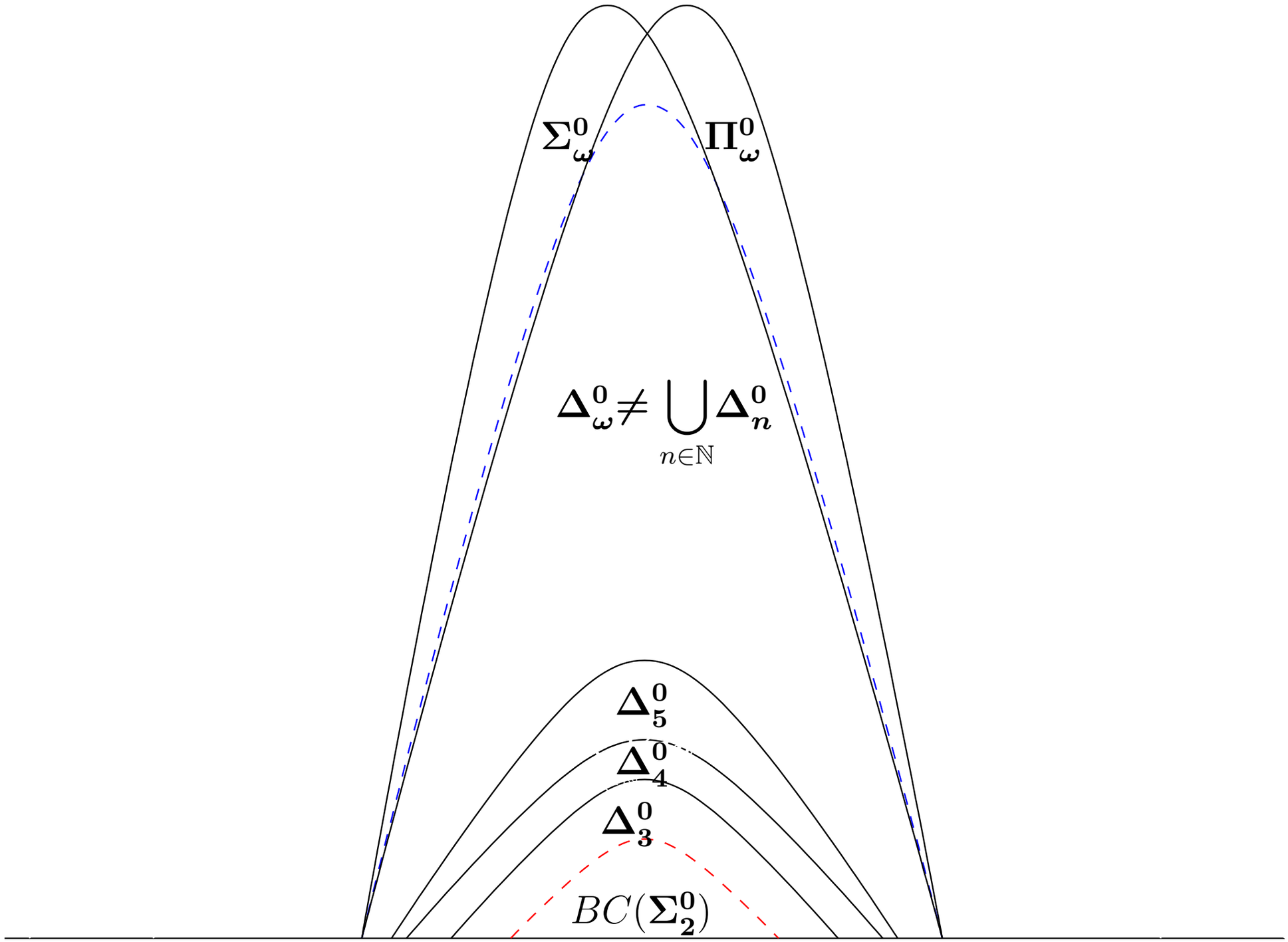
$$\Longleftrightarrow$$

$$\exists k\ \forall l\ \exists m \quad \underbrace{\rightarrow\rightarrow \ldots \rightarrow\rightarrow}_{k} \underbrace{\rightarrow\rightarrow \ldots \rightarrow\rightarrow}_{l} \underbrace{\overset{0}{\rightarrow}\overset{0}{\rightarrow} \ldots \overset{0}{\rightarrow}}_{m} \overset{1}{\rightarrow} \quad \in T$$

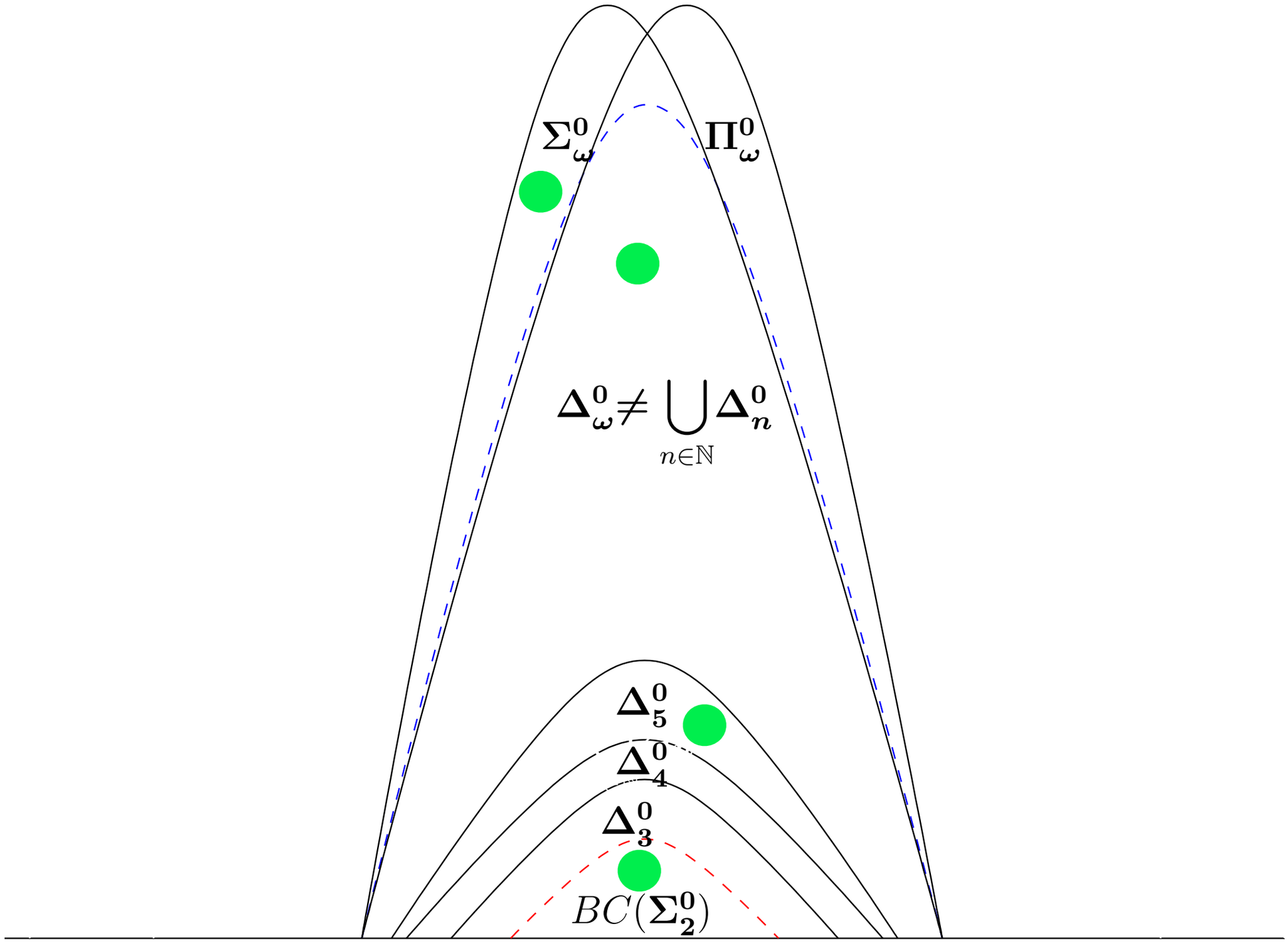




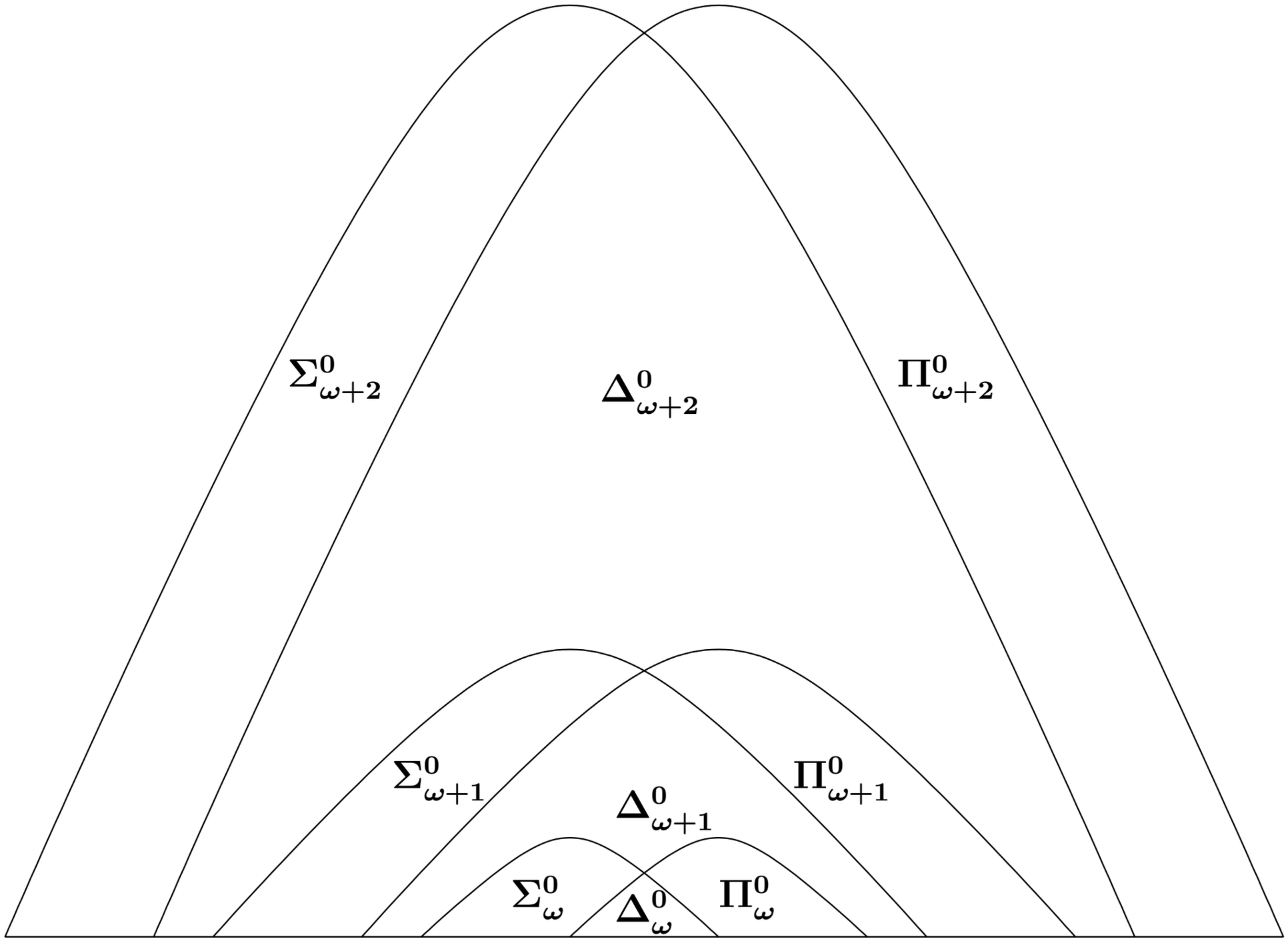
The Borel hierarchy



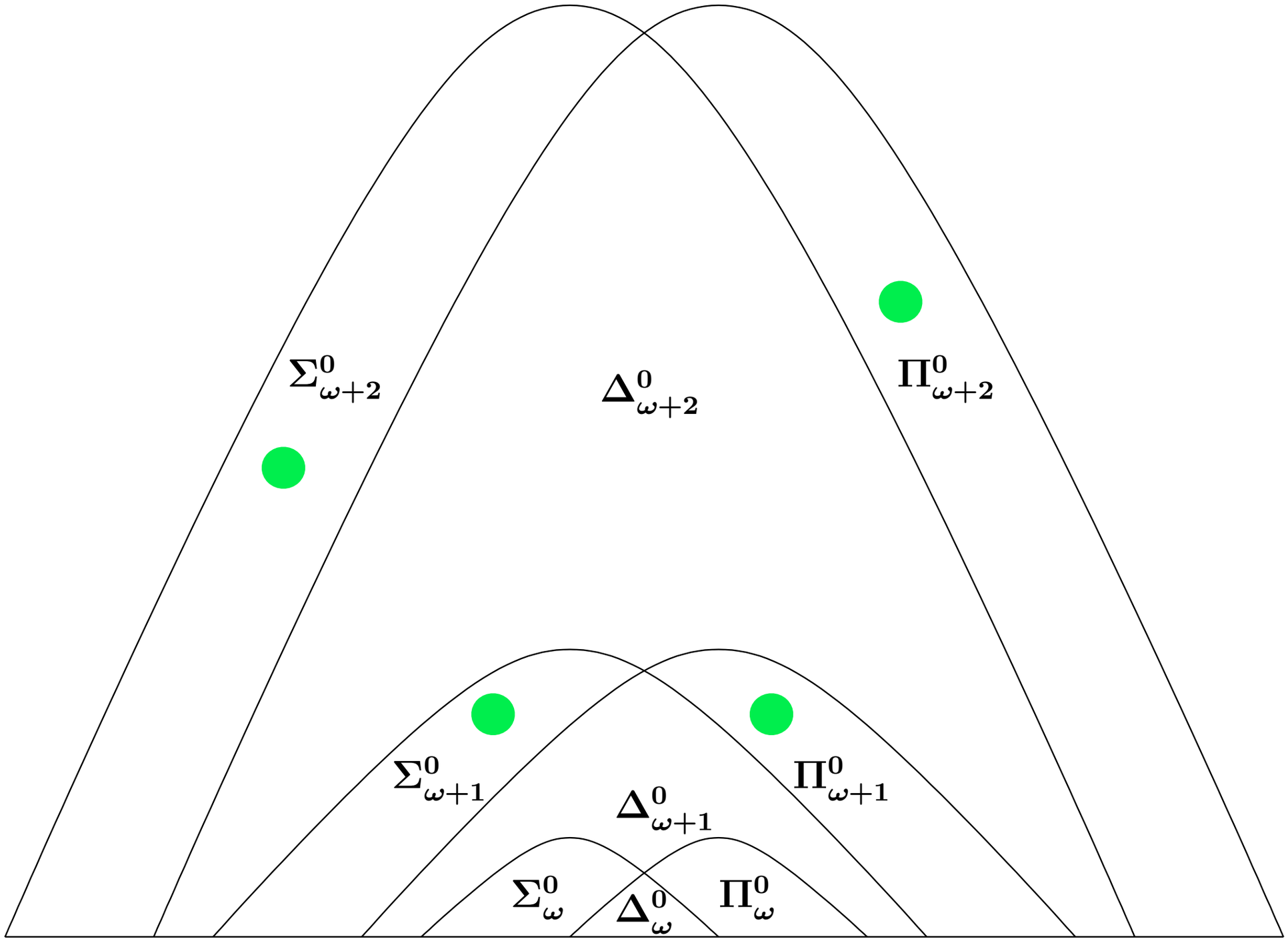
The Borel hierarchy



The Borel hierarchy



The Borel hierarchy





Variations on the Gale-Stewart game.....

Gale Stewart

$$\mathcal{G}(A)$$

A: any set of infinite words

$$\mathcal{G}(A)$$

I II

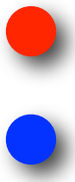
$\mathcal{G}(A)$

I II



$\mathcal{G}(A)$

I II



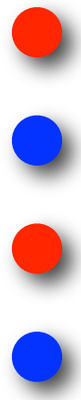
$\mathcal{G}(A)$

I II



$\mathcal{G}(A)$

I II



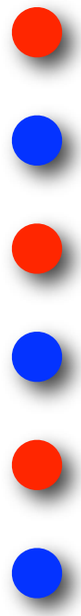
$\mathcal{G}(A)$

I II



$\mathcal{G}(A)$

I II



$\mathcal{G}(A)$

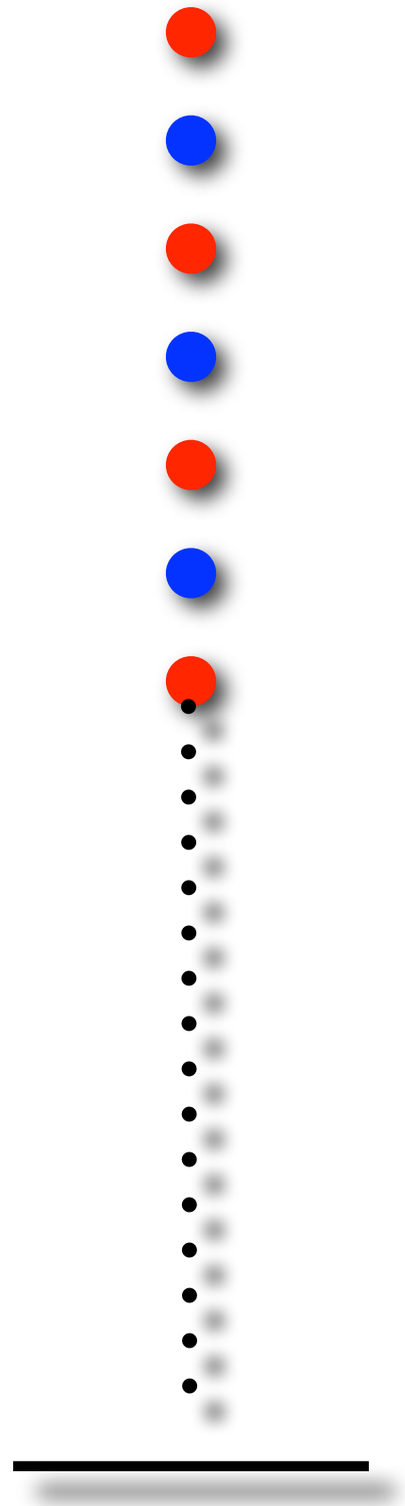
I II



I wins $\iff x \in A$

$\mathcal{G}(A)$

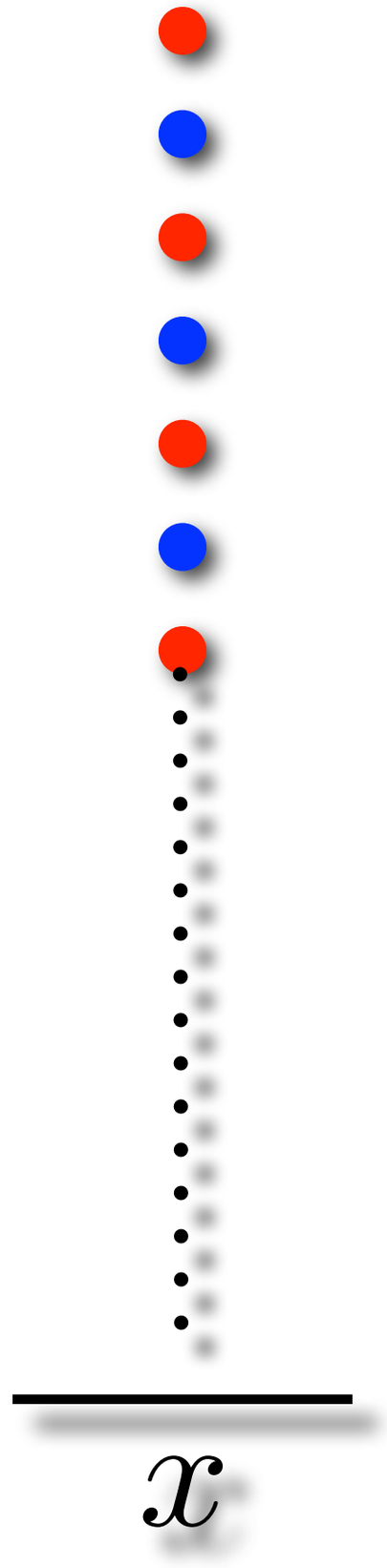
I II



I wins $\iff x \in A$

$\mathcal{G}(A)$

I II



x

I wins $\iff x \in A$

Banach Mazur



$BM(A)$

A: any set of infinite words

$$\mathcal{BM}(A)$$

I wins $\iff x \in A$

$$\mathcal{BM}(A)$$

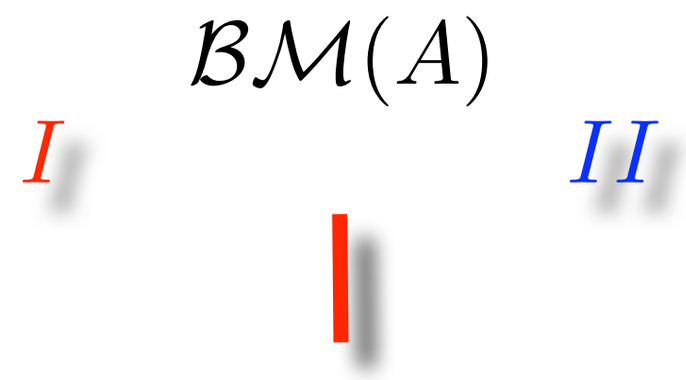
I

I wins $\iff x \in A$

$$\mathcal{BM}(A)$$

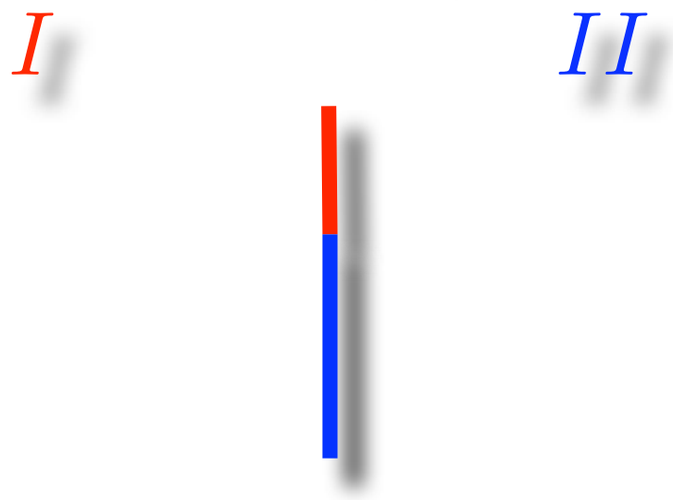
I $\qquad\qquad\qquad\qquad$ II

I wins $\iff x \in A$



I wins $\iff x \in A$

$\mathcal{BM}(A)$



I wins $\iff x \in A$

$\mathcal{BM}(A)$

I II



I wins $\iff x \in A$

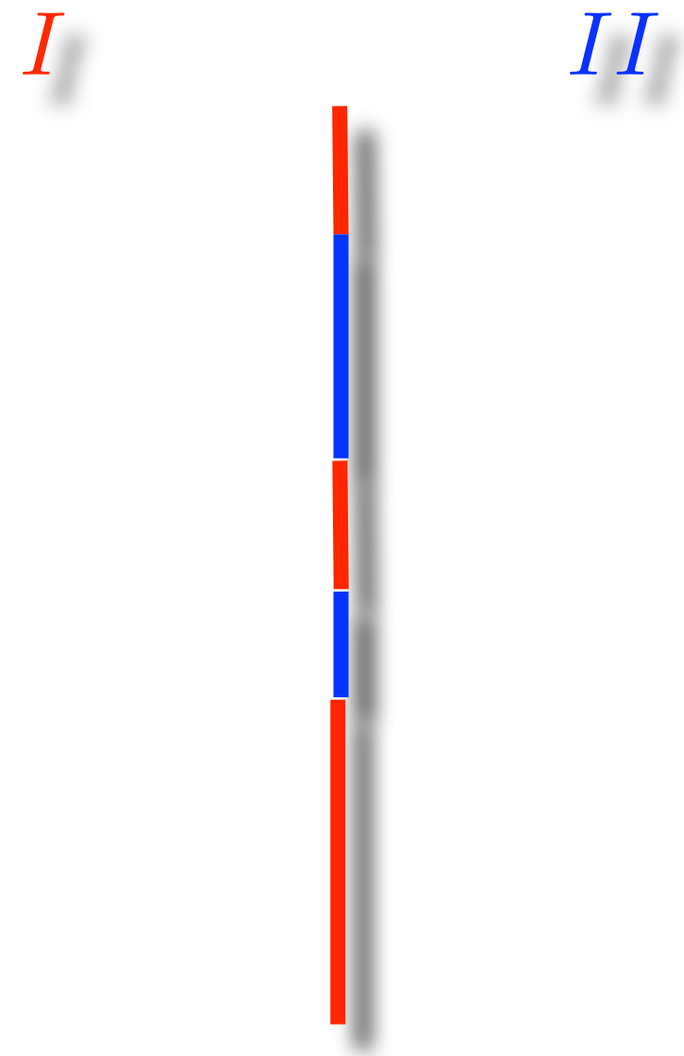
$\mathcal{BM}(A)$

I II



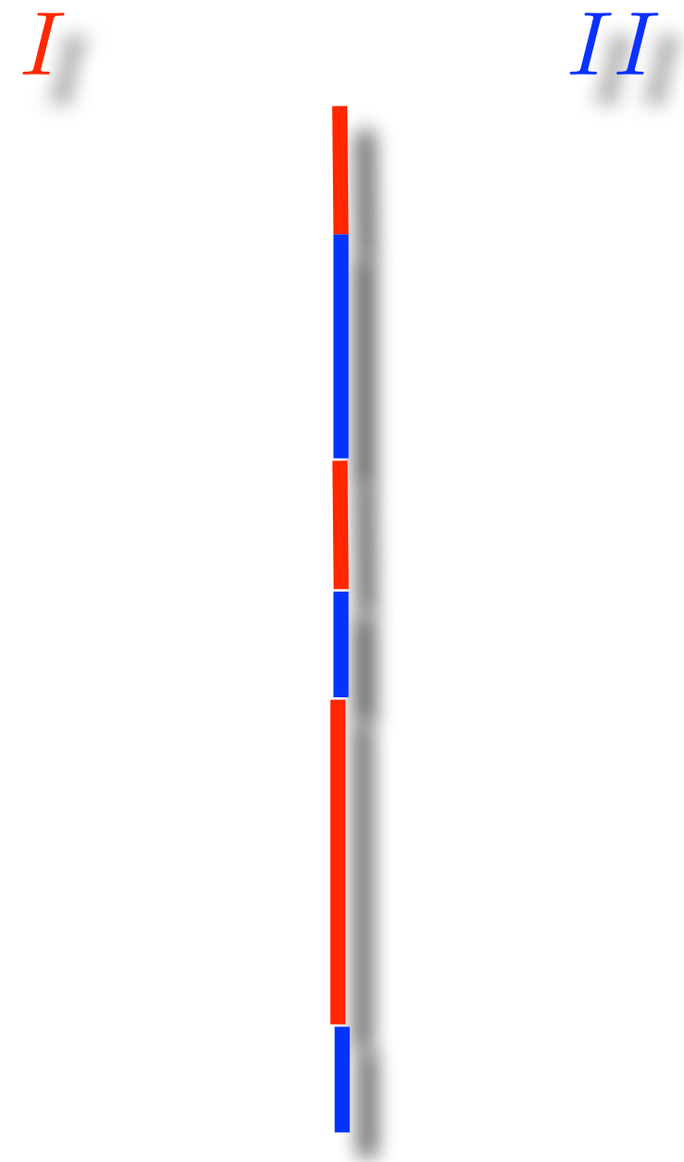
I wins $\iff x \in A$

$\mathcal{BM}(A)$



I wins $\iff x \in A$

$\mathcal{BM}(A)$



I wins $\iff x \in A$

$\mathcal{BM}(A)$

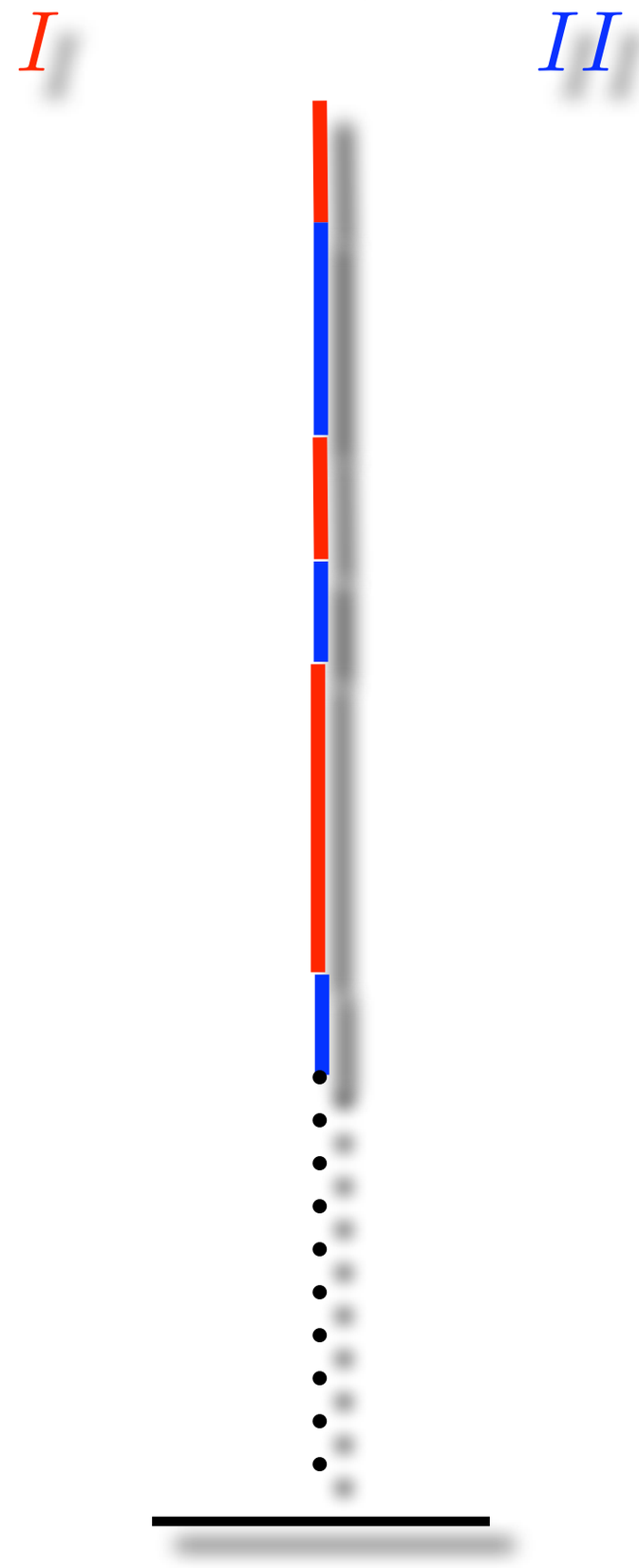
I $\quad\quad$ II



I wins $\iff x \in A$

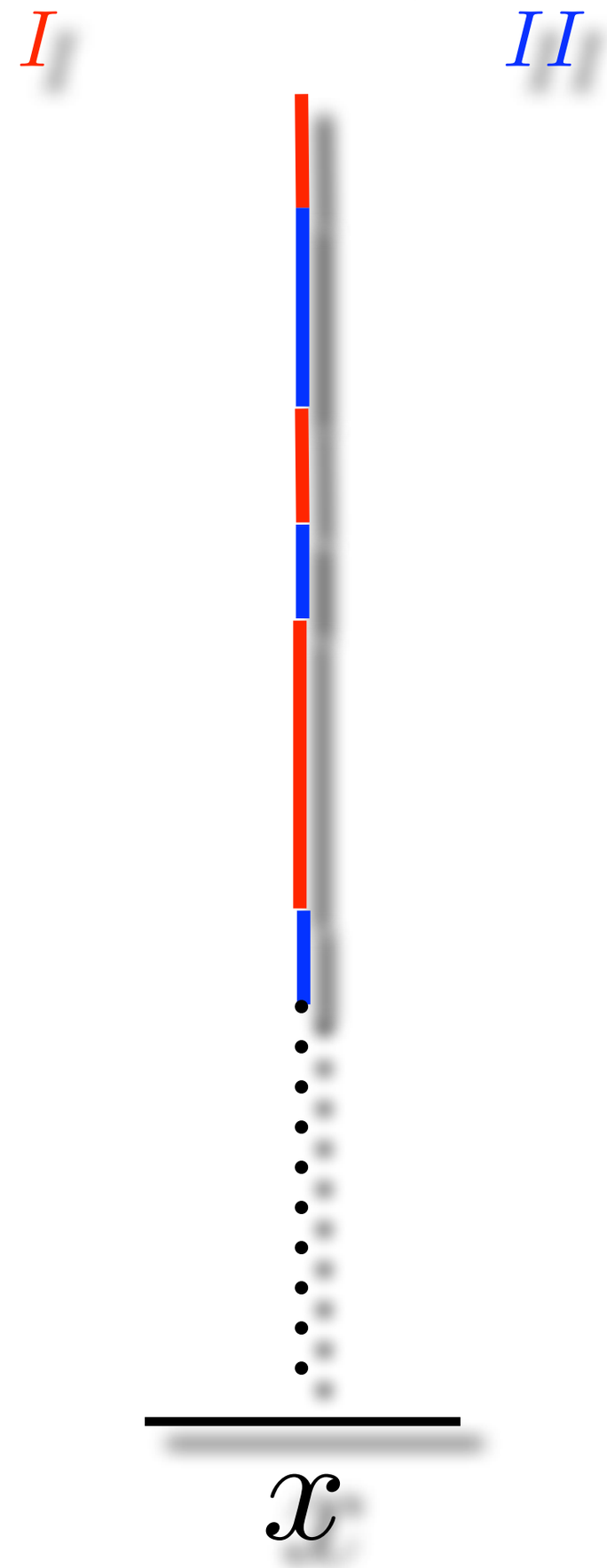
$\mathcal{BM}(A)$

I wins $\iff x \in A$



$\mathcal{BM}(A)$

I wins $\iff x \in A$



Perfect Set

$$\mathcal{PS}(A)$$

A: any set of infinite words

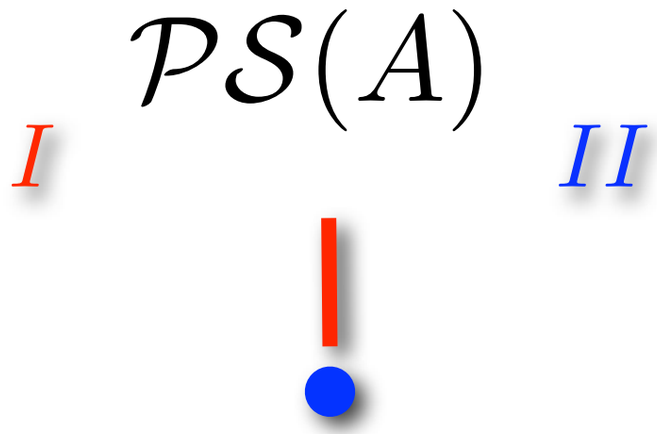
$$\mathcal{PS}(A)$$

I II

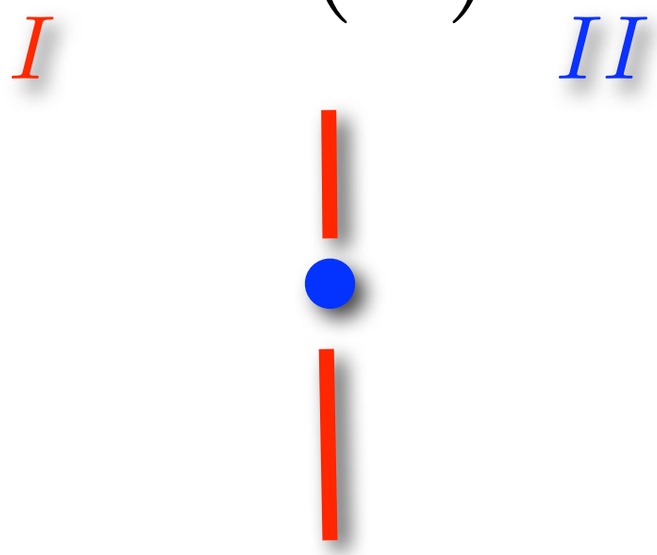
$$\mathcal{PS}(A)$$

I II



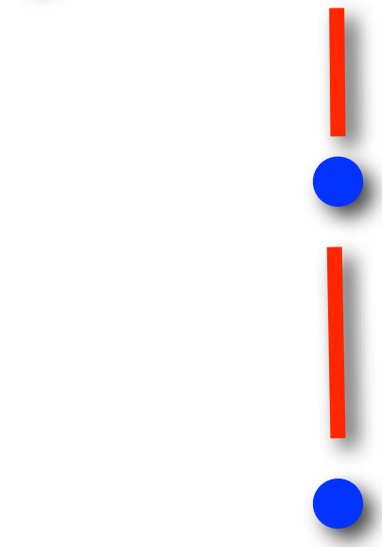

$\mathcal{PS}(A)$
I
II

$\mathcal{PS}(A)$

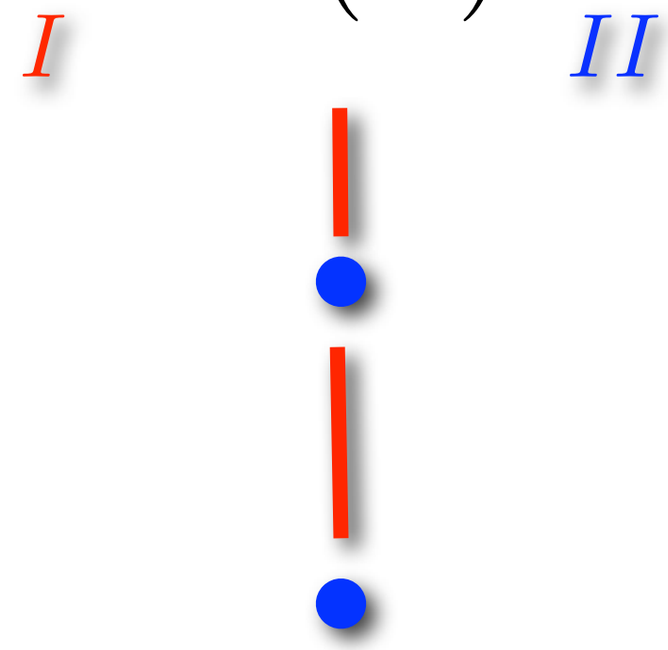


$\mathcal{PS}(A)$

I II



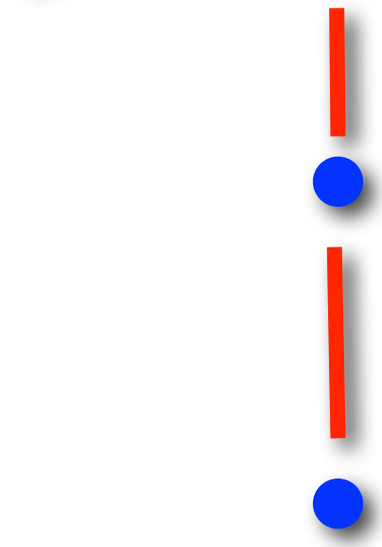
$$\mathcal{PS}(A)$$



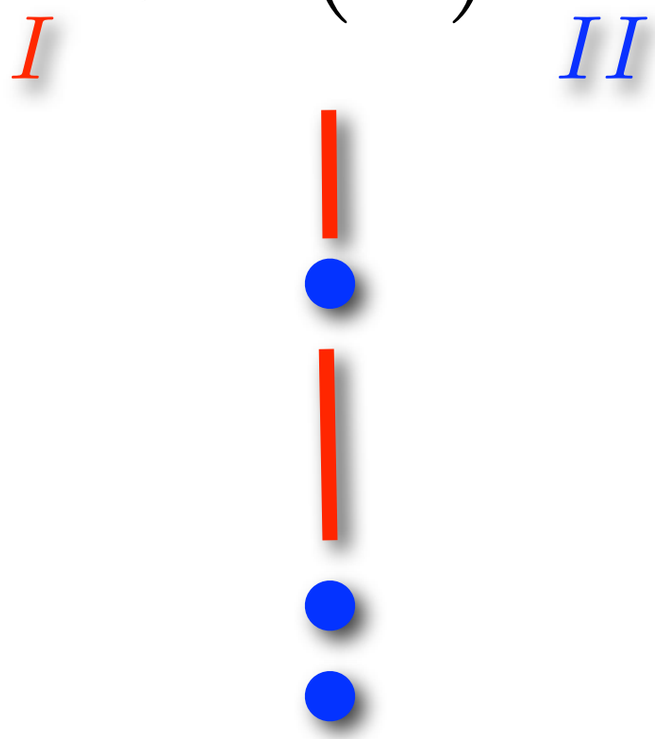
the empty sequence !

$\mathcal{PS}(A)$

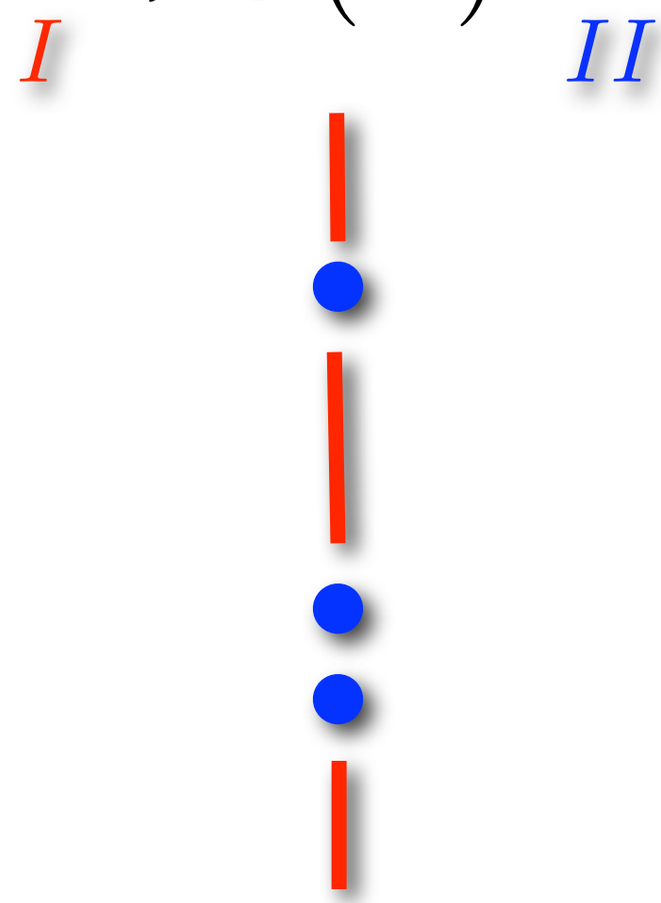
I II



$$\mathcal{PS}(A)$$

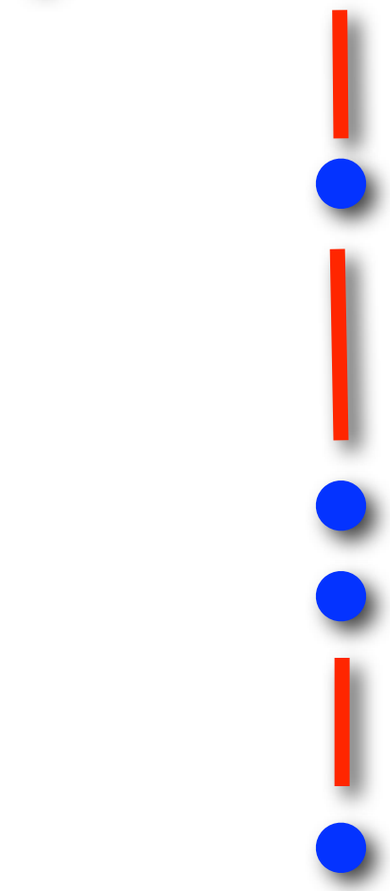


$\mathcal{PS}(A)$

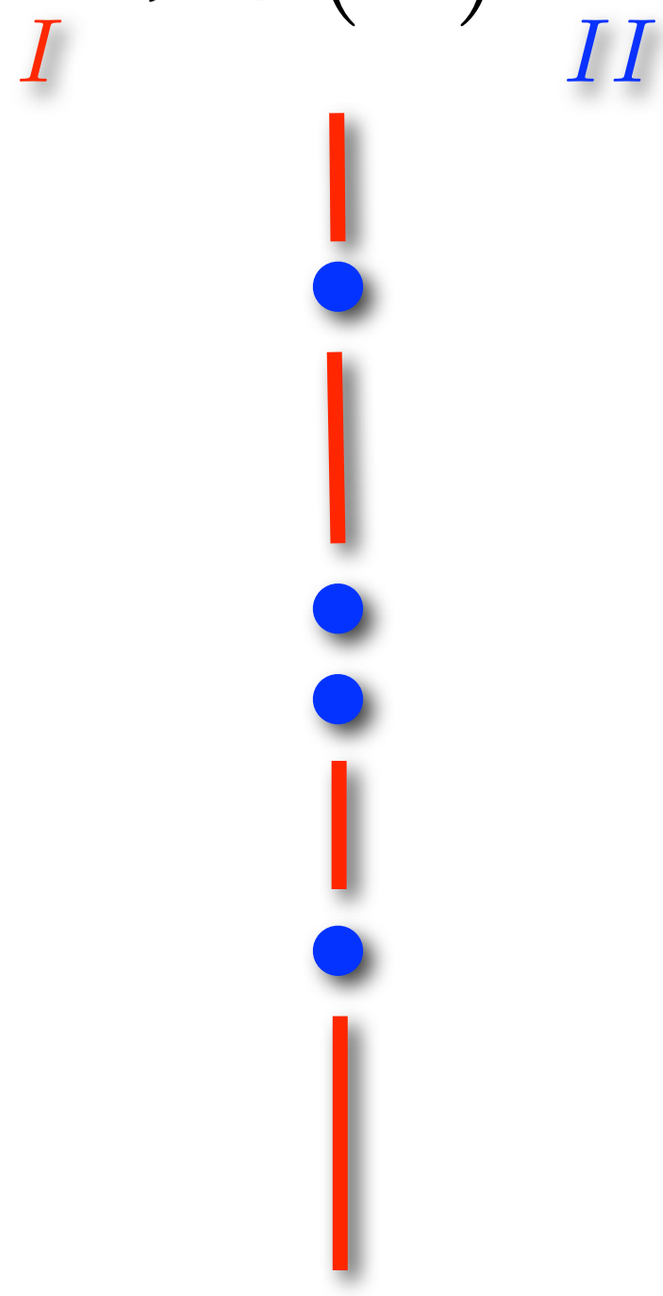


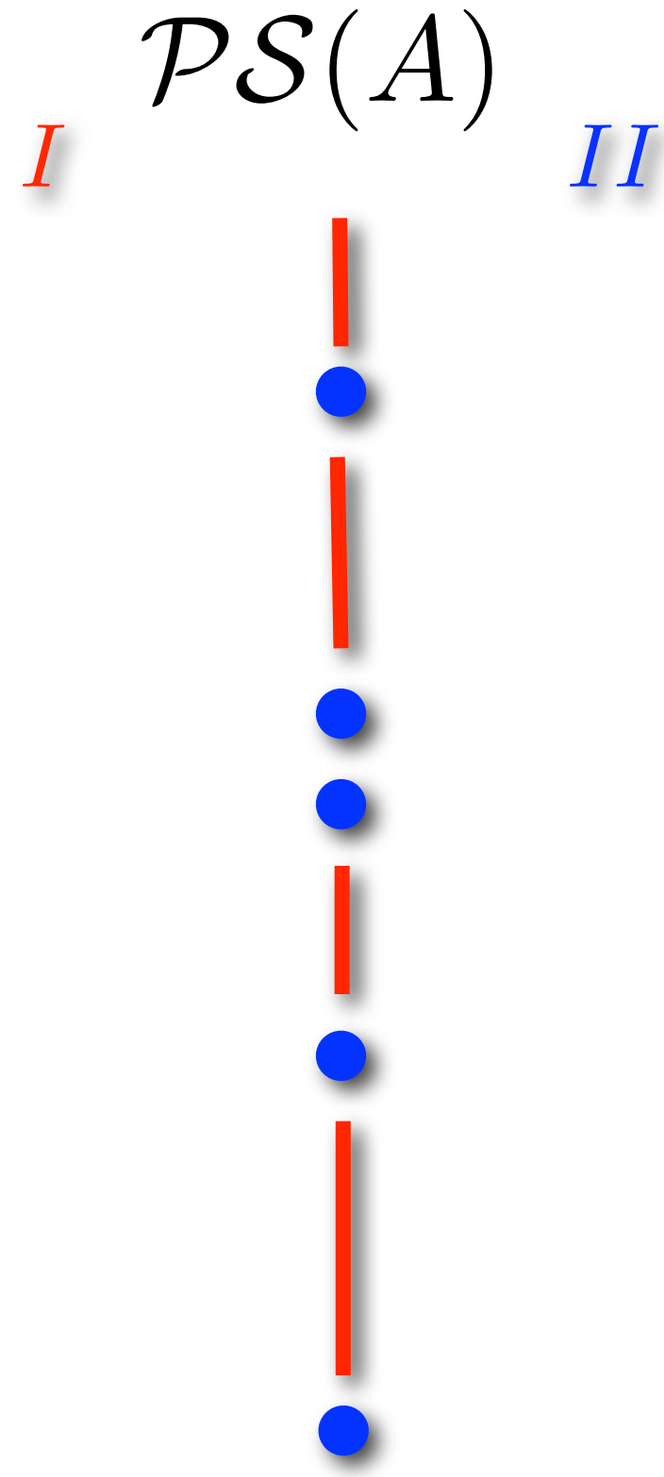
$\mathcal{PS}(A)$

I II

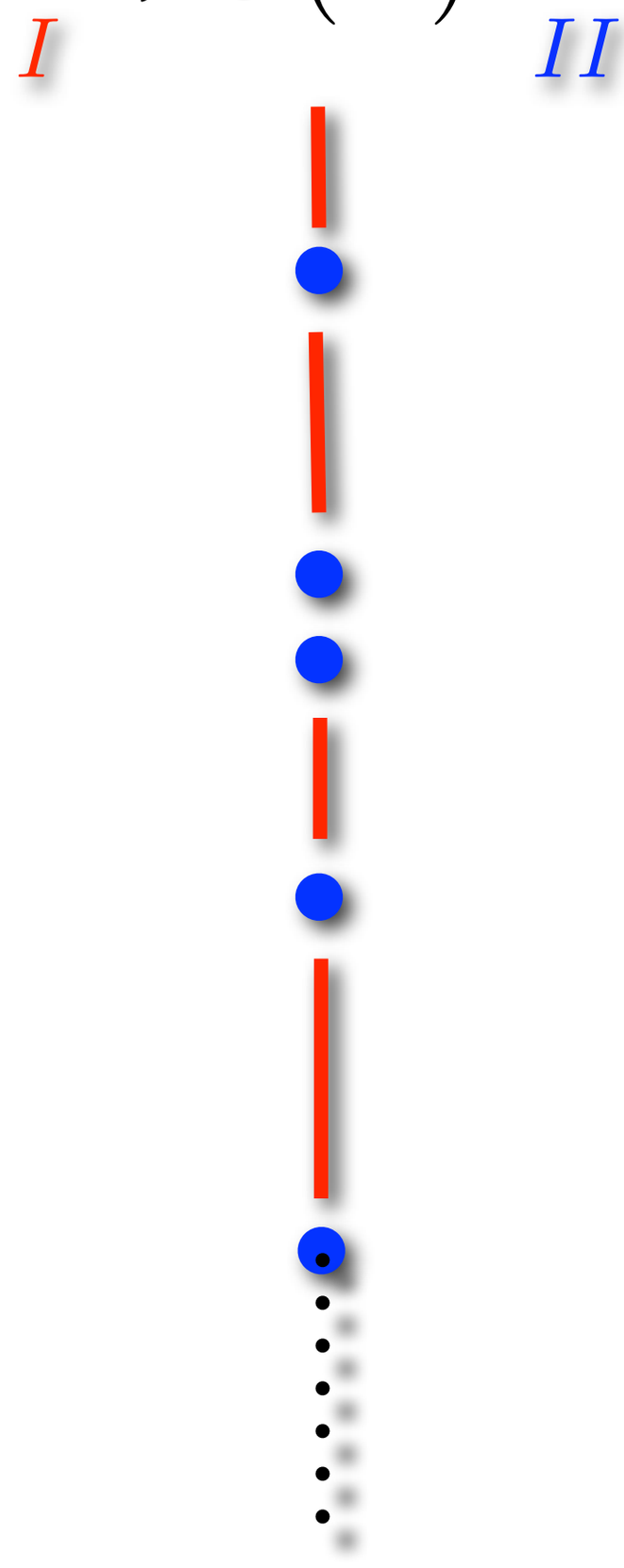


$\mathcal{PS}(A)$




$\mathcal{PS}(A)$
I
II

$$\mathcal{PS}(A)$$



$\mathcal{PS}(A)$

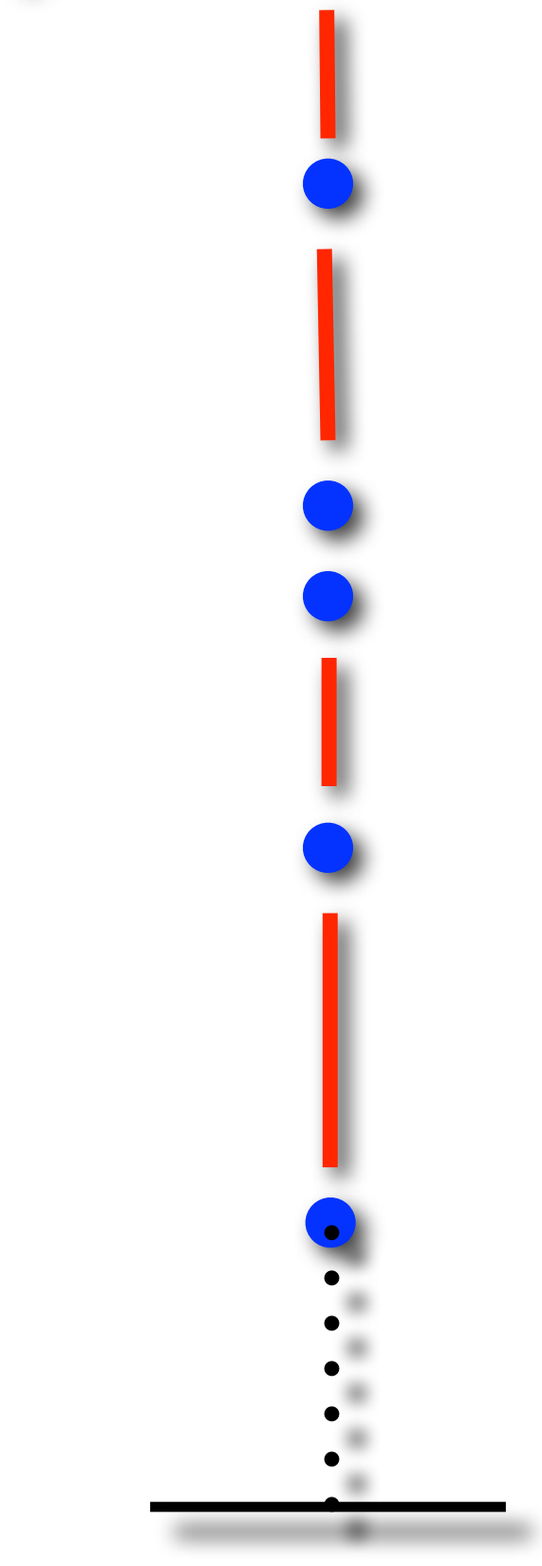
I II



I wins $\iff x \in A$

$\mathcal{PS}(A)$

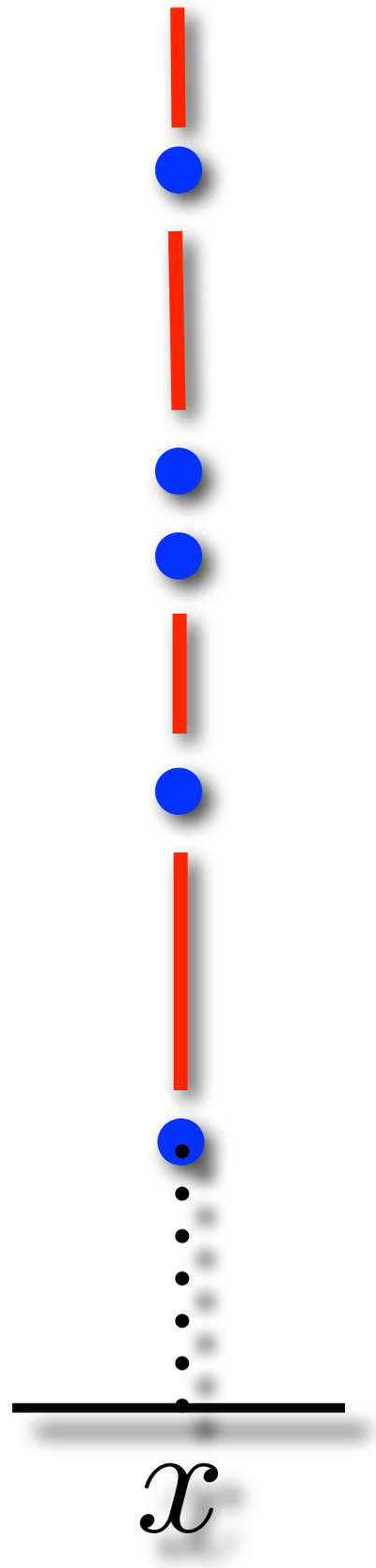
I II



I wins $\iff x \in A$

$\mathcal{PS}(A)$

I $\qquad\qquad$ II



I wins $\iff x \in A$

x

$$Det \;\; \mathcal{G}(A) \Longrightarrow \begin{cases} Det \;\; \mathcal{PS}(A) \\ \qquad\qquad \& \\ Det \;\; \mathcal{BM}(A) \end{cases}$$

$$Det \;\; \mathcal{G}(A) \not\Longleftarrow \begin{cases} Det \;\; \mathcal{PS}(A) \\ \qquad\qquad \& \\ Det \;\; \mathcal{BM}(A) \end{cases}$$

Banach Mazur



$BM(A)$

A: any set of infinite words

$$\mathcal{BM}(A)$$

I wins $\iff x \in A$

$\mathcal{BM}(A)$

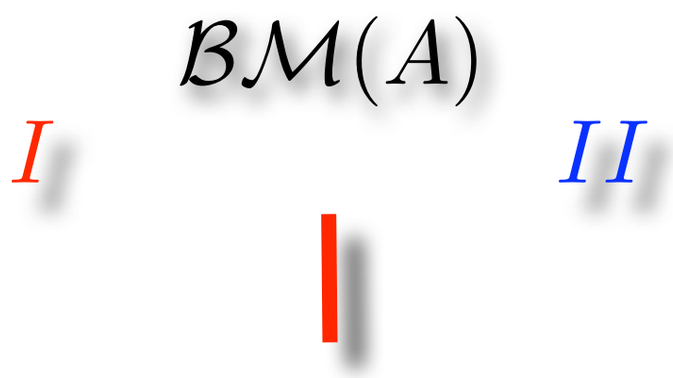
I

I wins $\iff x \in A$

$\mathcal{BM}(A)$

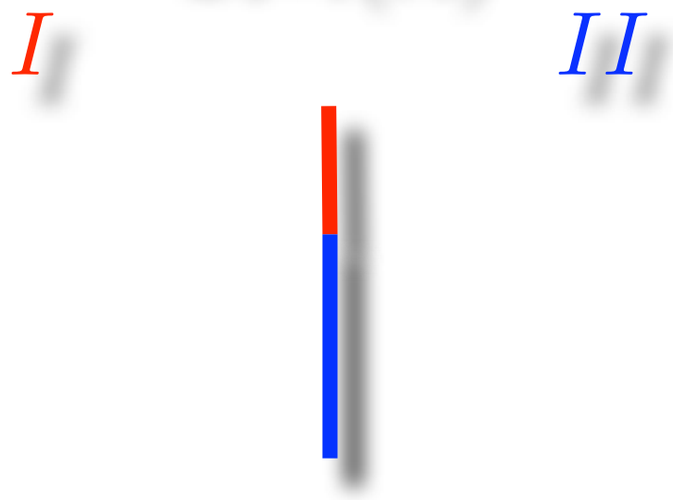
I II

I wins $\iff x \in A$



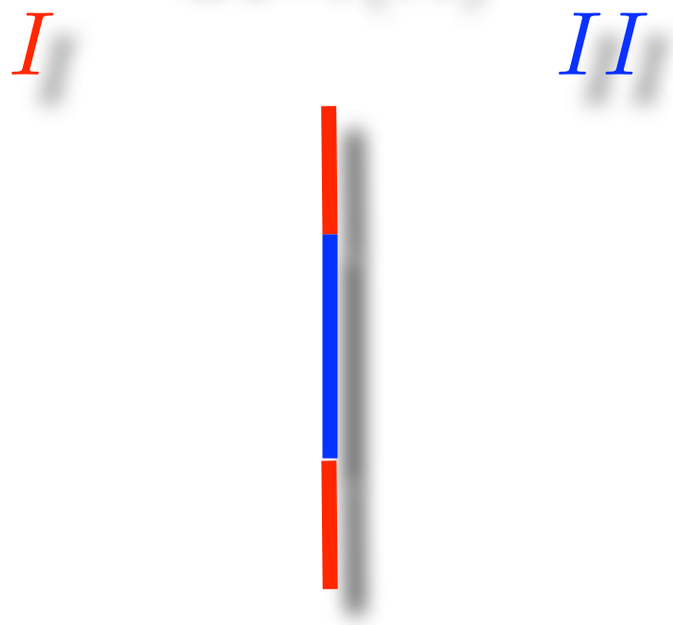
I wins $\iff x \in A$

$\mathcal{BM}(A)$



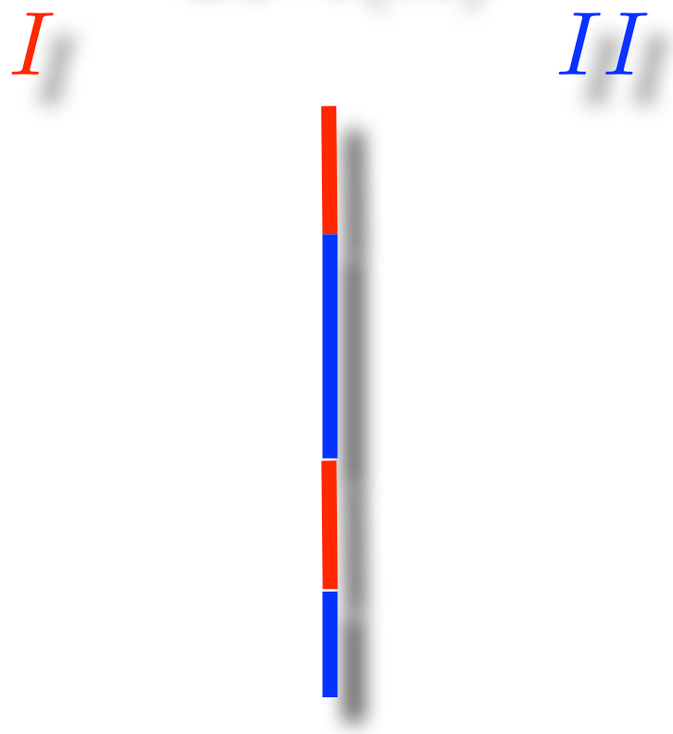
I wins $\iff x \in A$

$\mathcal{BM}(A)$



I wins $\iff x \in A$

$\mathcal{BM}(A)$



I wins $\iff x \in A$

$\mathcal{BM}(A)$

I II



I wins $\iff x \in A$

$\mathcal{BM}(A)$

I II



I wins $\iff x \in A$

$\mathcal{BM}(A)$

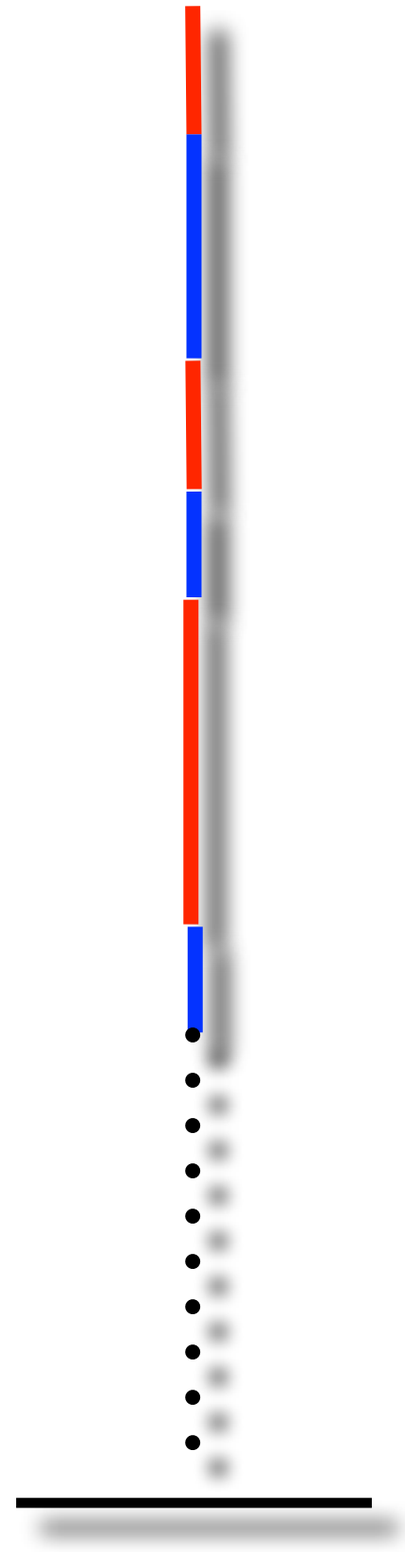
I II



I wins $\iff x \in A$

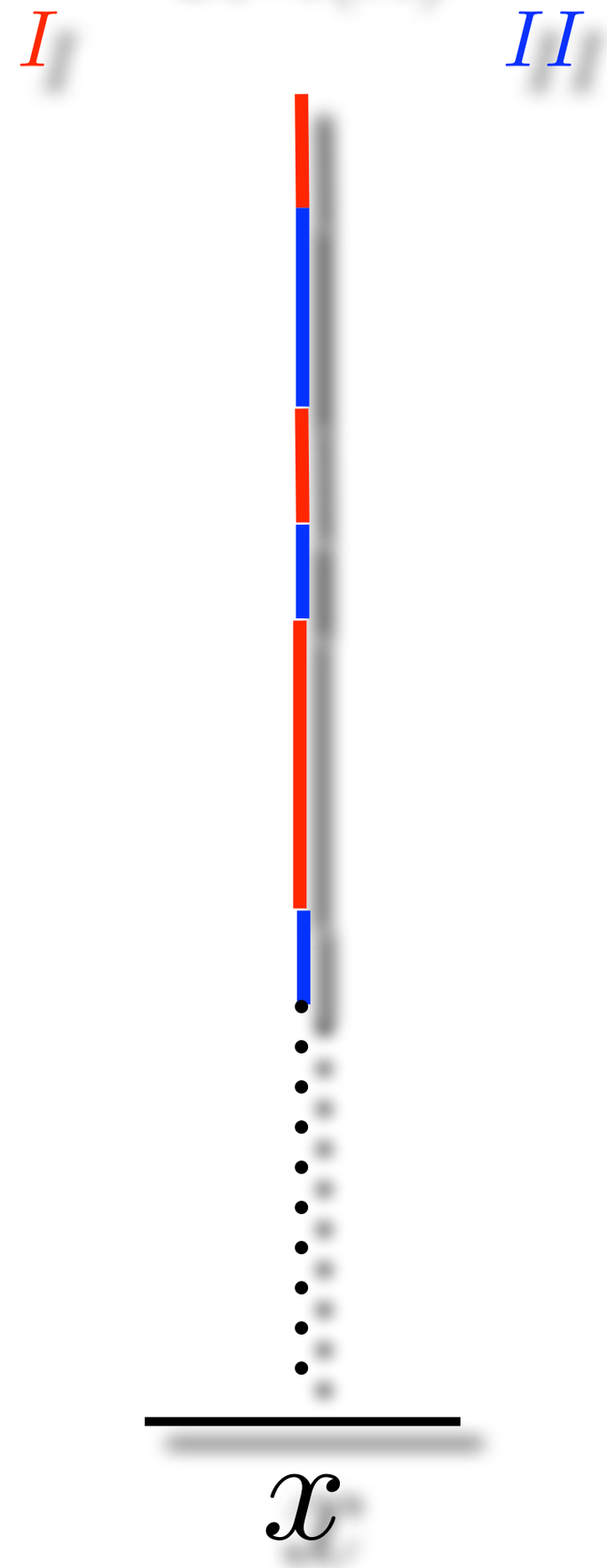
$\mathcal{BM}(A)$

I II



I wins $\iff x \in A$

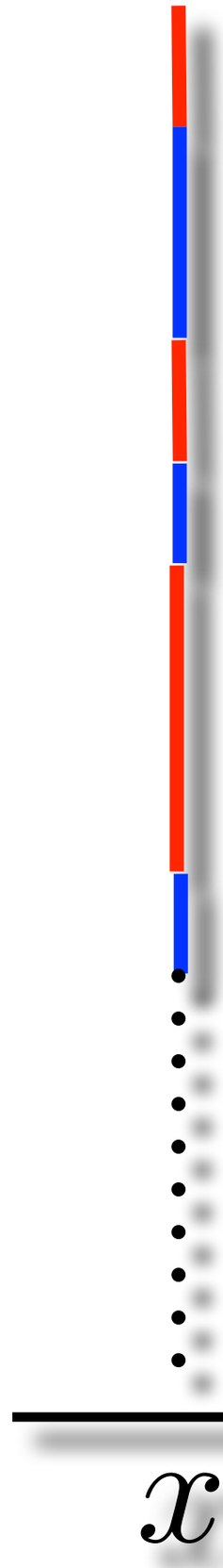
$\mathcal{BM}(A)$



I wins $\iff x \in A$

$\mathcal{BM}(A)$

I $\qquad\qquad$ II



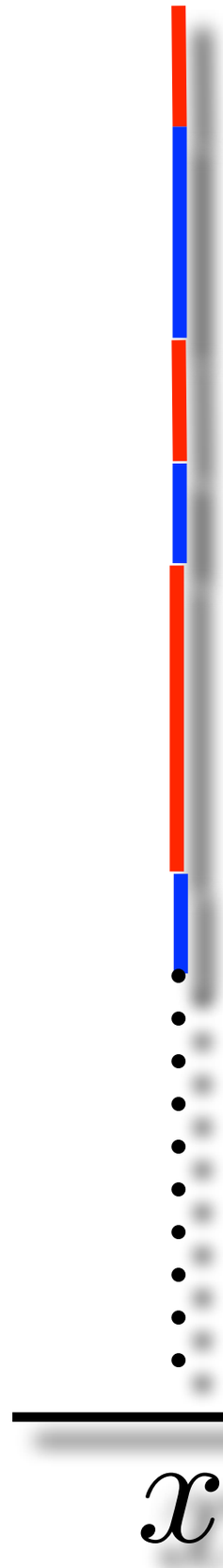
x

$Det\ \ \mathcal{BM}(A)$

I wins $\iff x \in A$

$\mathcal{BM}(A)$

I $\qquad\qquad$ II



$Det\ \ \mathcal{BM}(A)$

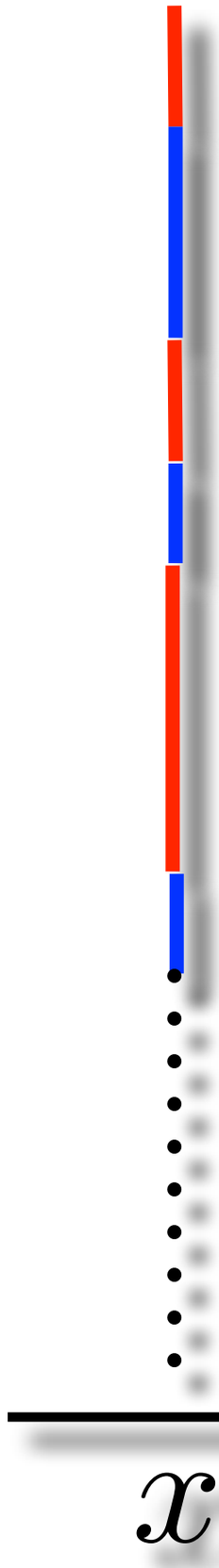
$\Longleftrightarrow$

I wins $\Longleftrightarrow$ $x \in A$

x

$\mathcal{BM}(A)$

I $\qquad\qquad$ II



$Det\ \ \mathcal{BM}(A)$

$\Longleftrightarrow$

I wins $\Longleftrightarrow$ $x \in A$

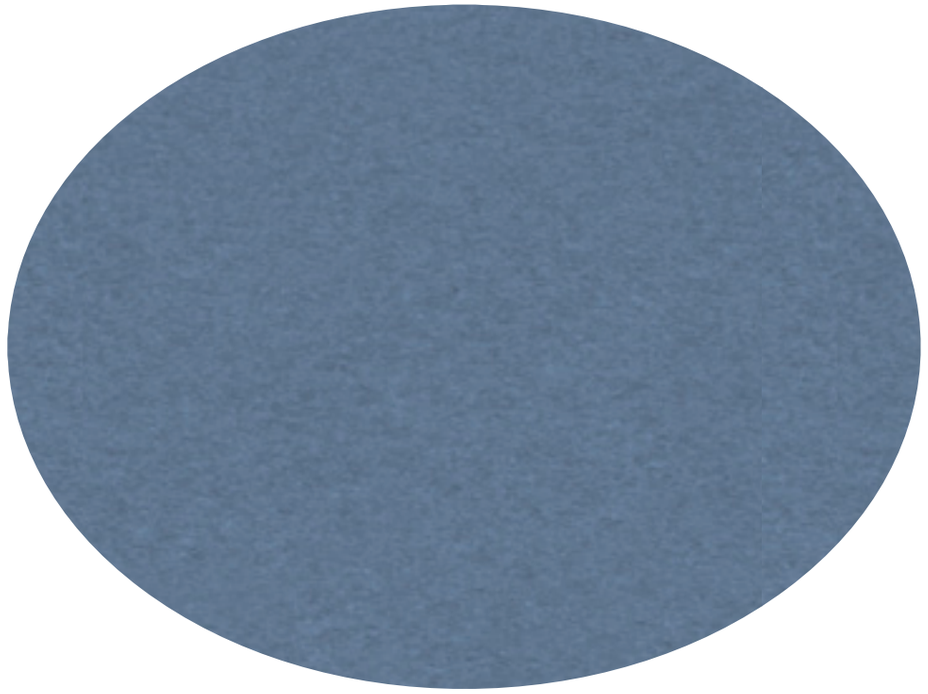
A has the Baire Property

x

$$Det \;\; \mathcal{BM}(A)$$

$$\Longleftrightarrow$$

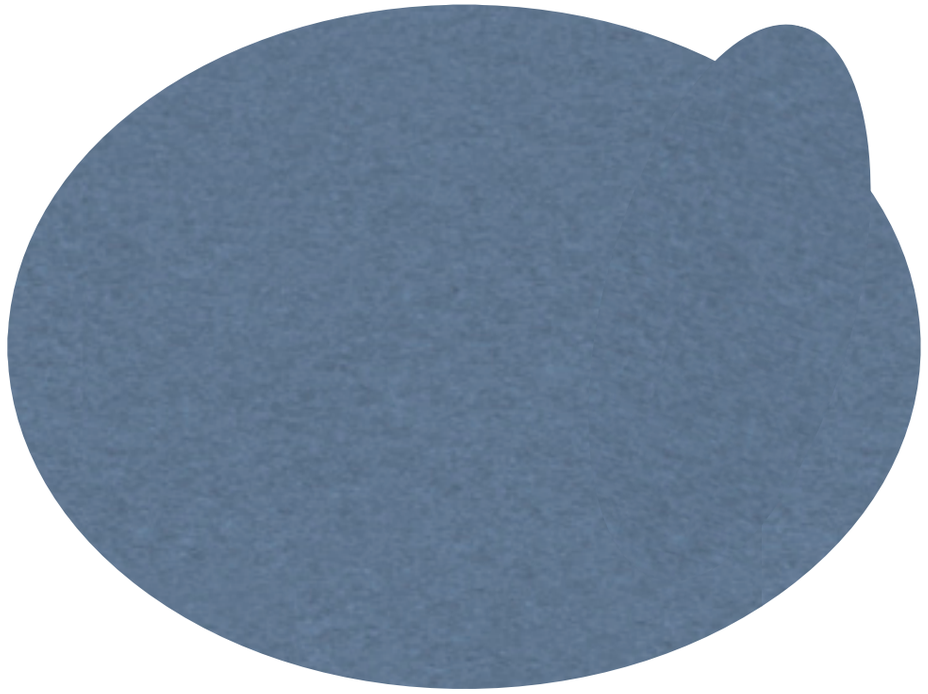
A has the Baire Property



$$Det\ \ \mathcal{BM}(A)$$

$$\Longleftrightarrow$$

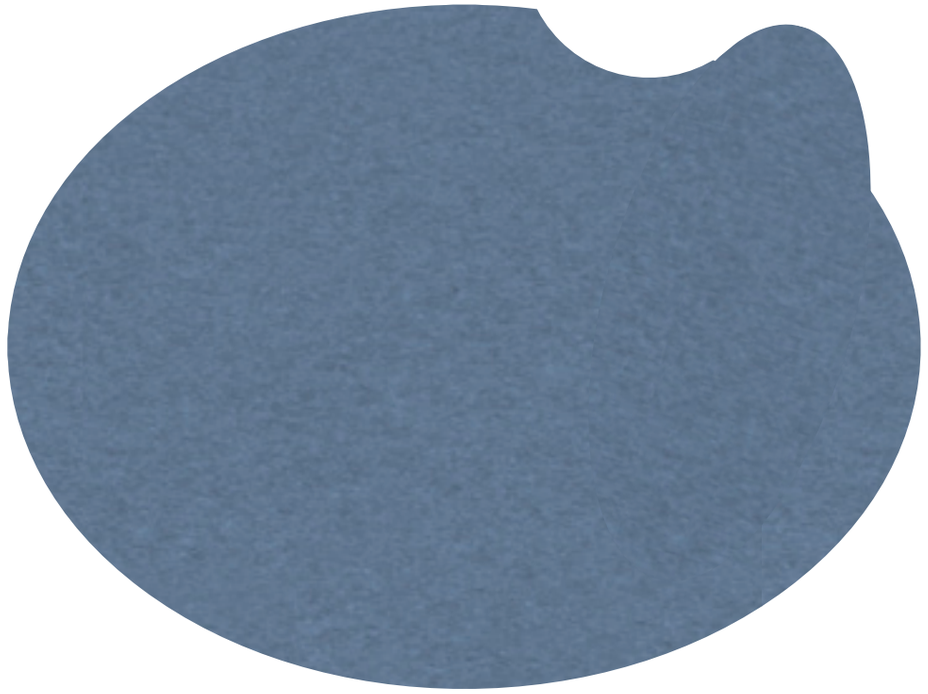
A has the Baire Property



$$Det \;\; \mathcal{BM}(A)$$

$$\Longleftrightarrow$$

A has the Baire Property



Perfect Set

$$\mathcal{PS}(A)$$

A: any set of infinite words

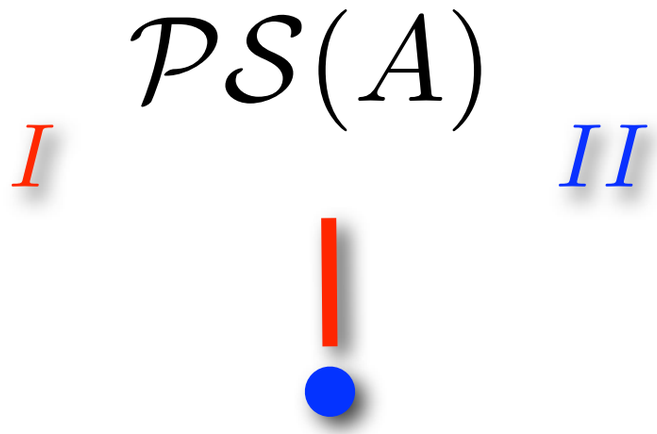
$$\mathcal{PS}(A)$$

I II

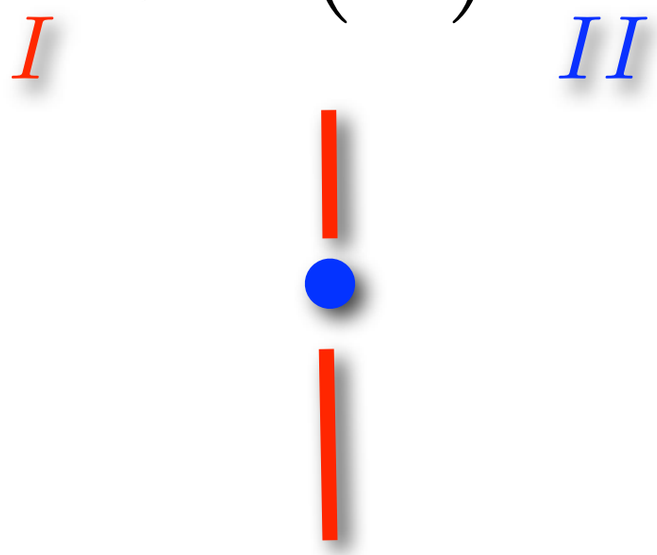
$$\mathcal{PS}(A)$$

I II



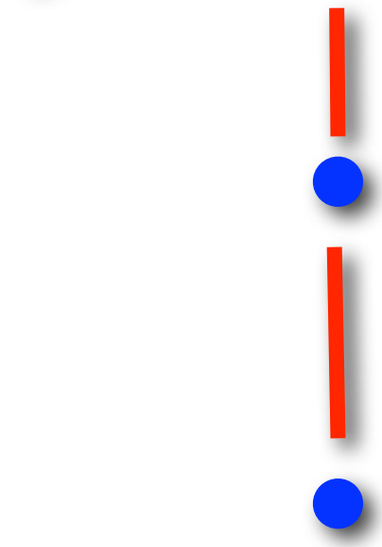

$\mathcal{PS}(A)$
I
II

$\mathcal{PS}(A)$

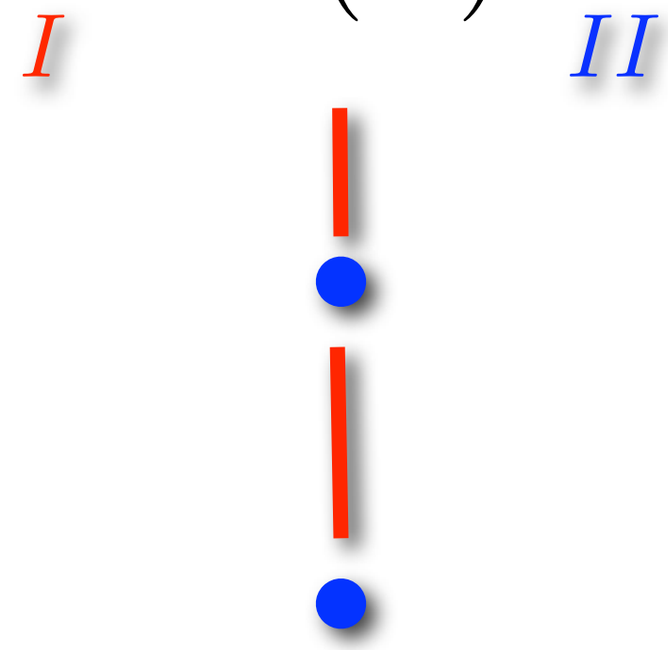


$\mathcal{PS}(A)$

I II



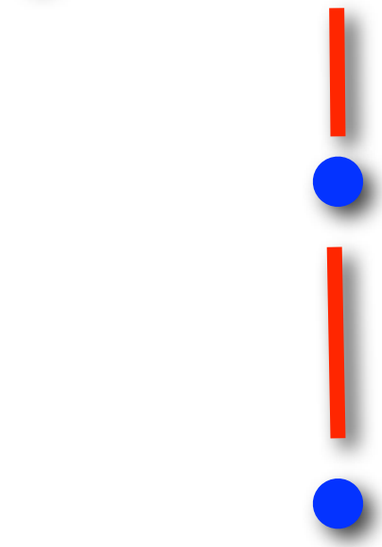
$$\mathcal{PS}(A)$$



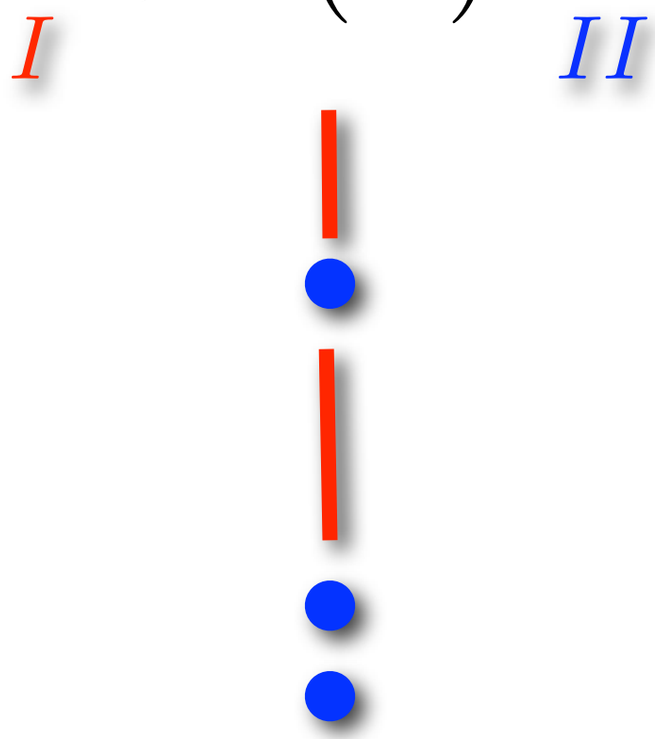
the empty sequence !

$\mathcal{PS}(A)$

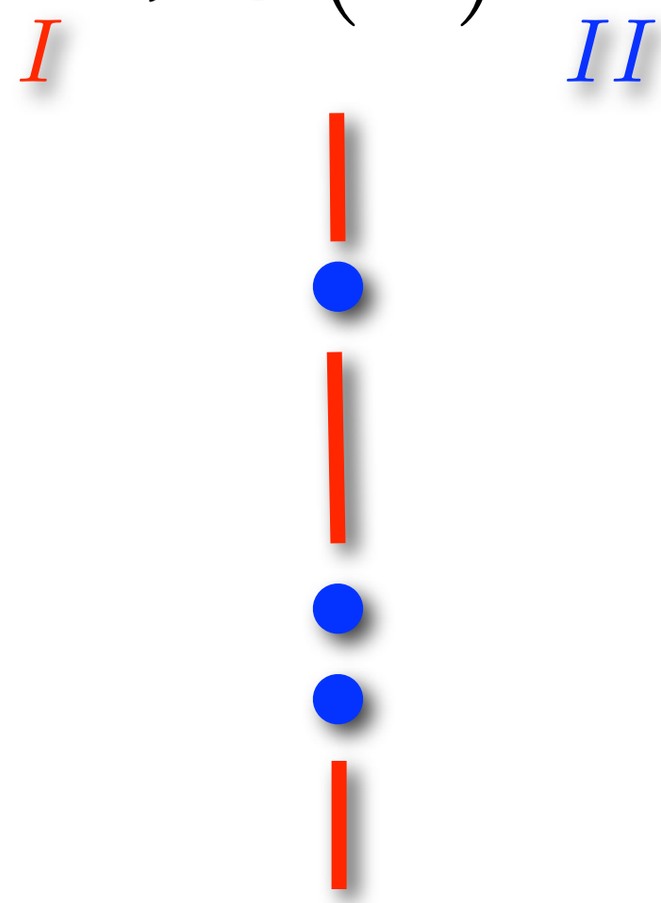
I II



$$\mathcal{PS}(A)$$

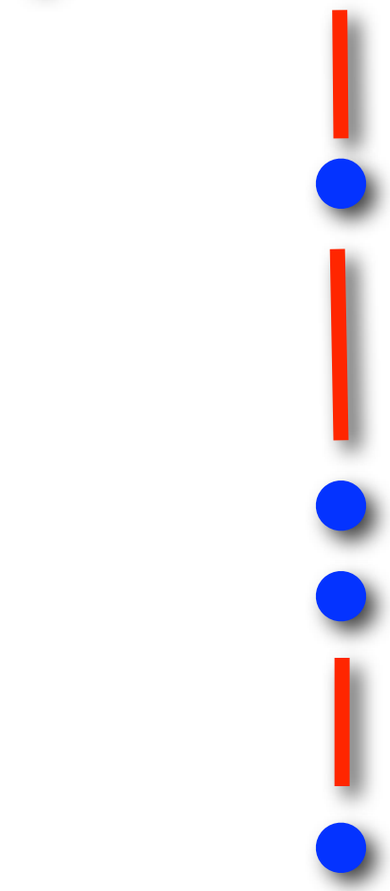


$\mathcal{PS}(A)$

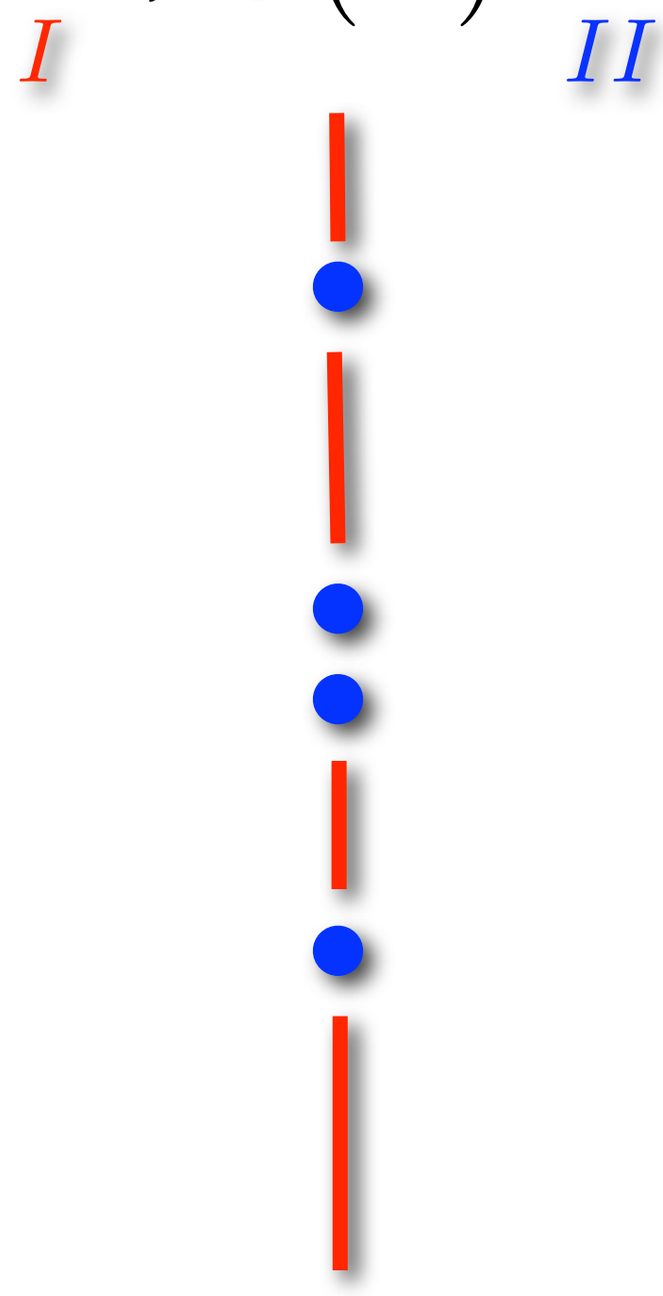


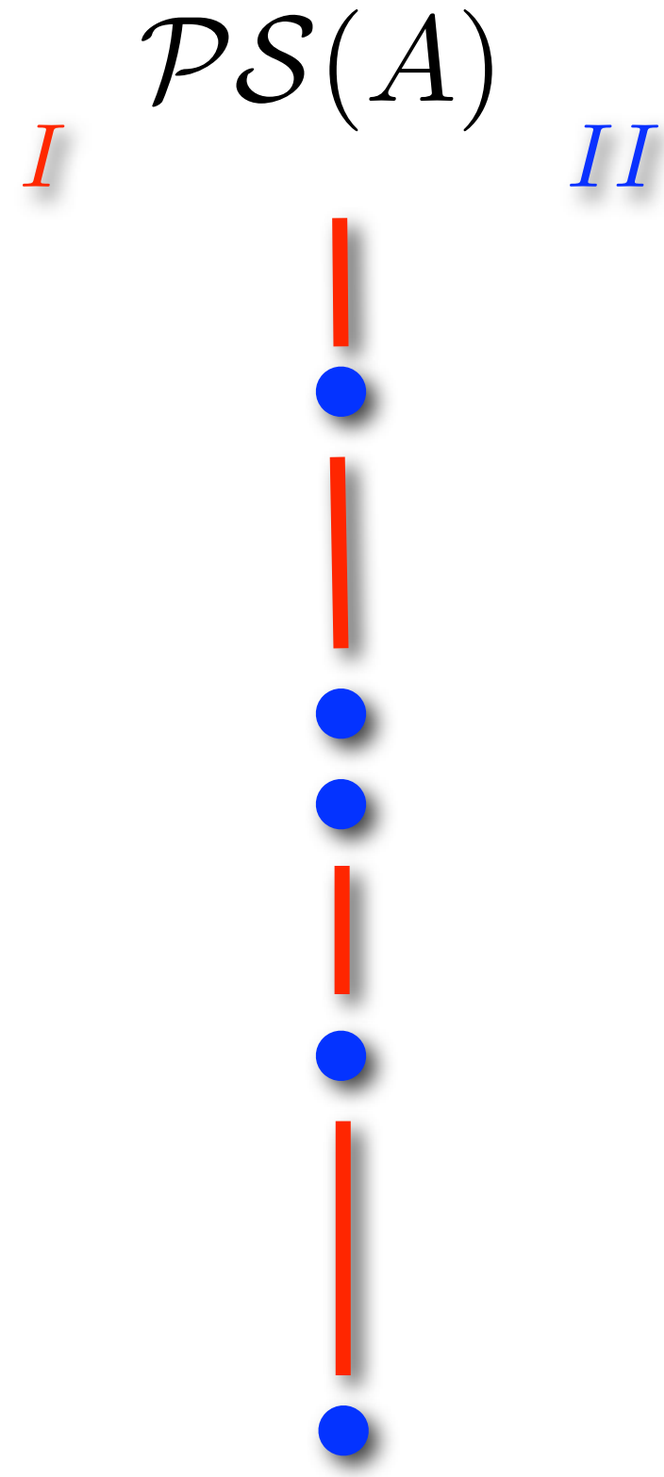
$\mathcal{PS}(A)$

I II

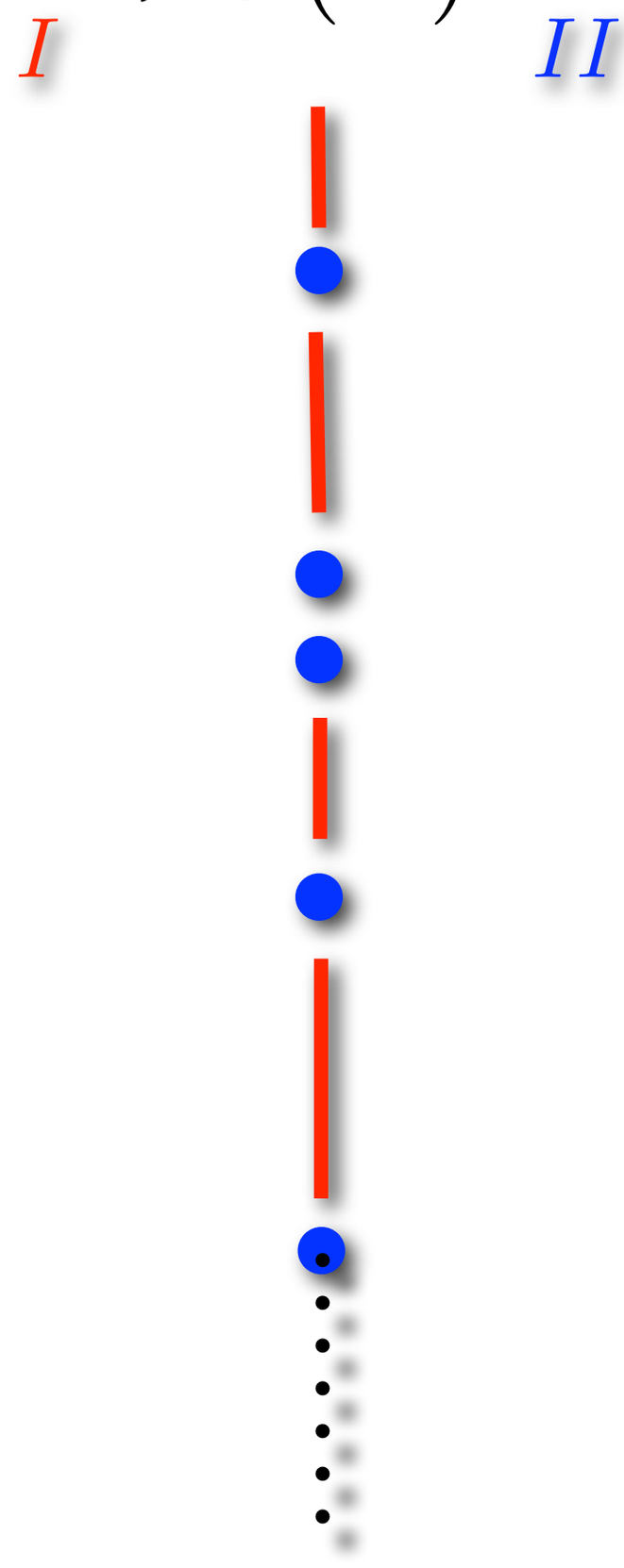


$\mathcal{PS}(A)$




$\mathcal{PS}(A)$
I
II

$$\mathcal{PS}(A)$$



$\mathcal{PS}(A)$

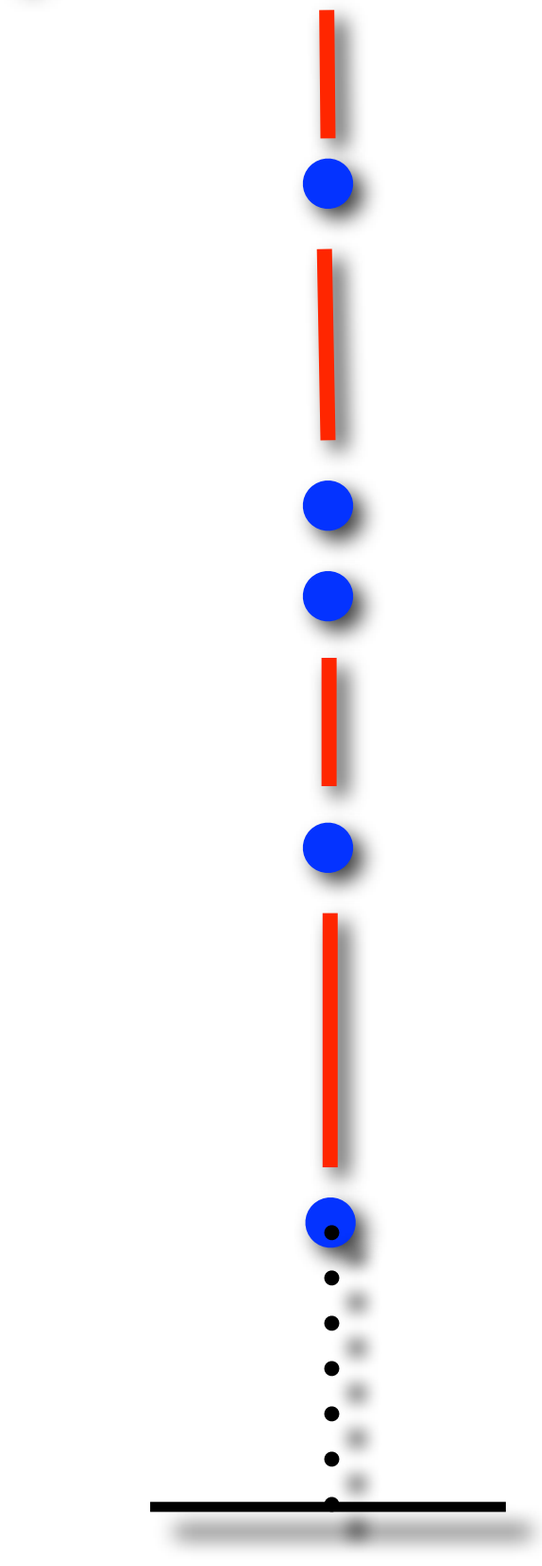
I II



I wins $\iff x \in A$

$\mathcal{PS}(A)$

I II

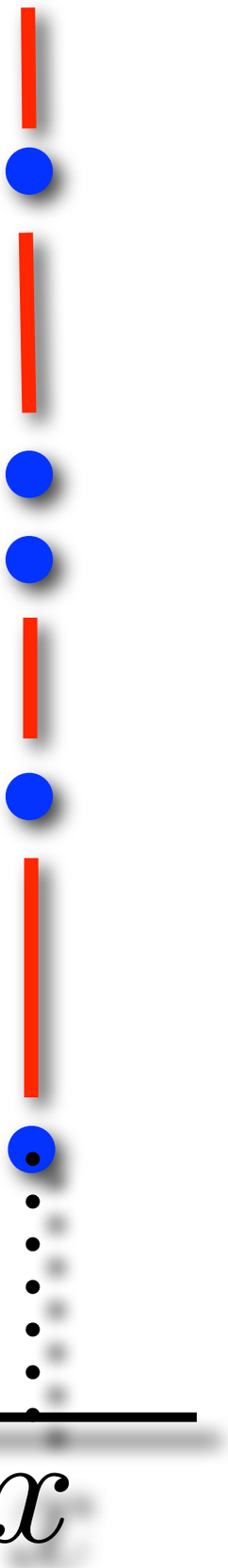


I wins $\iff x \in A$

$\mathcal{PS}(A)$

I *II*

A has the Perfect Set Property



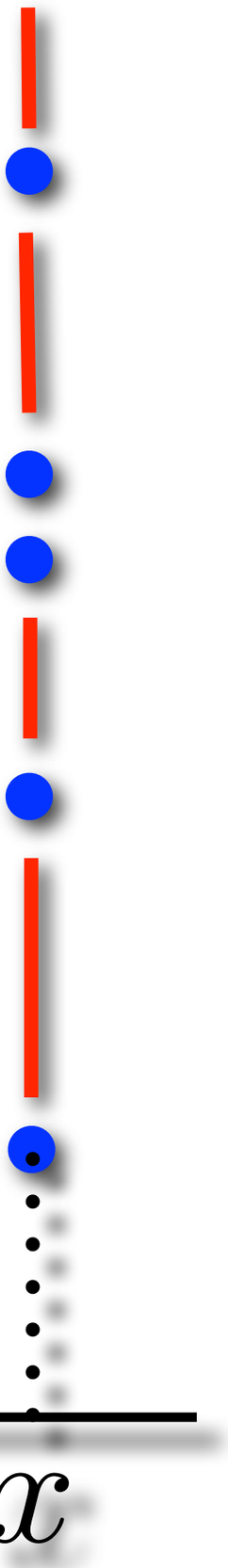
I wins $\iff x \in A$

x

$\mathcal{PS}(A)$

I *II*

A has the Perfect Set Property

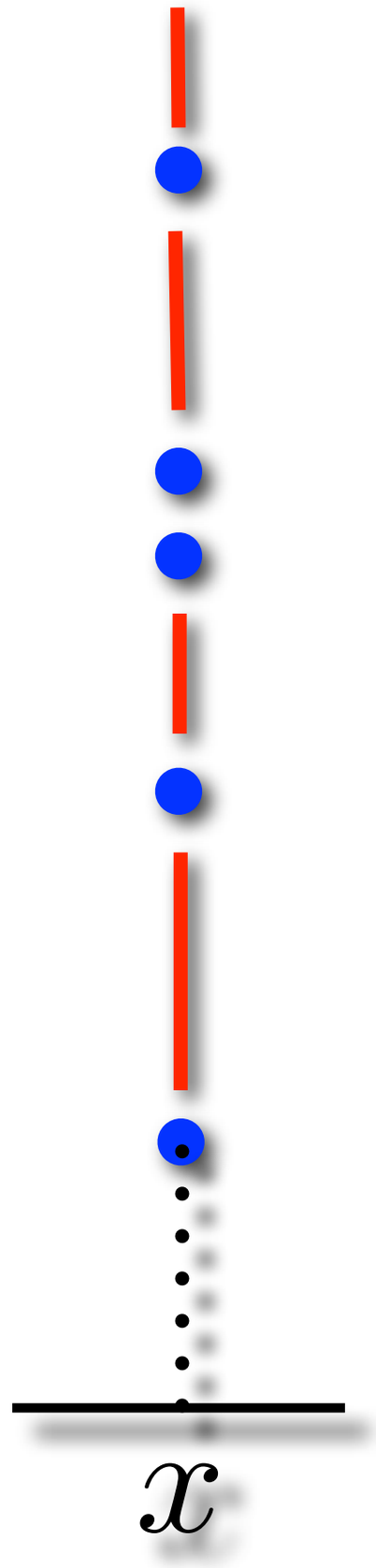


I wins $\iff x \in A$

x

$\mathcal{PS}(A)$

I $\qquad\qquad$ II



I wins $\iff x \in A$

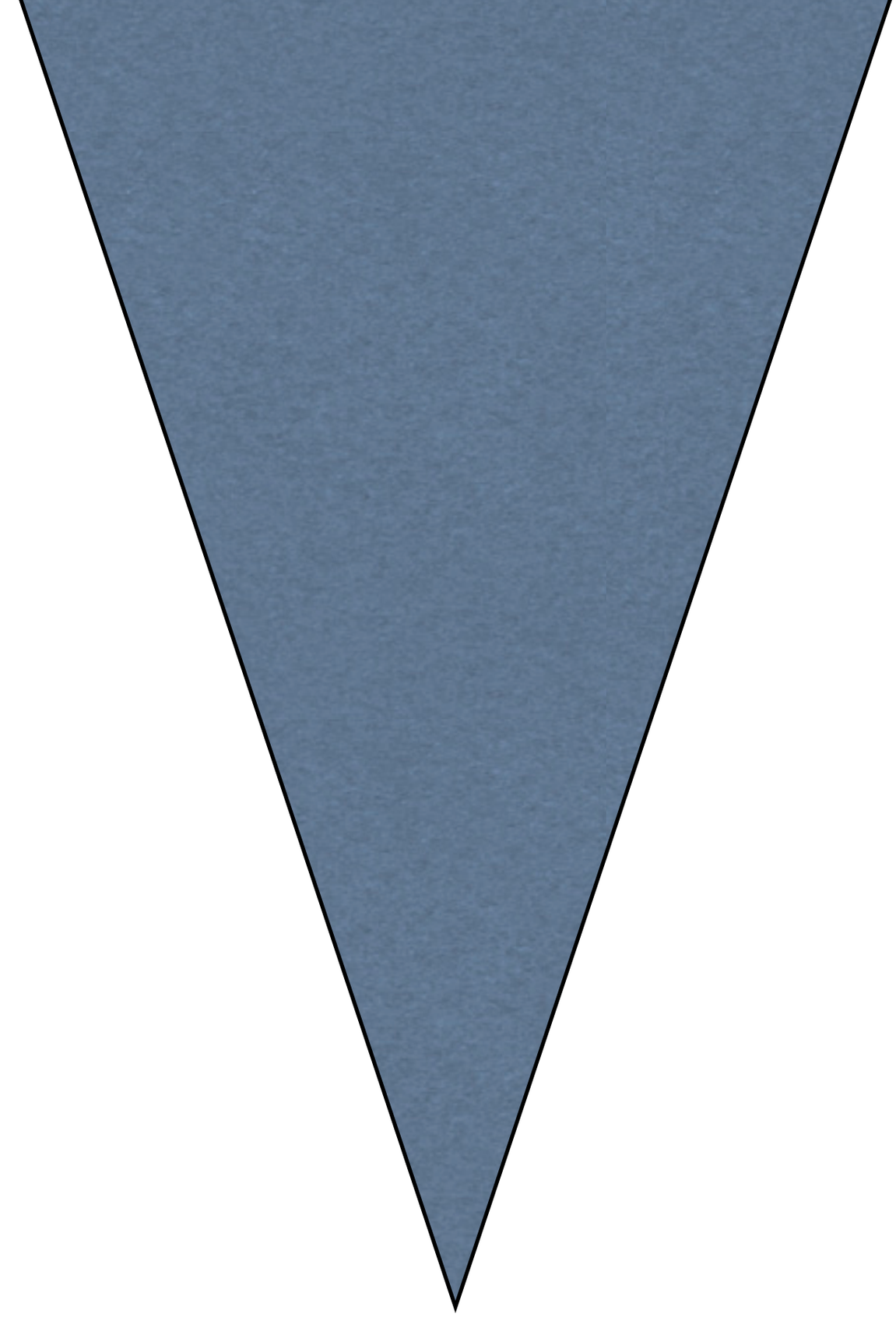
x

$$A \text{ has the Perfect Set Property}$$

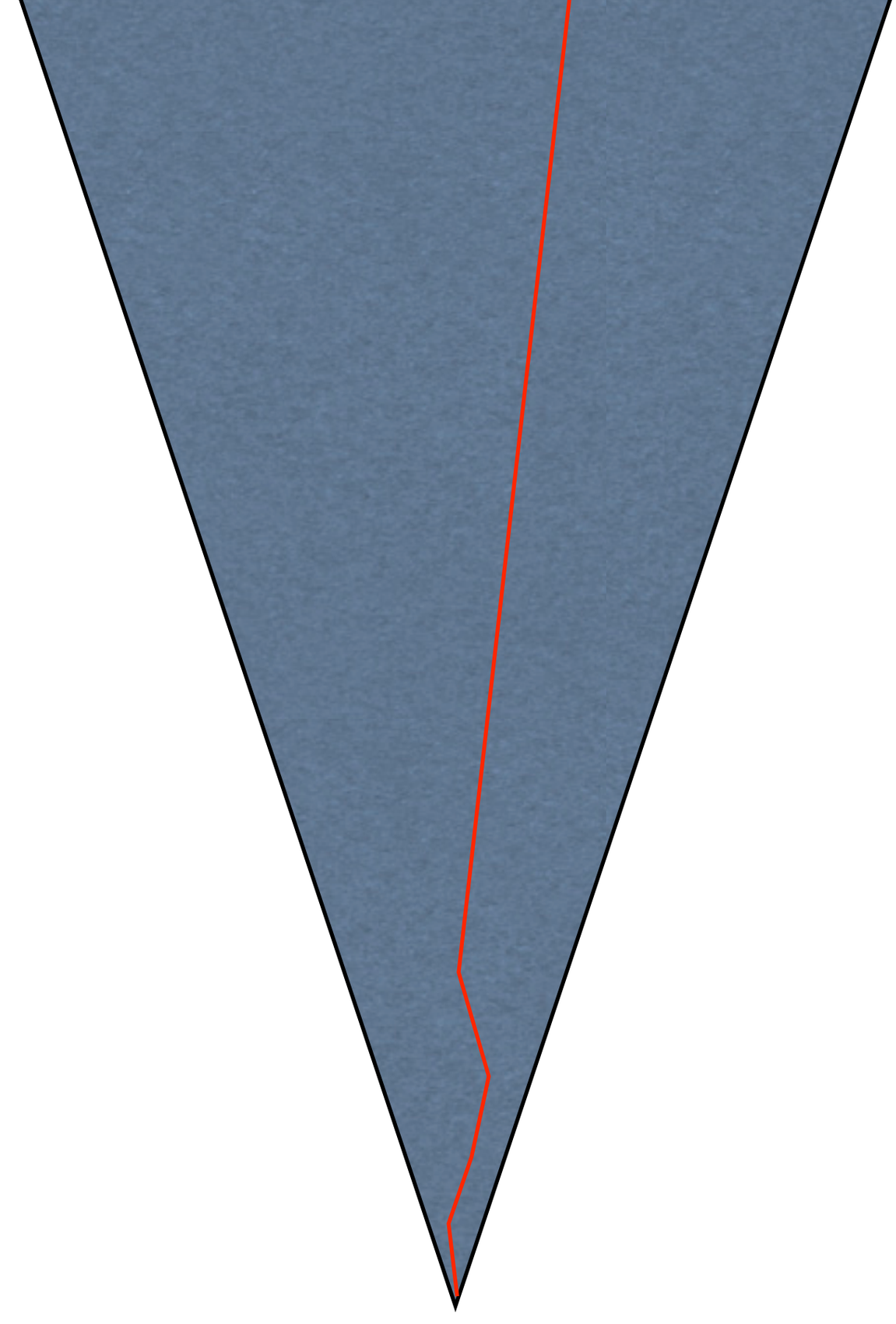
$$\iff \begin{cases} \text{either } A \text{ is countable} \\ \text{or} \\ A \text{ contains a perfect set} \end{cases}$$

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

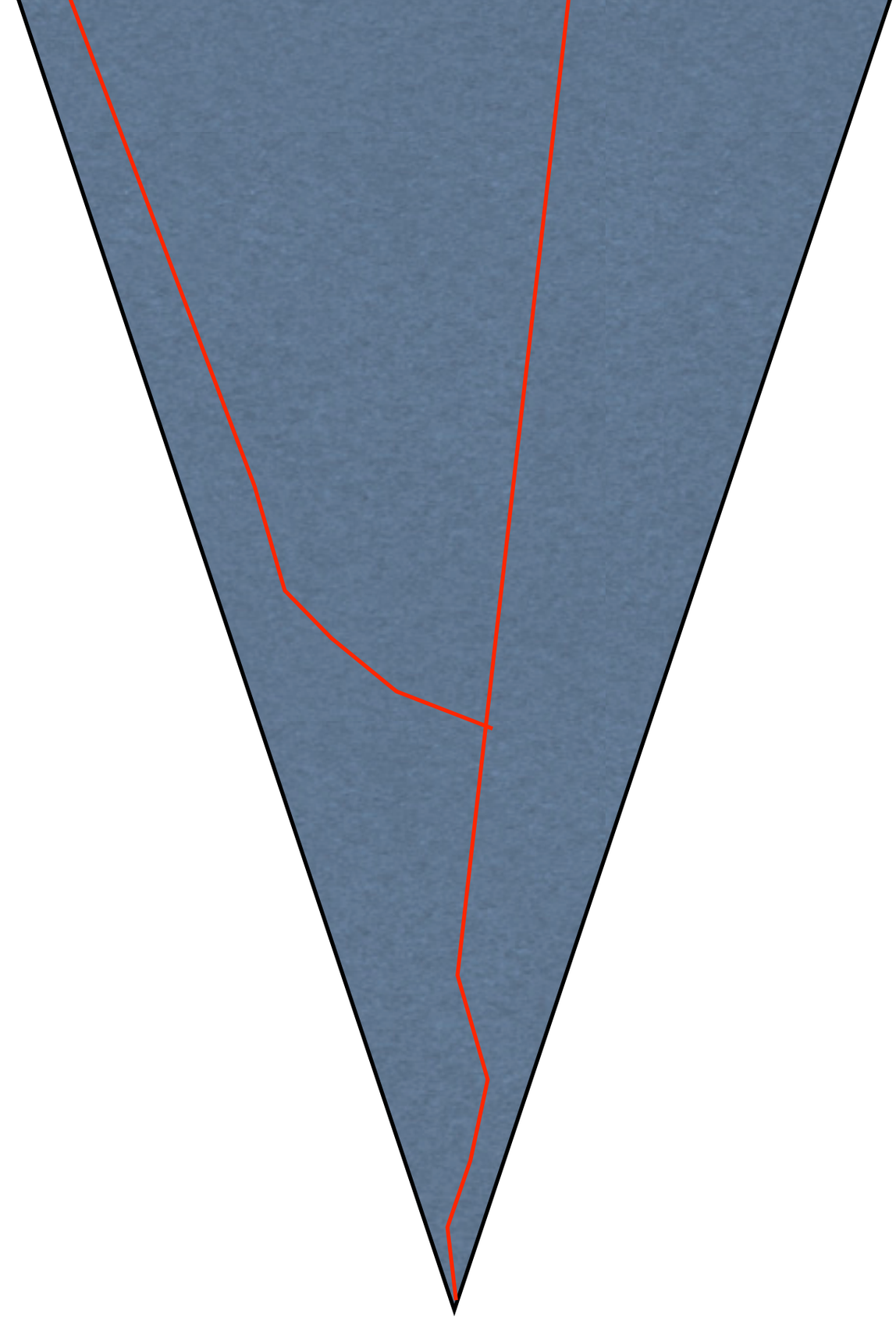
$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



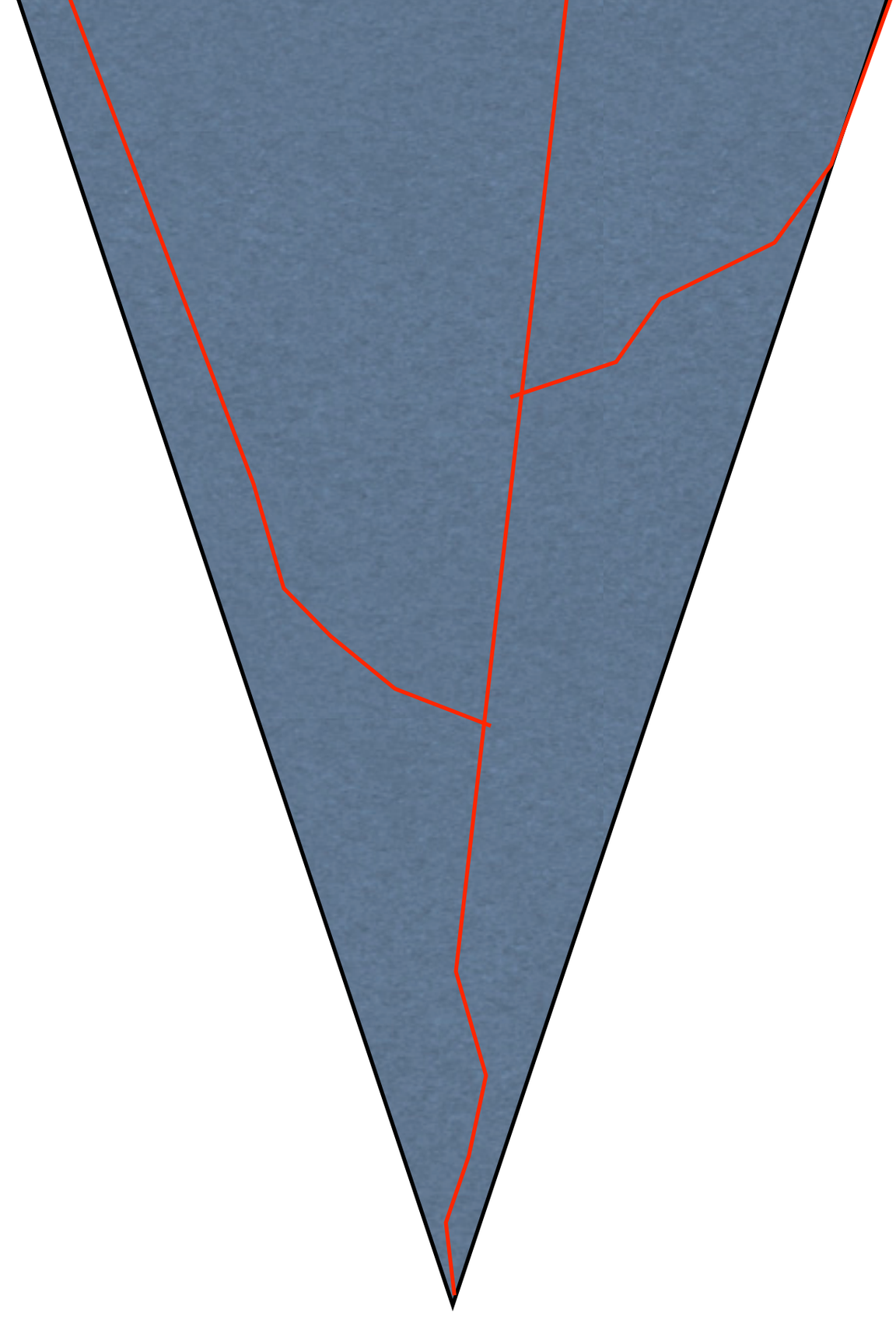
$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



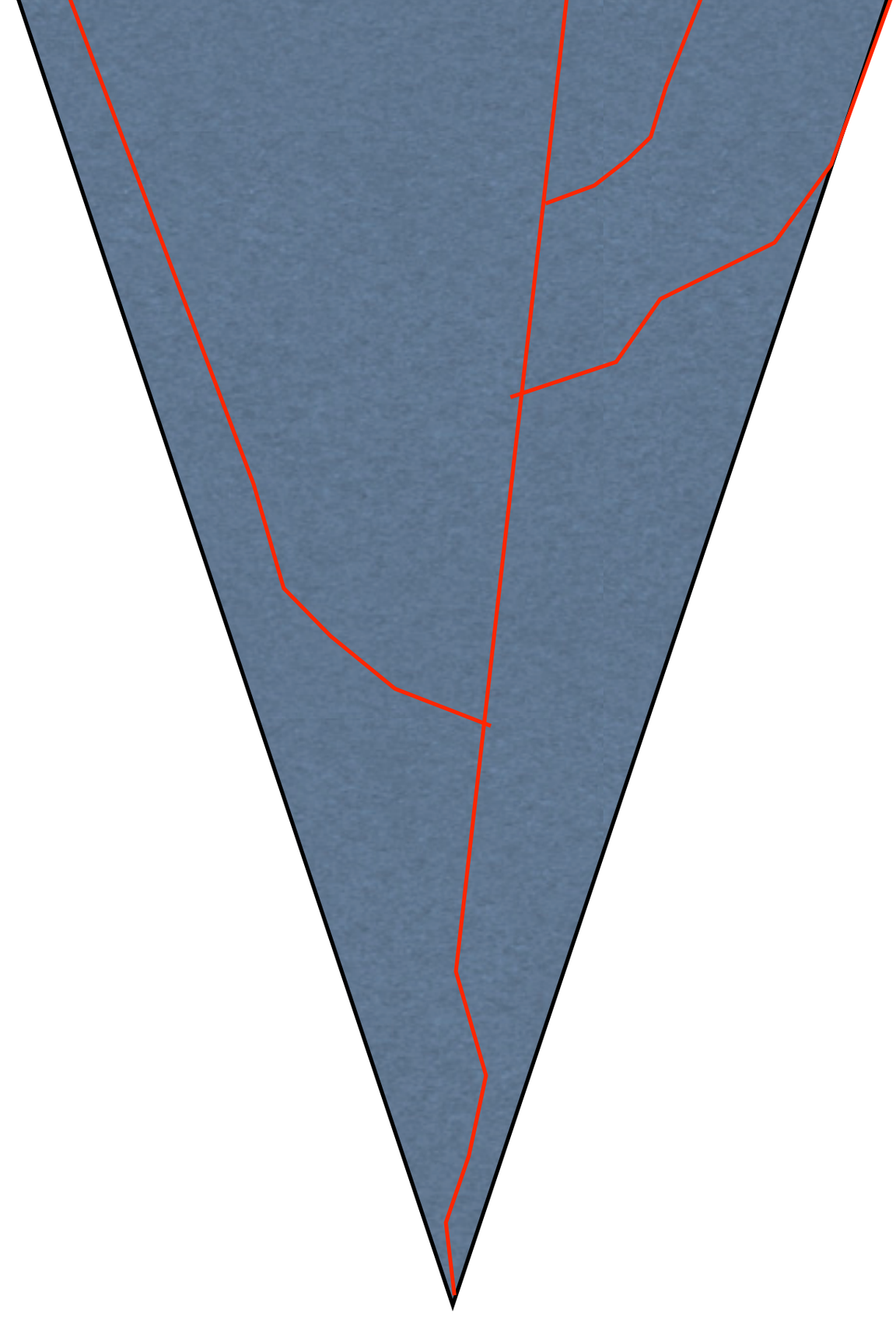
$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



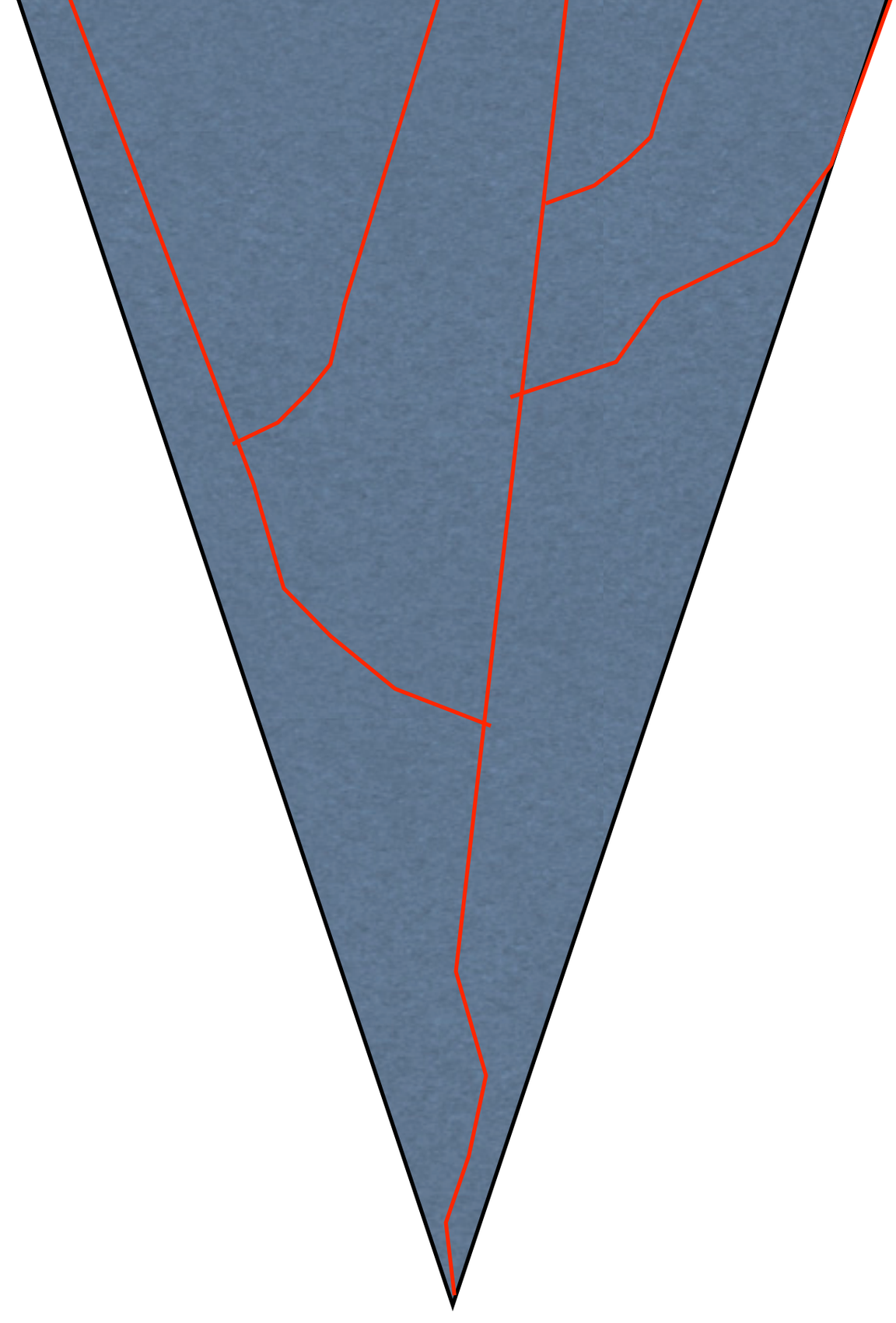
$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



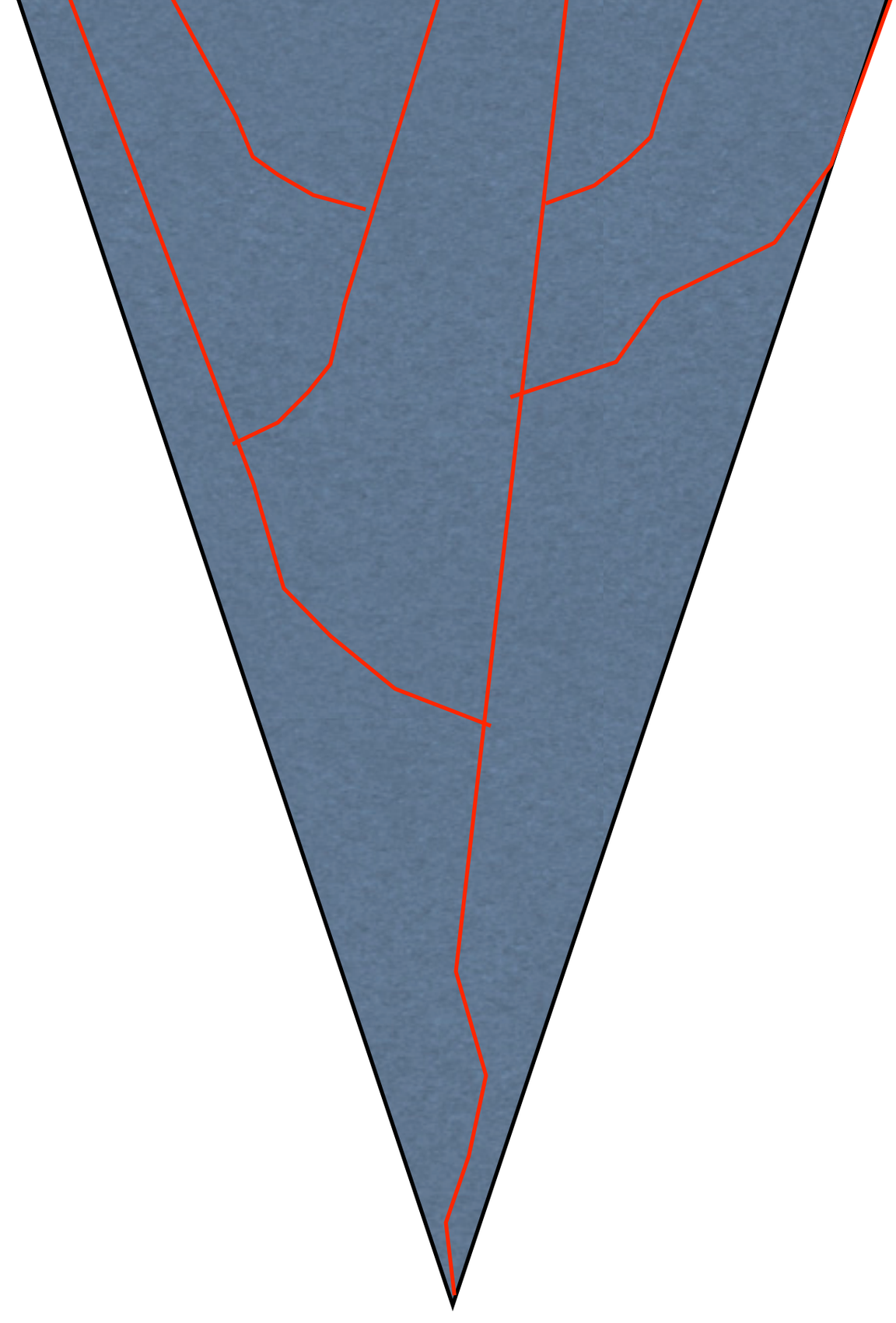
$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



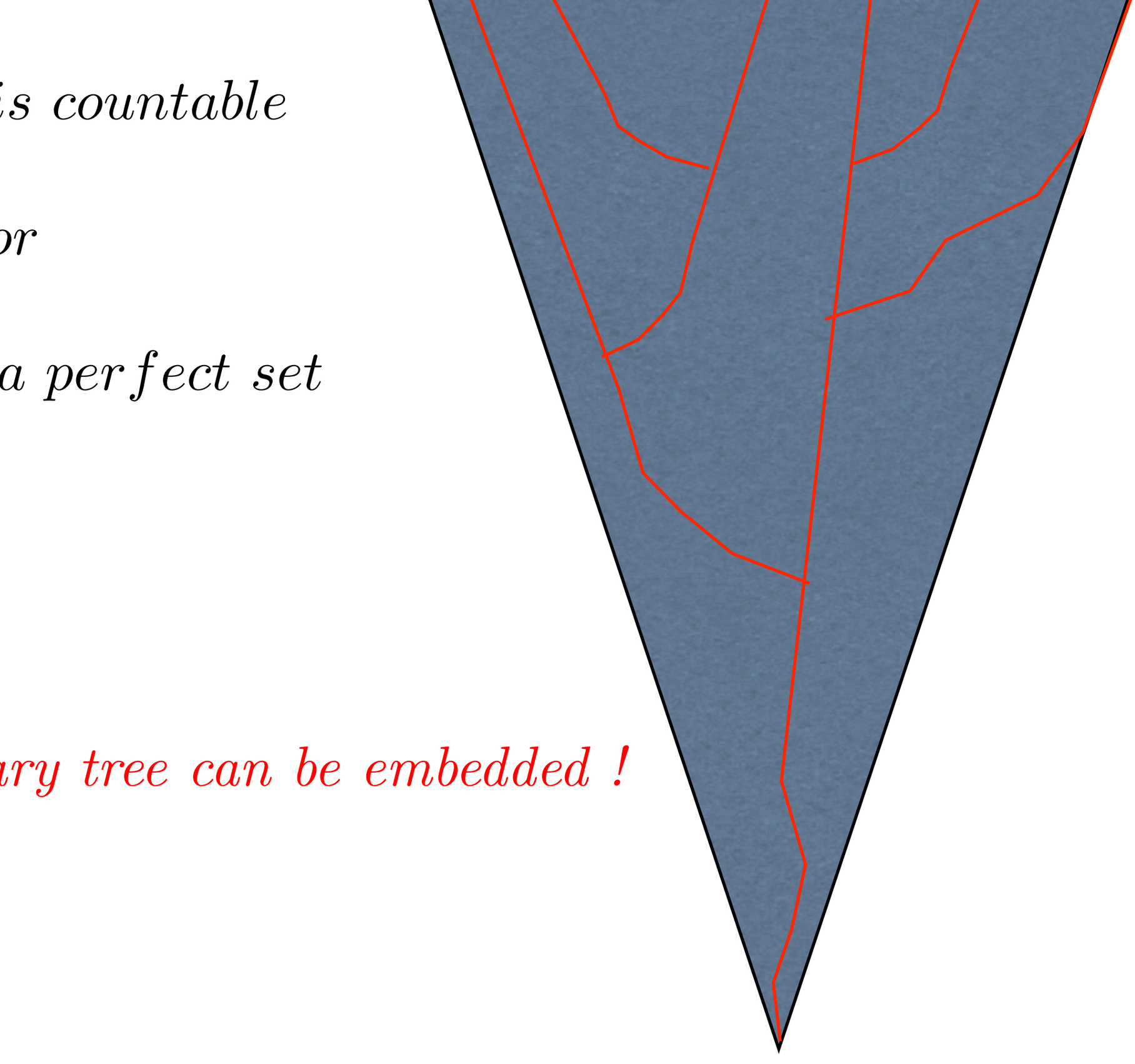
$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$



the whole binary tree can be embedded !

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$\omega = \aleph_0$

1

0

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$\omega_1 = \aleph_1$

$\omega = \aleph_0$

1

0

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1$

$\omega = \aleph_0$

1

0

$$\omega_3 = \aleph_3$$

$$\begin{cases} \textit{either } A \textit{ is countable} \\ \\ \textit{or} \\ \\ A \textit{ contains a perfect set} \end{cases}$$

$$\omega_2 = \aleph_2$$

$$\omega_1 = \aleph_1$$

$$\omega = \aleph_0$$

$$\begin{matrix} 1 \\ 0 \end{matrix}$$

$\omega_3 = \aleph_3$

$\left\{ \begin{array}{c} \textit{either } A \textit{ is countable} \\ \\ \textit{or} \\ \\ A \textit{ contains a perfect set} \end{array} \right.$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1 \quad 2^{\aleph_0}$

$\omega = \aleph_0$

1

0

$$\omega_3 = \aleph_3 2^{\aleph_0}$$

$$\begin{cases} \textit{either } A \textit{ is countable} \\ \\ \textit{or} \\ \\ A \textit{ contains a perfect set} \end{cases}$$

$$\omega_2 = \aleph_2$$

$$\omega_1 = \aleph_1$$

$$\omega = \aleph_0$$

$$1$$
$$0$$

$\omega_3 = \aleph_3$

$$\begin{cases} \textit{either } A \textit{ is countable} \\ \\ \textit{or} \\ \\ A \textit{ contains a perfect set} \end{cases}$$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1$ $2^{\aleph_0}$

$\omega = \aleph_0$

1

0

$$\omega_3 = \aleph_3$$

$$\begin{cases} \textit{either } A \textit{ is countable} \\ \\ \textit{or} \\ \\ A \textit{ contains a perfect set} \end{cases}$$

$$\omega_2 = \aleph_2$$

$$\omega_1 = \aleph_1$$

$$\omega = \aleph_0$$

$$\begin{matrix} 1 \\ 0 \end{matrix}$$

$$\begin{cases} \text{either } A \text{ is countable} \\ \\ \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$\omega_3 = \aleph_3$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1$

$\omega = \aleph_0$

1

0

$$\begin{cases} \text{either } A \text{ is countable} \\ \qquad\qquad\qquad card(A) = \aleph_0 \\ \qquad\quad \text{or} \\ \\ A \text{ contains a perfect set} \end{cases}$$

$\omega_3 = \aleph_3$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1$

$\omega = \aleph_0$

1

0

$$\begin{cases} \text{either } A \text{ is countable} \\ \qquad\qquad card(A) = \aleph_0 \\ \text{or} \\ A \text{ contains a perfect set} \\ \qquad\qquad card(A) = 2^{\aleph_0} \end{cases}$$

$\omega_3 = \aleph_3$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1$

$\omega = \aleph_0$

1

0

$$\begin{cases} \text{either } A \text{ is countable} \\ \qquad\qquad card(A) = \aleph_0 \\ \qquad \text{or} \\ A \text{ contains a perfect set} \\ \qquad\qquad card(A) = 2^{\aleph_0} \end{cases}$$

$$CH := \quad \text{"}2^{\aleph_0} = \aleph_1\text{"}$$

$\omega_3 = \aleph_3$

$\omega_2 = \aleph_2$

$\omega_1 = \aleph_1 \qquad 2^{\aleph_0}$

$\omega = \aleph_0$

1

0

Annals of Mathematics, **102** (1975), 363-371

Borel determinacy

By DONALD A. MARTIN

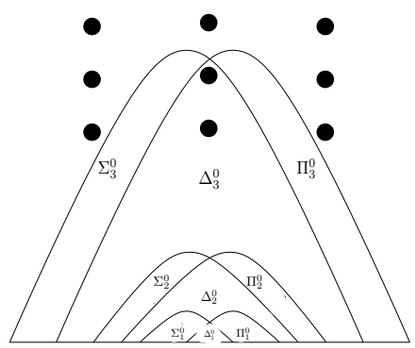
Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots\rangle$ all of whose finite initial segments belong to Y. For each $A \subseteq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0\rangle \in Y$, II picks y_1 with $\langle y_0, y_1\rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2\rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega\rangle \in A$. ($\omega$ = the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements of Y of even length such that always $\langle y_0, \cdots, y_{2n-1}, s(\langle y_0, \cdots, y_{2n-1}\rangle)\rangle \in Y$. A play $\langle y_0, y_1, \cdots\rangle$ of $\mathcal{G}(A, Y)$ is a *play according to* the strategy s if $(\forall n)(y_{2n} = s(\langle y_0, \cdots, y_{2n-1}\rangle))$. The notions of a *strategy for* II and a *play according to* a strategy for II are similarly defined. s is a winning strategy for I (II) if every play according to s is a win for I (II). $\mathcal{G}(A, Y)$ is *determined* if one of the players has a winning strategy.

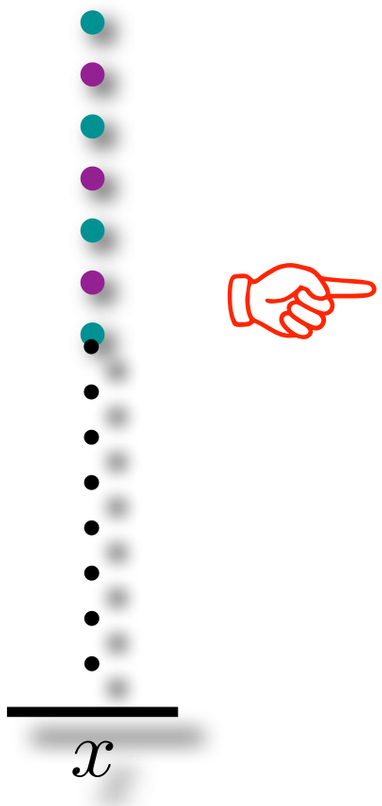


An example: evaluation game for 1st order logic

non determined game

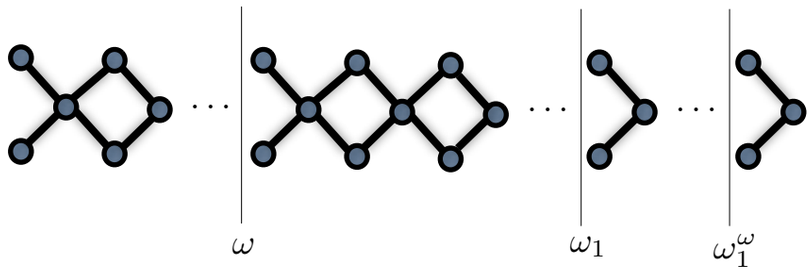


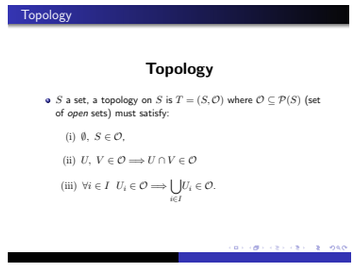
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

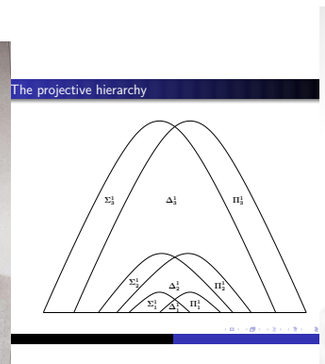
Games that construct mappings



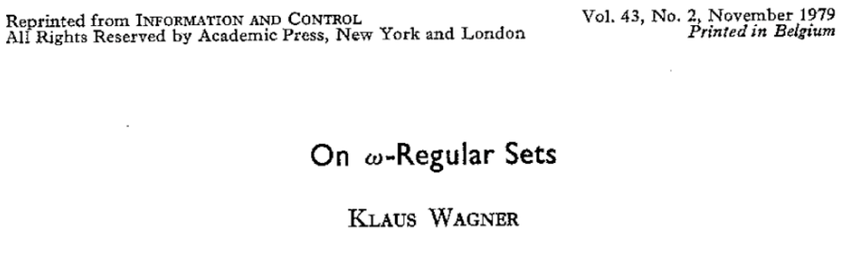










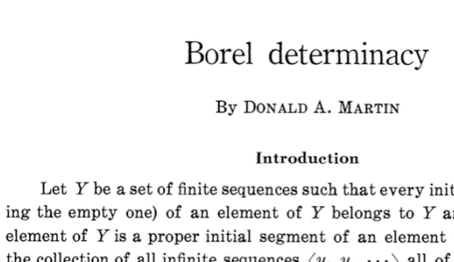


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

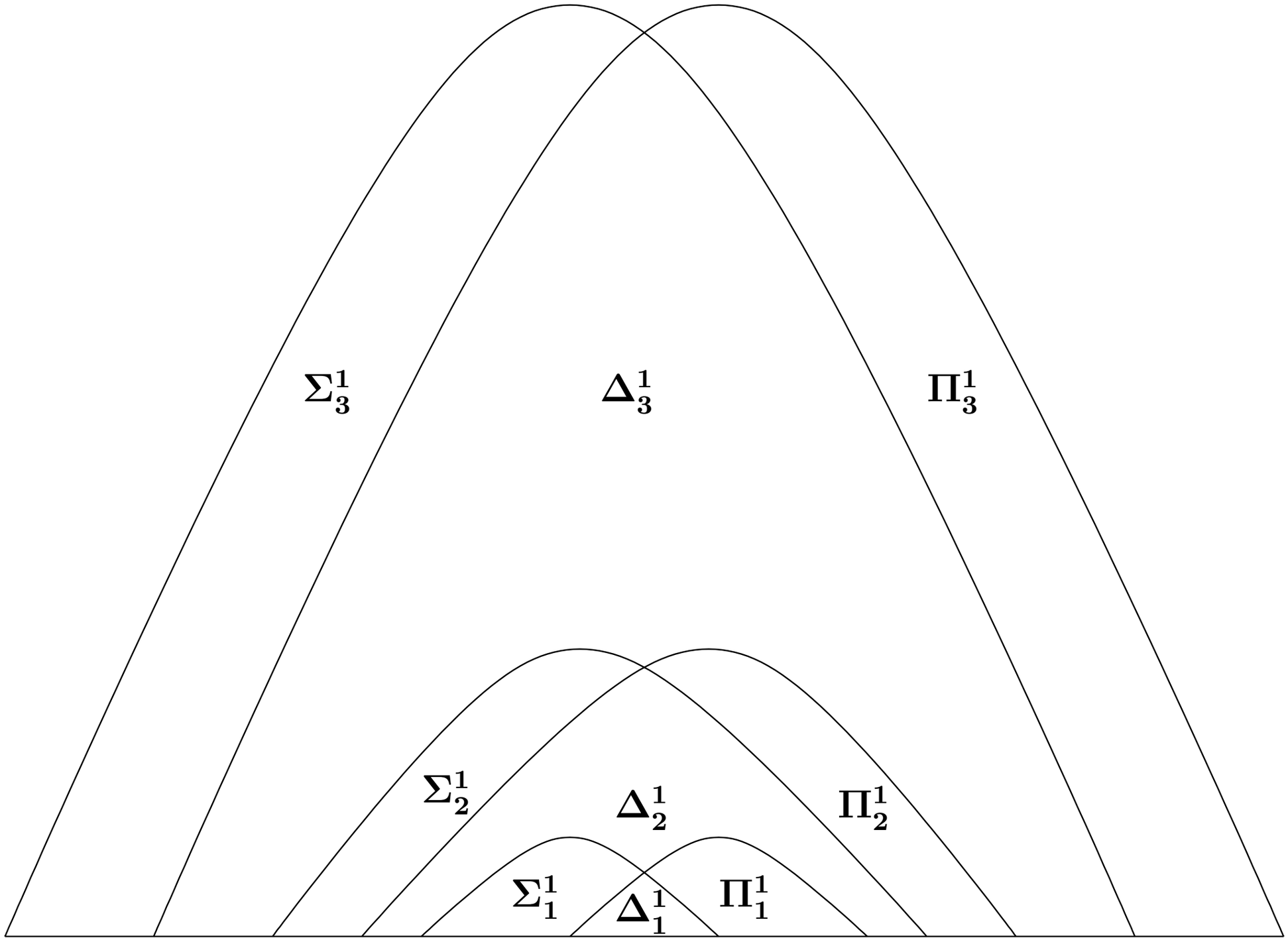
Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

The projective hierarchy



The Projective Hierarchy

Definition

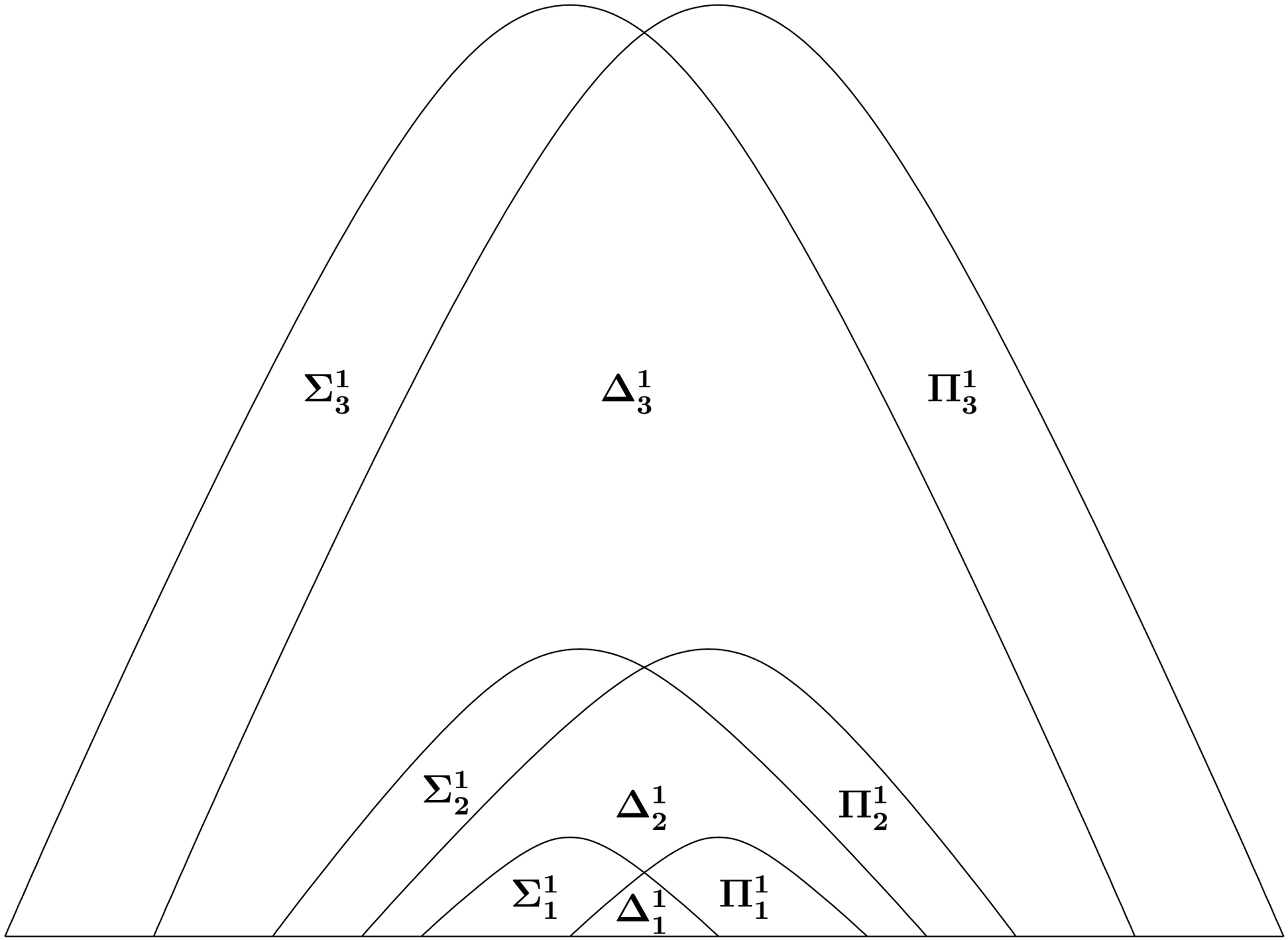
$A \subseteq S$ is analytic iff $x \in A \Leftrightarrow \exists y\ (x, y) \in B$, where B is closed in $S \times \omega^\omega$ (equivalently Borel)

1. $\mathbf{\Sigma^1_1} = \{\text{Analytic sets}\}$
2. $\mathbf{\Pi^1_1} = \{S \setminus A : A \in \mathbf{\Sigma^1_n}\}$
3. $\mathbf{\Delta^1_n} = \mathbf{\Sigma^1_n} \cap \mathbf{\Pi^1_n}$
4. $\mathbf{\Sigma^1_{n+1}} = \{A \subseteq S : x \in A \Leftrightarrow \exists y\ (x, y) \in B$, for some$B \subseteq S \times \omega^\omega, B \in \mathbf{\Pi^1_n}\}$

Theorem (Suslin)

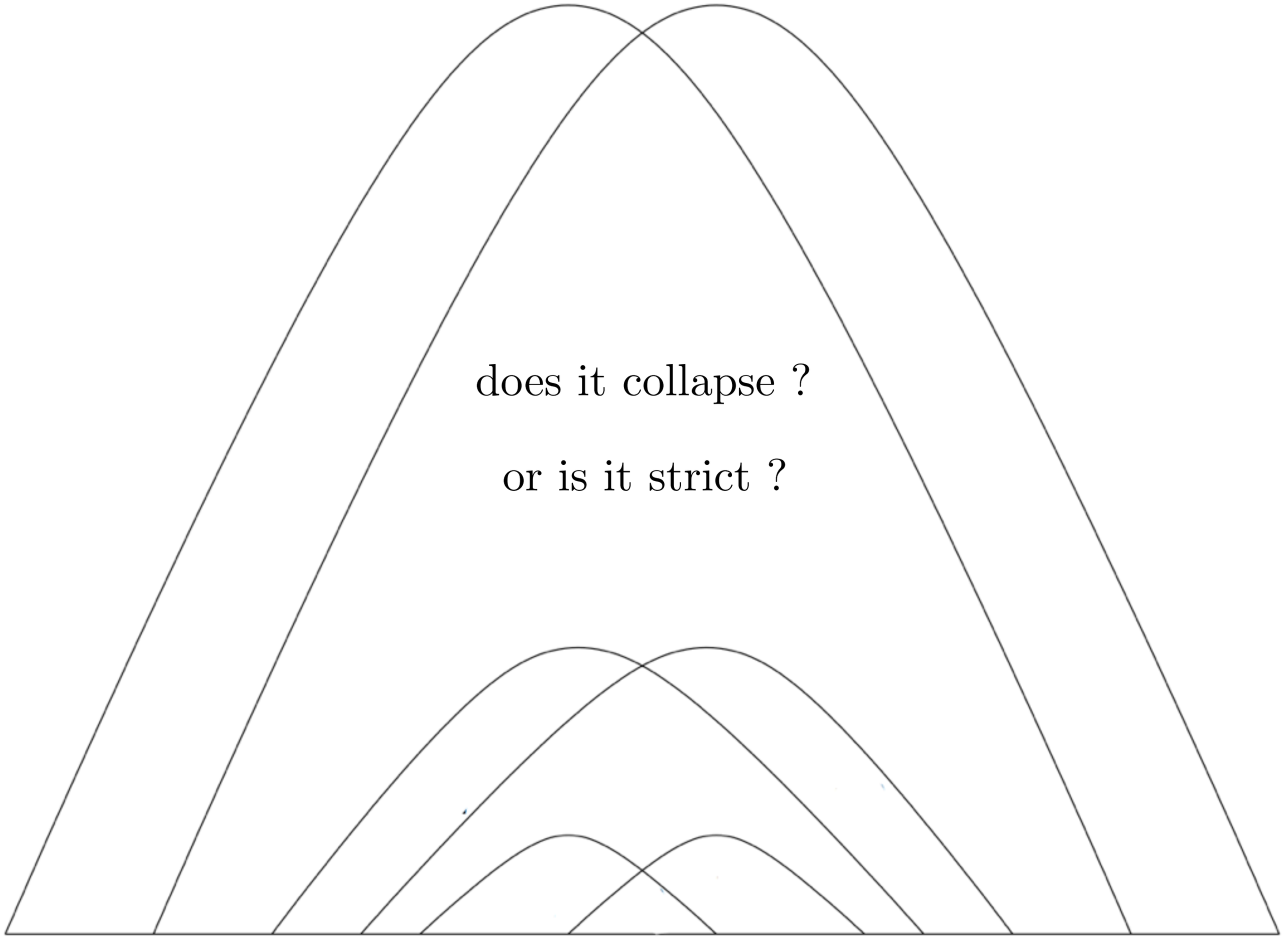
$$Borel = \Delta^1_1$$

The projective hierarchy

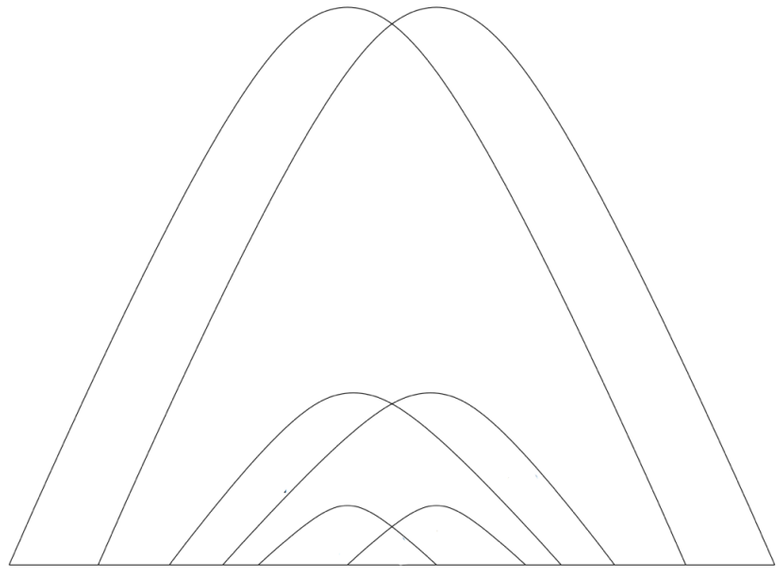


does it collapse ?

or is it strict ?



universal sets $\oplus$ diagonalization



Wadge Games

AD and large cardinals

Theorem (Martin 75)

$$ZF + DC \vdash Det(Borel)$$

Theorem (Martin 69)

$$ZF + DC \vdash 0^\sharp \ exists \implies Det(\Sigma^1_1)$$

Theorem (Harrington 78)

$$ZF + DC \vdash 0^\sharp \ exists \impliedby Det(\Sigma^1_1)$$

Theorem

$$ZF + DC \vdash Det(\mathbf{\Sigma^1_1}) \iff \forall a \in \mathbb{R} \ a^\sharp \ exists$$

AD and large cardinals

Theorem (Martin-Steel 88)

$ZF + DC \vdash$ there are n Woodin cardinals + a measurable above $\Longrightarrow Det(\mathbf{\Sigma^1_{n+1}})$

- $Det(\mathbf{\Delta^1_{n+1}}) \Longrightarrow$ there exists an inner model with n Woodin cardinals

Theorem (Martin-Steel-Woodin)

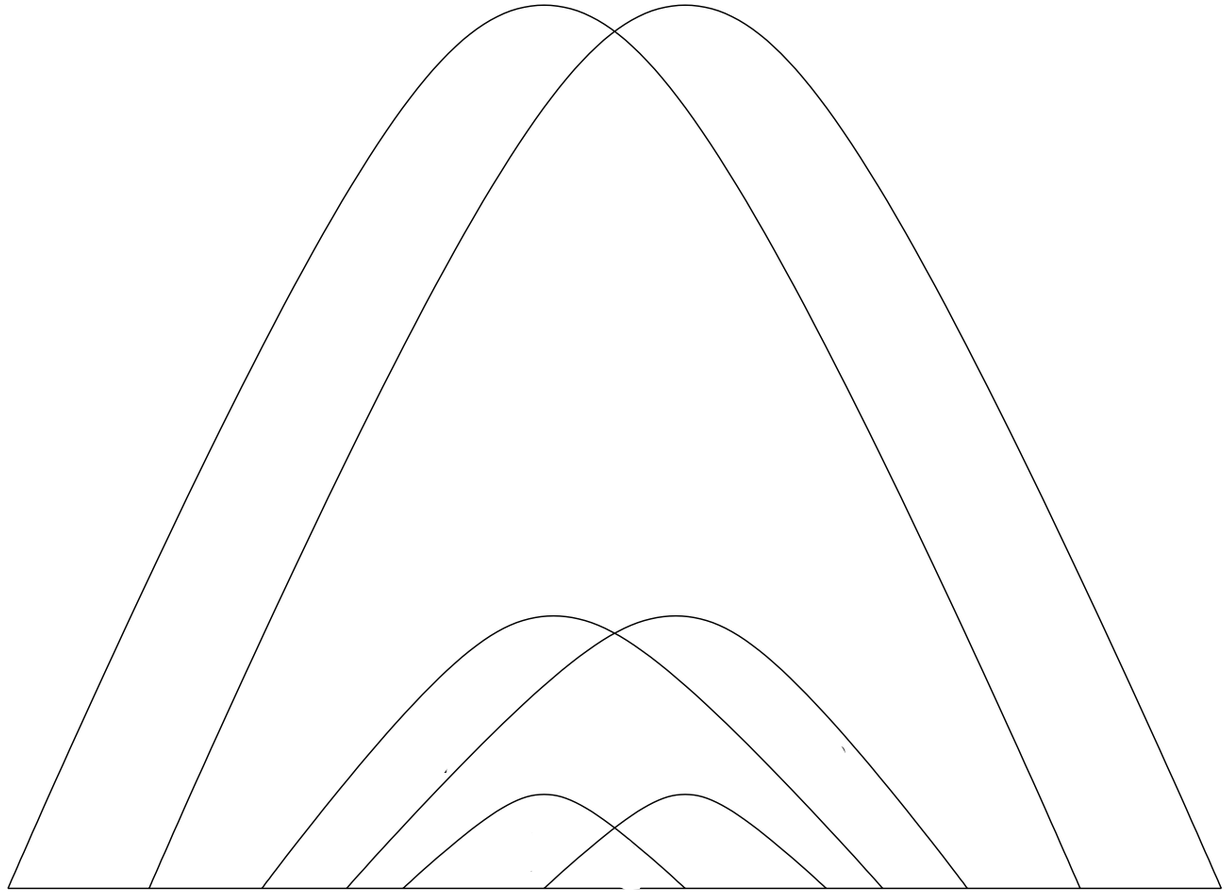
There exist infinitely many Woodin cardinals and a measurable above $\Longrightarrow (AD)^{L(\mathbb{R})}$

AD and large cardinals

Theorem (Woodin)

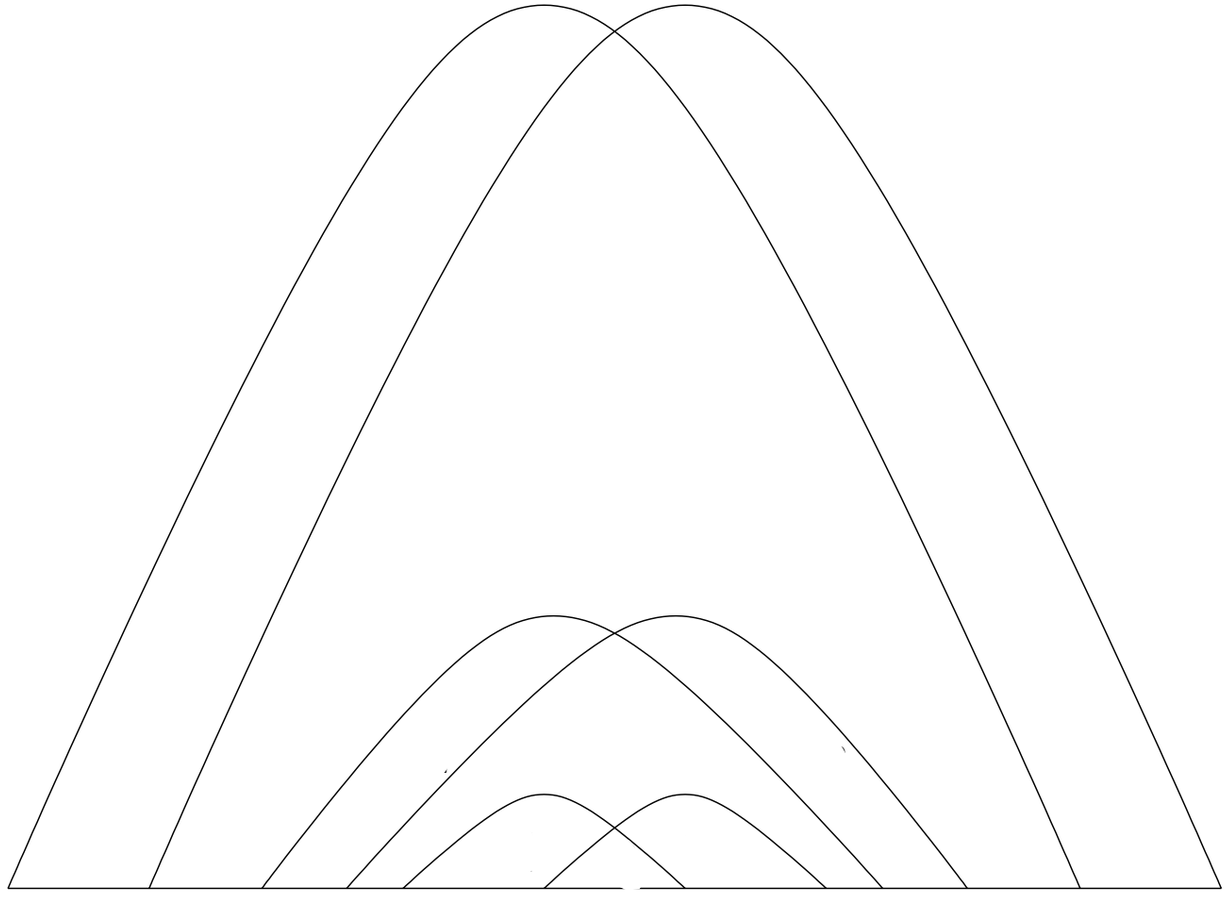
The following are equivalent:

- $ZFC+$ *"there exist infinitely many Woodin cardinals"*
- $ZF + AD$



UNIL | Université de Lausanne

Why should we care ?



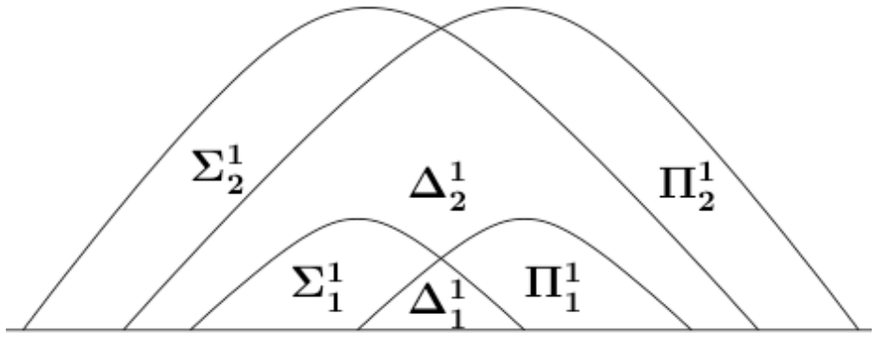
UNIL | Université de Lausanne

- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$ $\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$

$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$

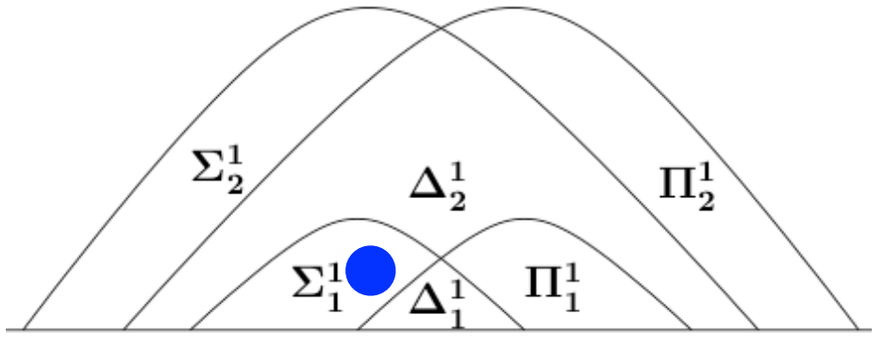
- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$ $\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$

$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$



- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$ $\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$

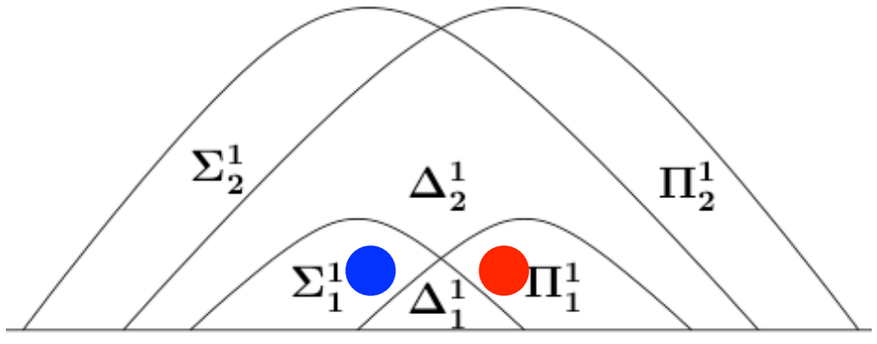
$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$

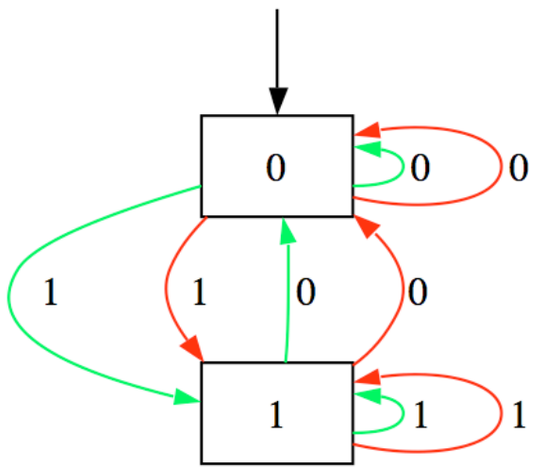


- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$

$$\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$$

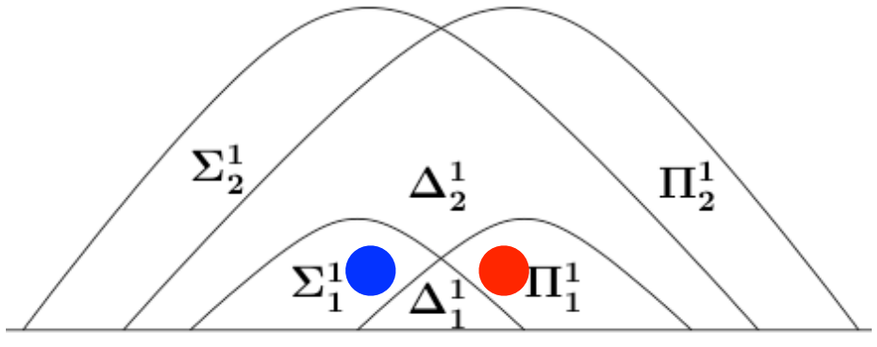
$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$





- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$ $\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$

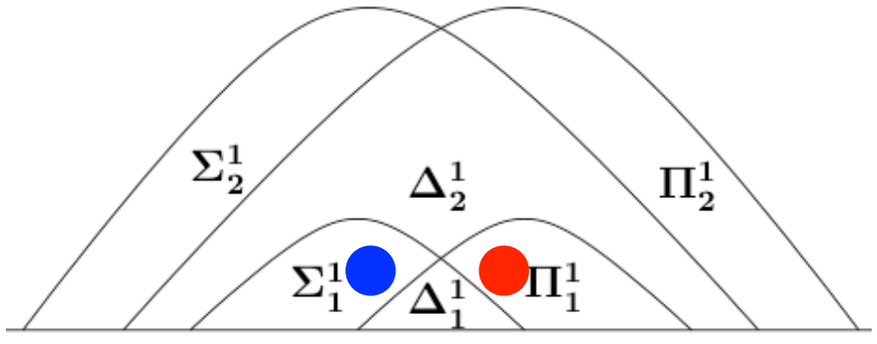
$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$



- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$

$$\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$$

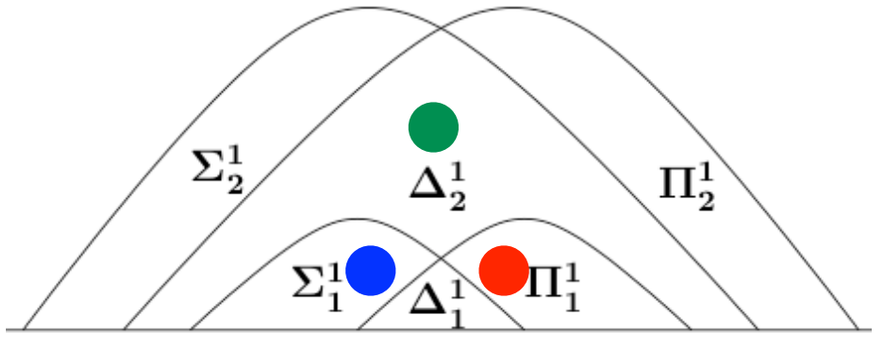
$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$



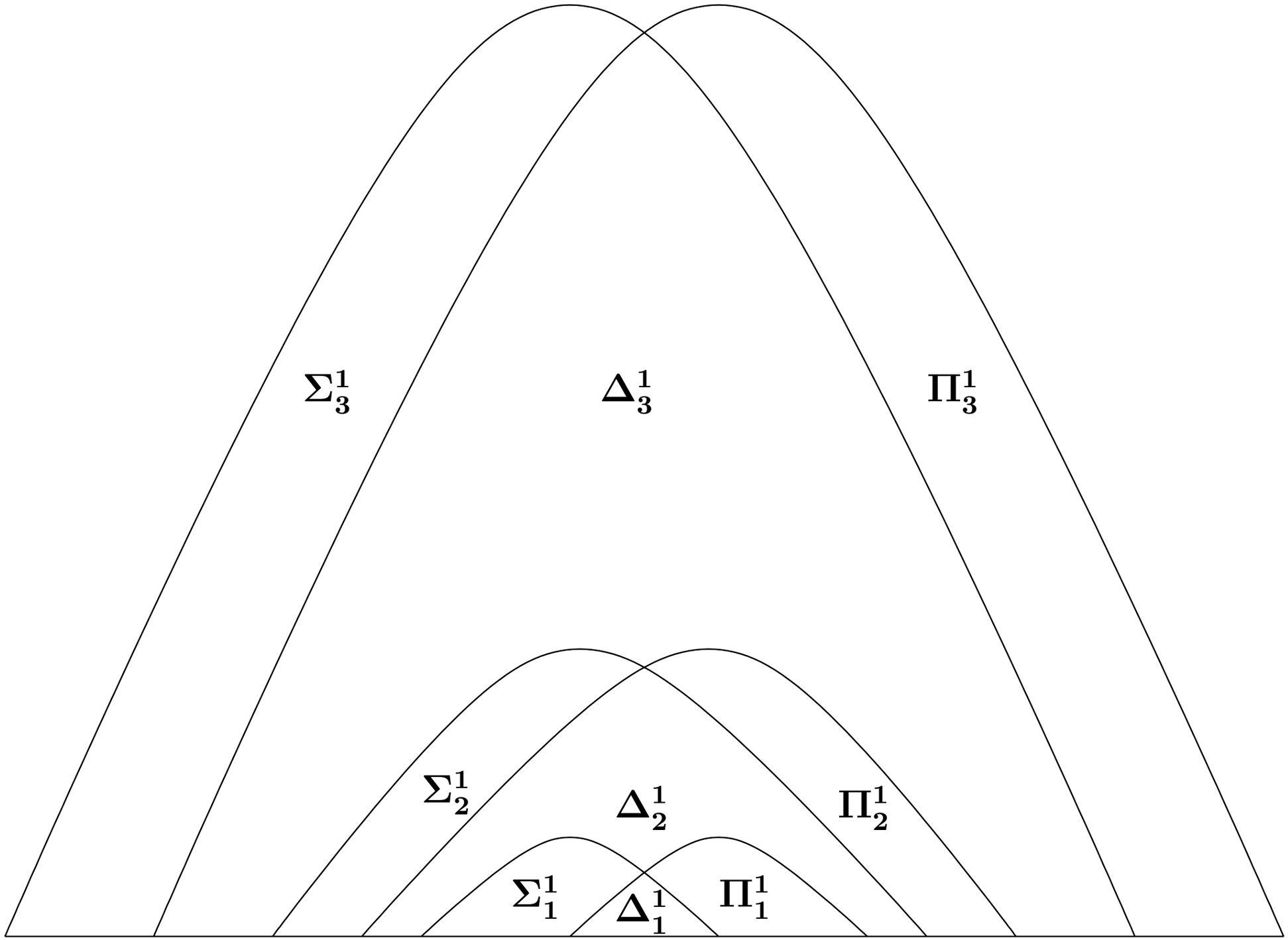
- Non. Det. Turing Machine $\leftrightsquigarrow$ $\exists^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Sigma}_1^1$
- Det. Tree Automaton $\leftrightsquigarrow$ $\forall^{\mathbb{R}}\mathcal{B}$ $\leftrightsquigarrow$ $\mathbf{\Pi}_1^1$
- Alt. Tree Automaton $\leftrightsquigarrow$

$$\begin{cases} \exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B} \leftrightsquigarrow \mathbf{\Sigma}_2^1 \\ \neg\left(\exists^{\mathbb{R}}\ \forall^{\mathbb{R}}\mathcal{B}\right) \leftrightsquigarrow \mathbf{\Pi}_2^1 \end{cases}$$

$\leftrightsquigarrow$ $\mathbf{\Delta}_2^1$



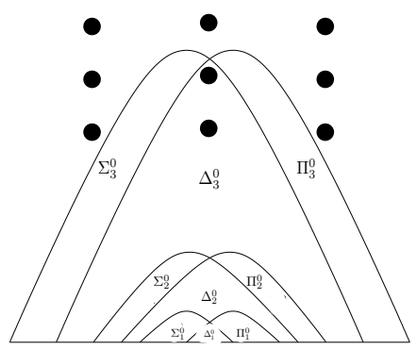
The projective hierarchy



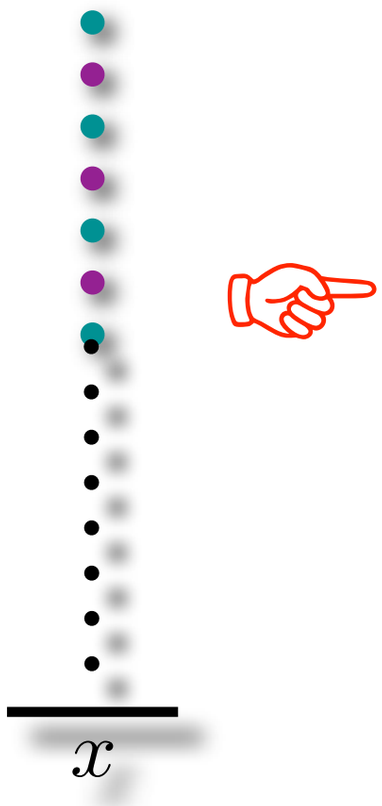


An example: evaluation game for 1st order logic

non determined game

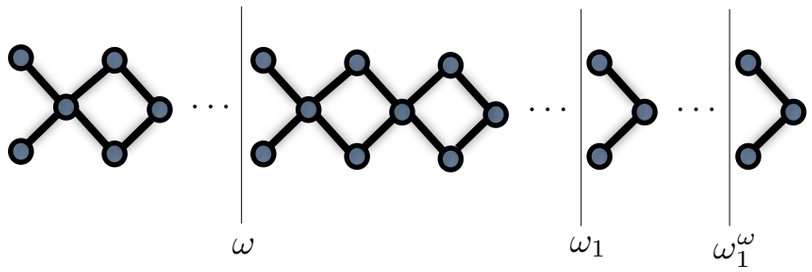


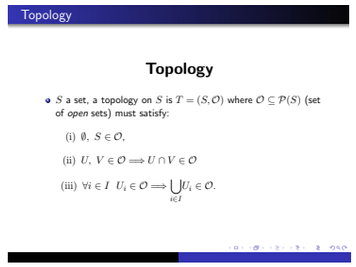
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

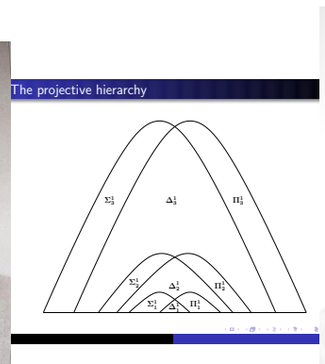
Games that construct mappings



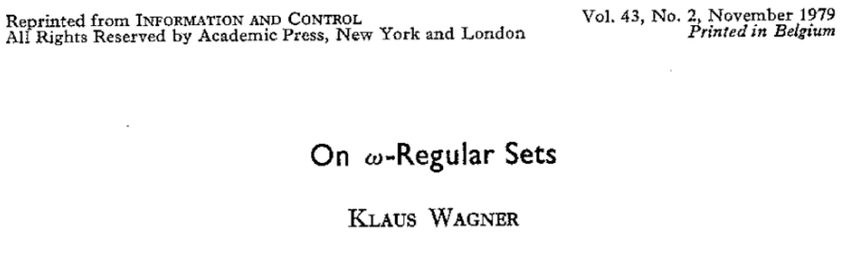










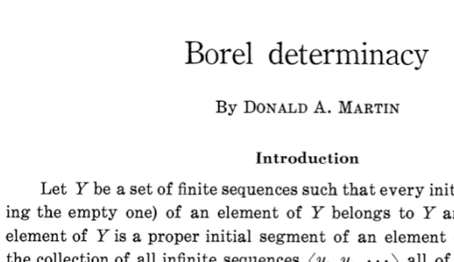


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of



Games that construct mappings

Completeness

Completeness

inside the arithmetical hierarchy

a problem (language, set) A is complete for a class Γ if

$A \in \Gamma$

$\forall B \in \Gamma \quad B \leq A$

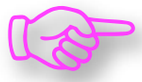
☞ $\forall \mathcal{B} \in \Gamma \quad \mathcal{B} \leq \mathcal{A}$

☞ $\forall B \in \Gamma \quad B \leq A$

☞ there exists a mapping f (of some kind...) s.t.

$$\forall x \ (x \in B \Rightarrow f(x) \in A)$$

$$\forall x \ (x \notin B \Rightarrow f(x) \notin A)$$

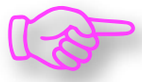


$\forall B \in \Gamma \quad B \leq A$



there exists a mapping f (of some kind...) s.t.

$\forall x \quad (x \in B \Leftrightarrow f(x) \in A)$



$$\forall B \in \Gamma \quad B \leq A$$



there exists a mapping f (of some kind...) s.t.

$$f^{-1} B = A$$



Games that construct mappings

$$\mathbb{G}(f)$$

f: any mapping from

infinite words to infinite words

$$x \longmapsto f(x)$$

$$\mathbb{G}(f)$$

I $I\!I$

$$\mathbb{G}(f)$$

I II



$$\mathbb{G}(f)$$

I II



$$\mathbb{G}(f)$$



$$\mathbb{G}(f)$$



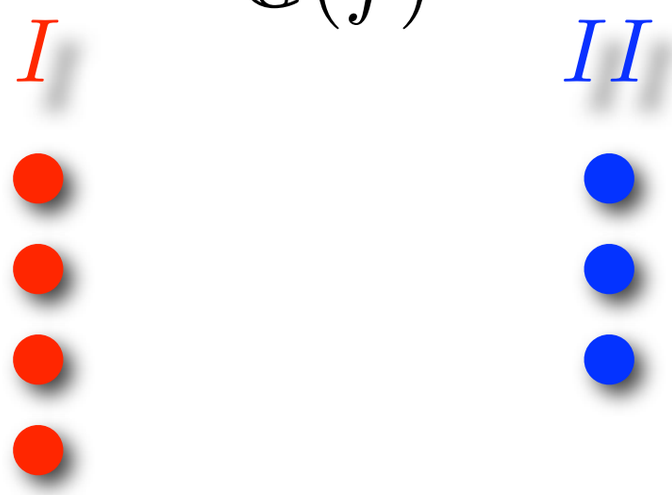
$$\mathbb{G}(f)$$



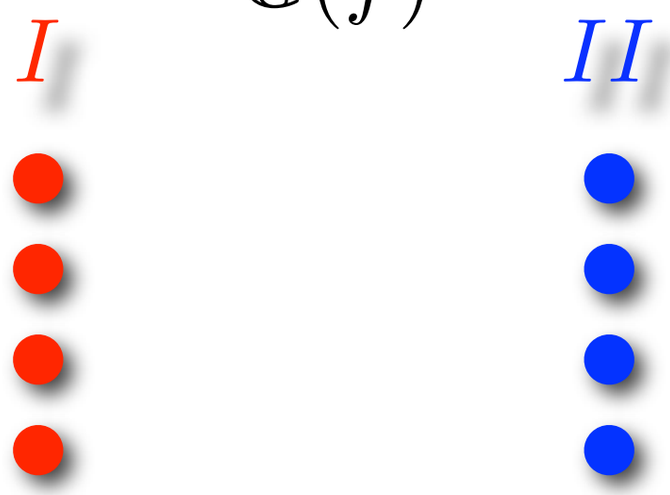
$$\mathbb{G}(f)$$



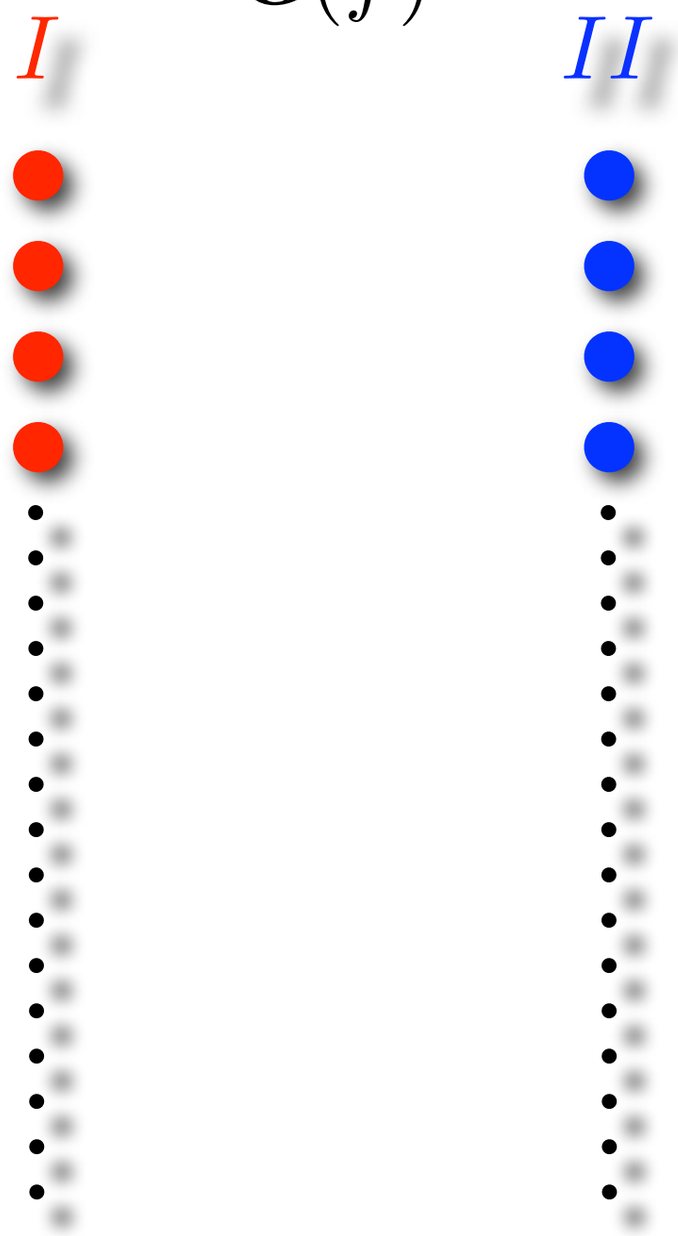
$$\mathbb{G}(f)$$



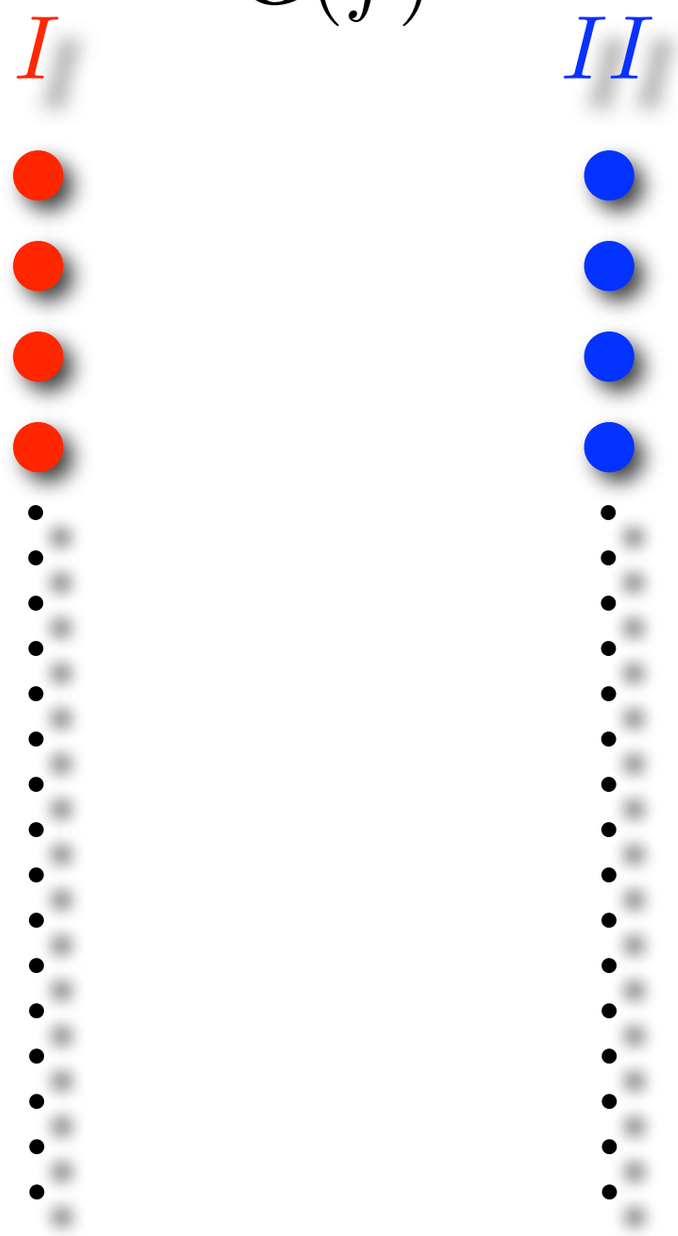
$$\mathbb{G}(f)$$

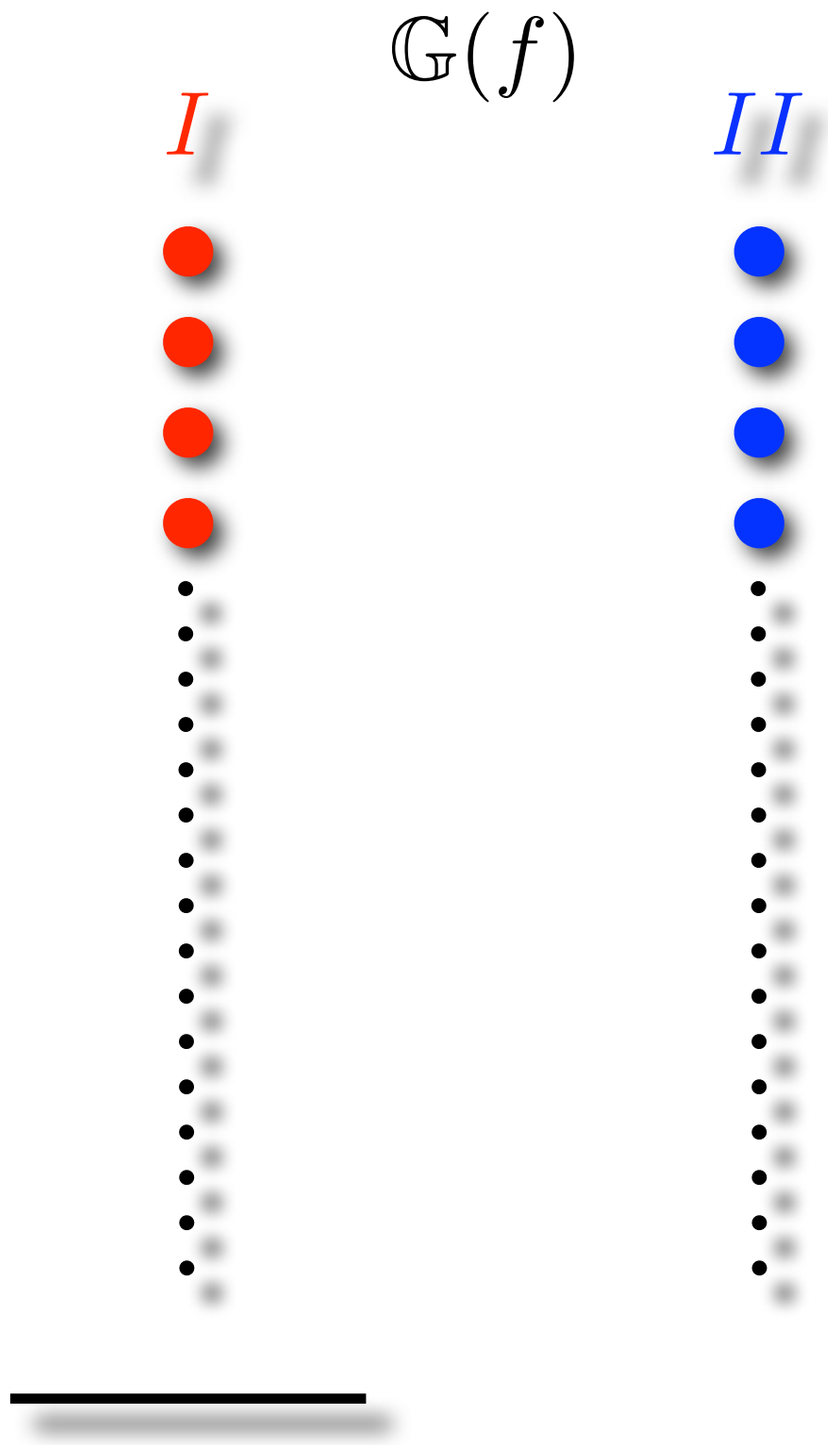


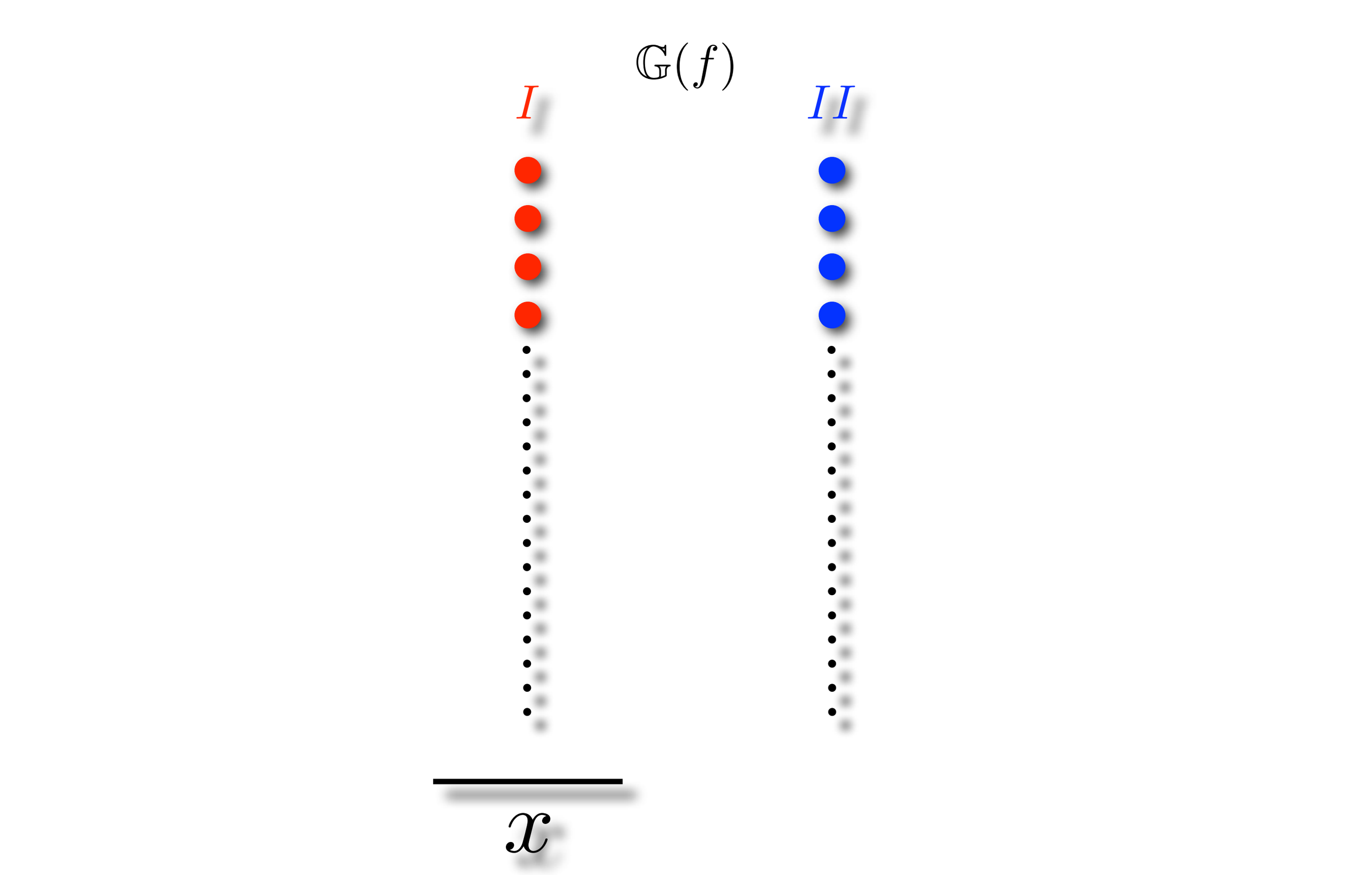
$\mathbb{G}(f)$

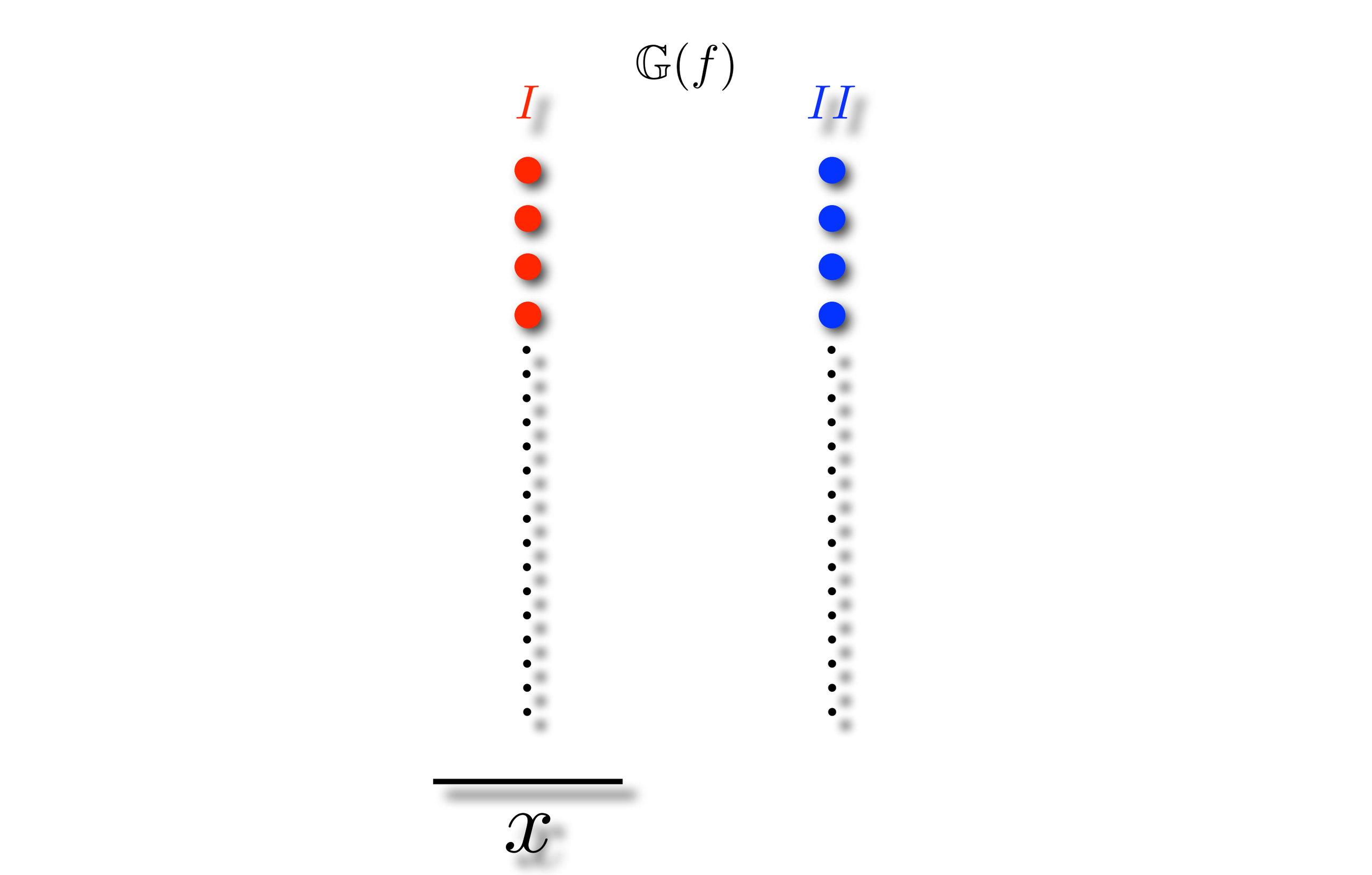


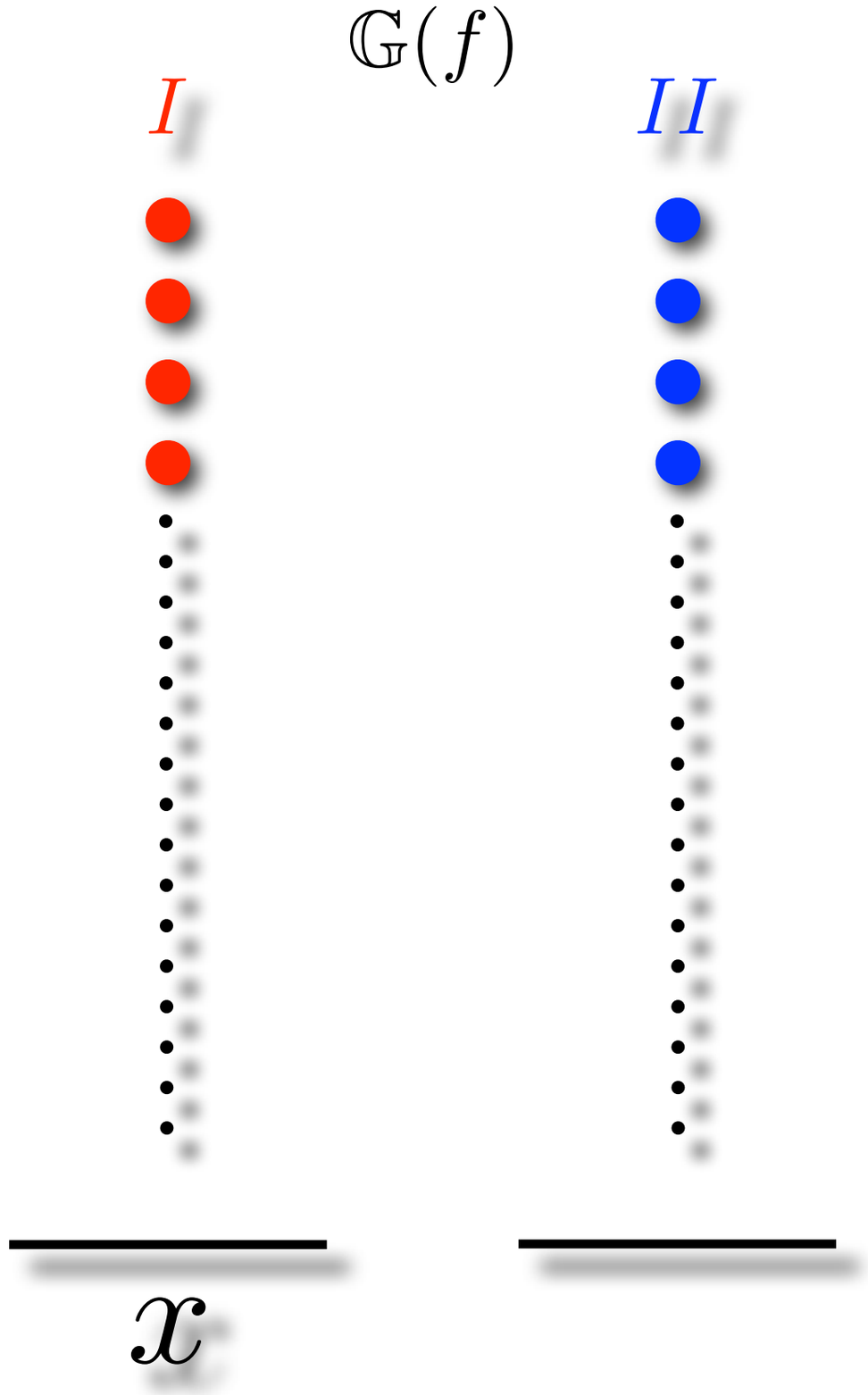
$\mathbb{G}(f)$

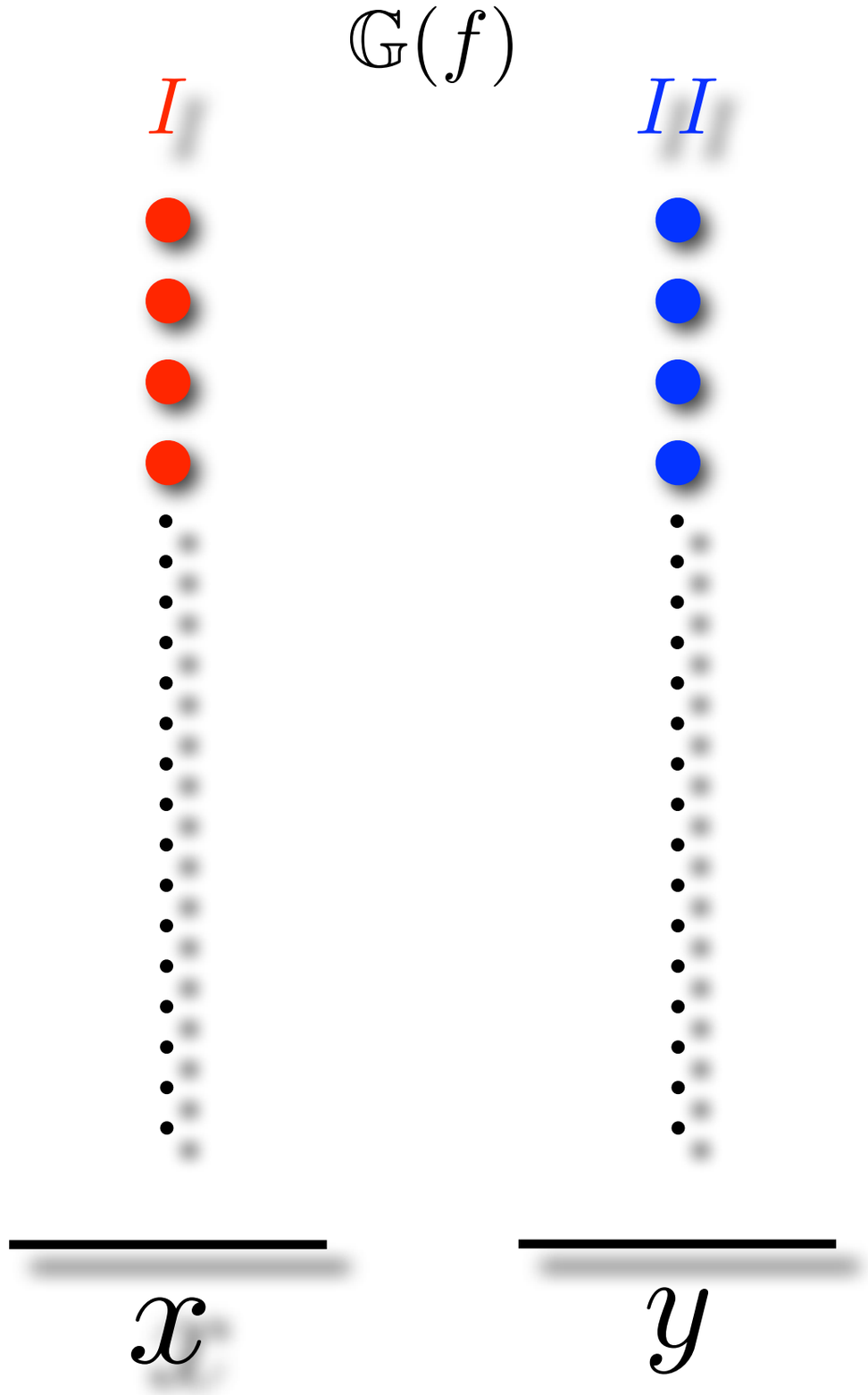



$\mathbb{G}(f)$
I
II

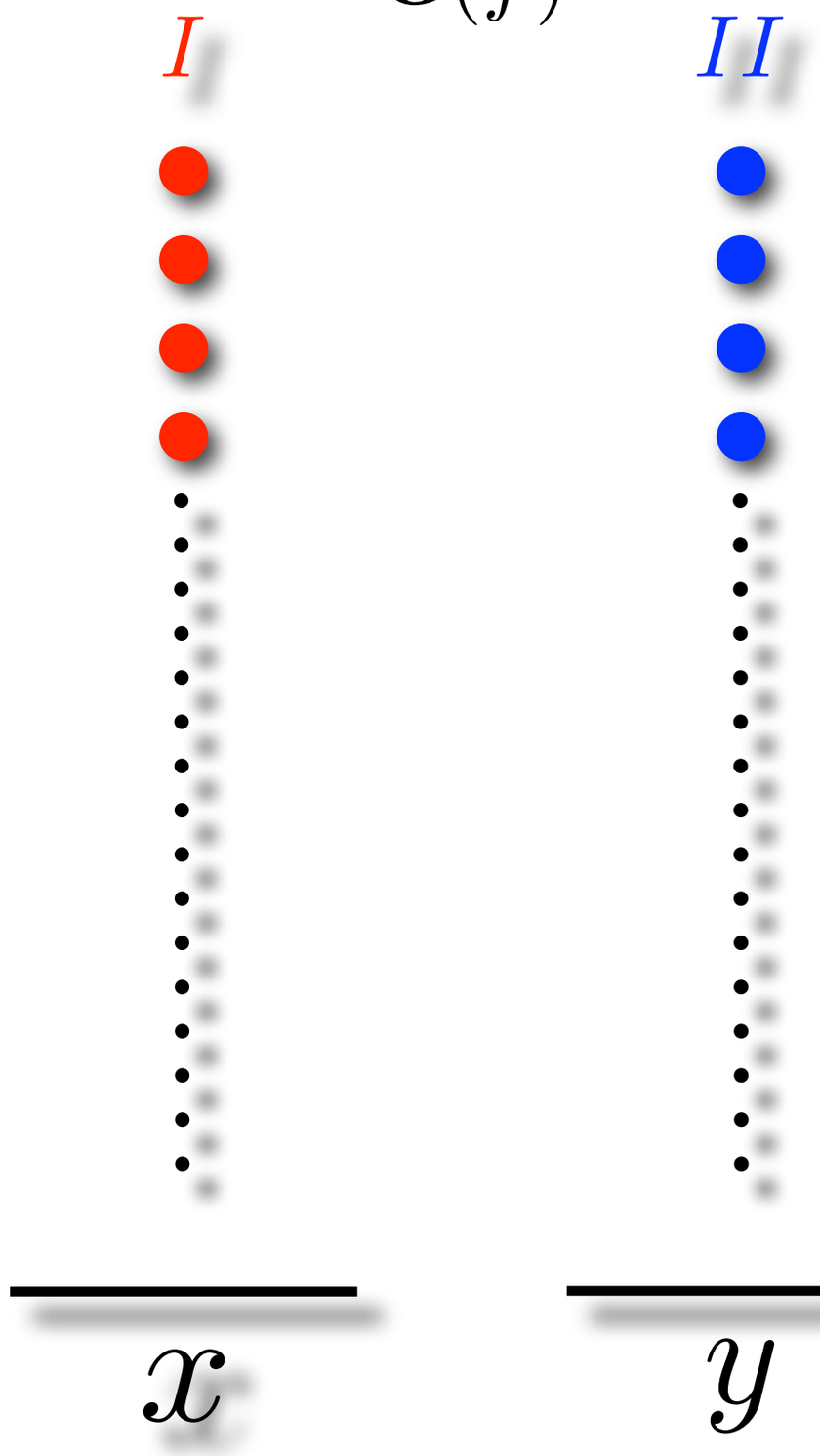

$\mathbb{G}(f)$
I
II
x


$\mathbb{G}(f)$
I
II
x


$\mathbb{G}(f)$
I
II
x


$\mathbb{G}(f)$
I
II
x
y

$\mathbb{G}(f)$



II wins $\iff f(x) = y$

$$\mathbb{G}_{skip}(f)$$

f: any mapping from

infinite words to infinite words

$$x \longmapsto f(x)$$

$$\mathbb{G}_{skip}(f)$$

I II

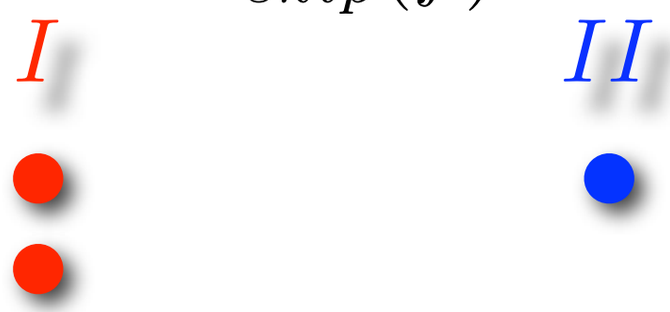
$$\mathbb{G}_{skip}(f)$$



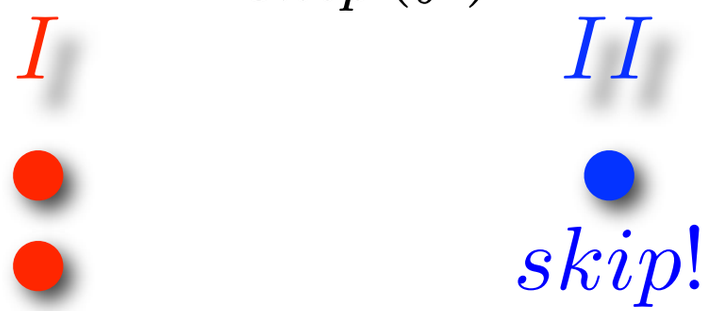
$$\mathbb{G}_{skip}(f)$$



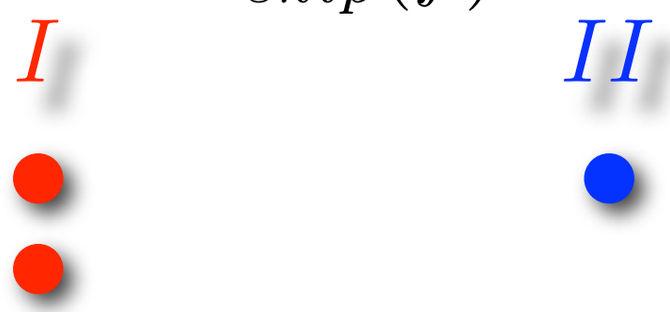
$$\mathbb{G}_{skip}(f)$$



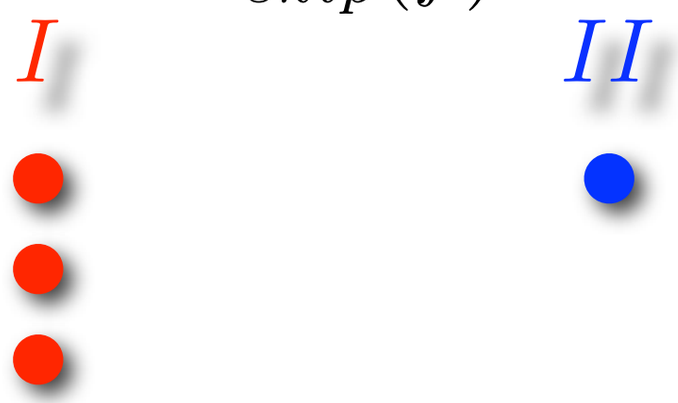
$$\mathbb{G}_{skip}(f)$$



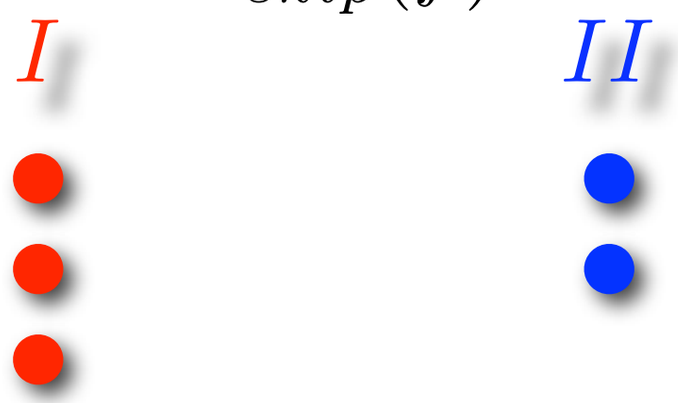
$$\mathbb{G}_{skip}(f)$$



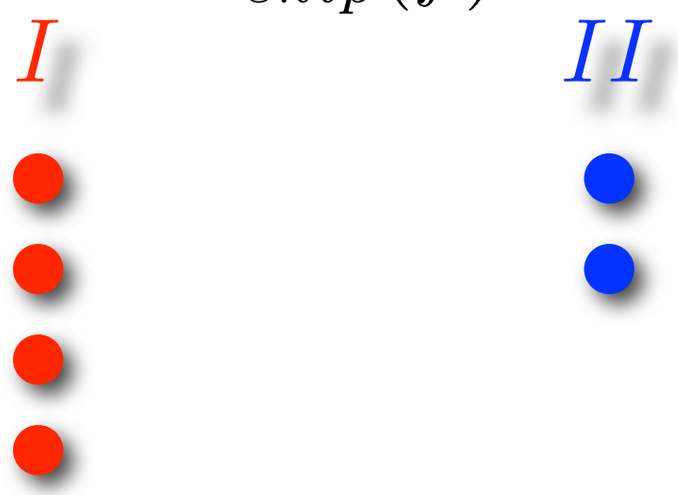
$$\mathbb{G}_{skip}(f)$$



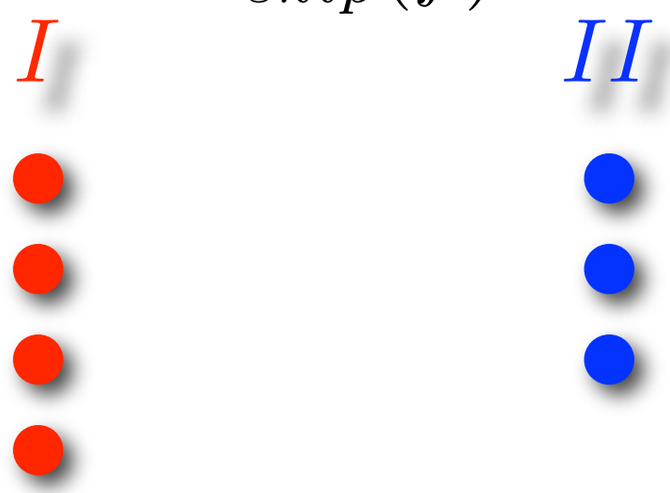
$$\mathbb{G}_{skip}(f)$$



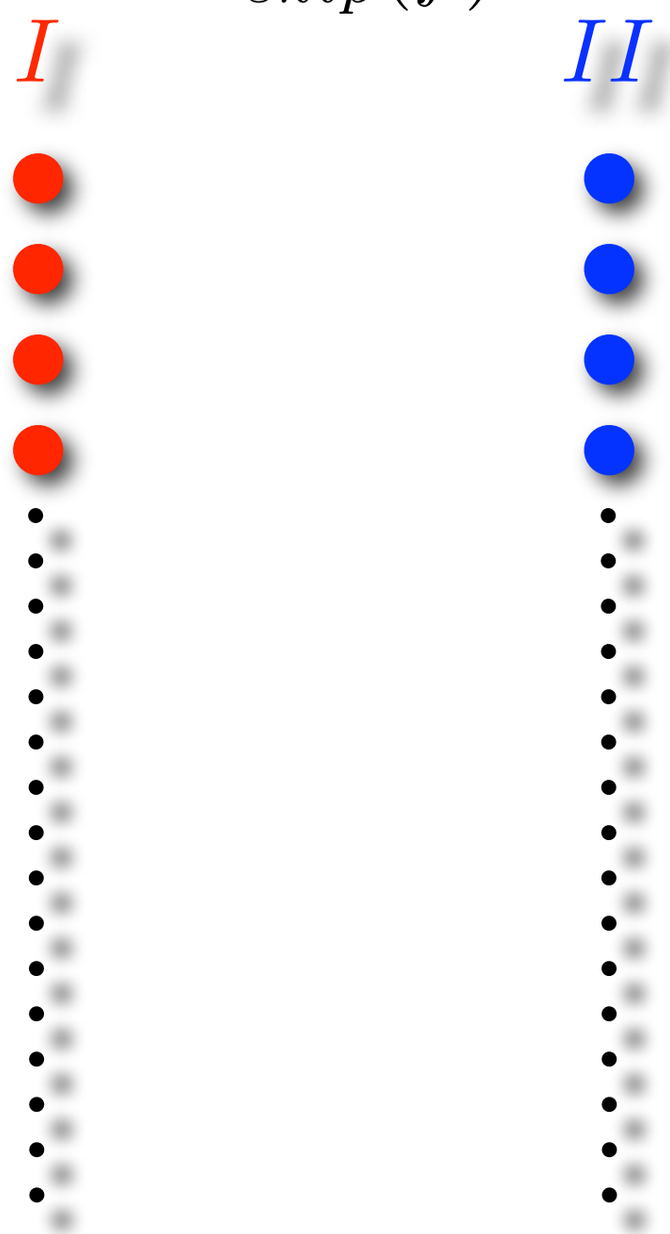
$$\mathbb{G}_{skip}(f)$$



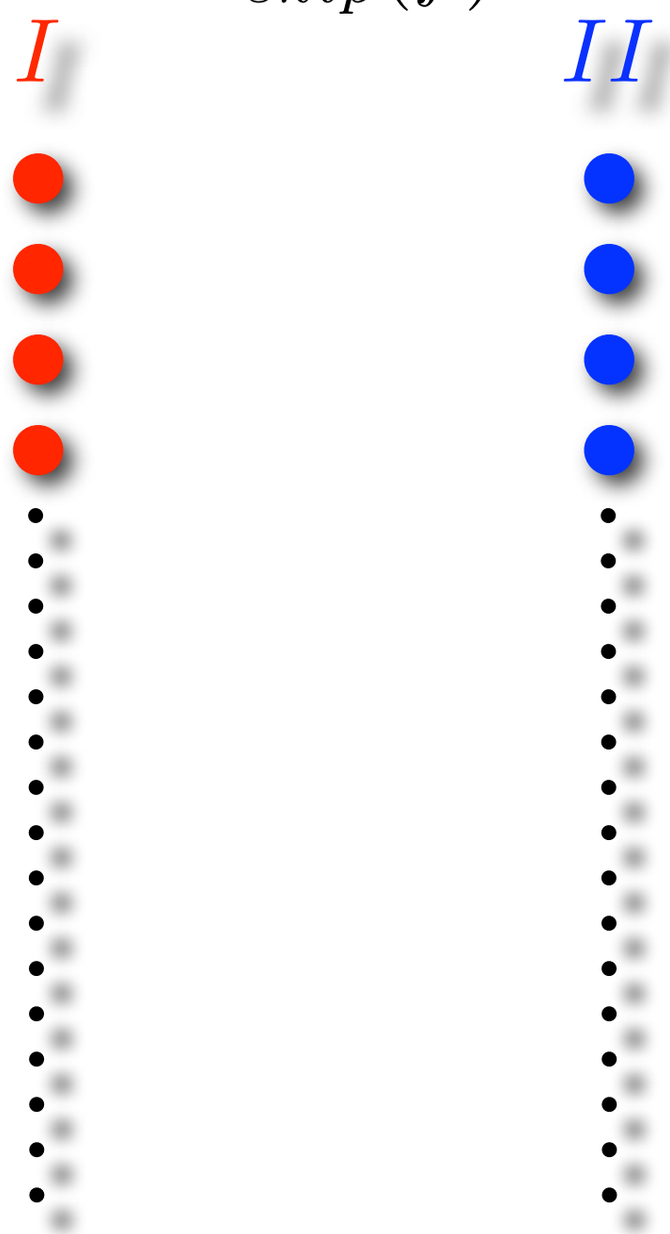
$$\mathbb{G}_{skip}(f)$$



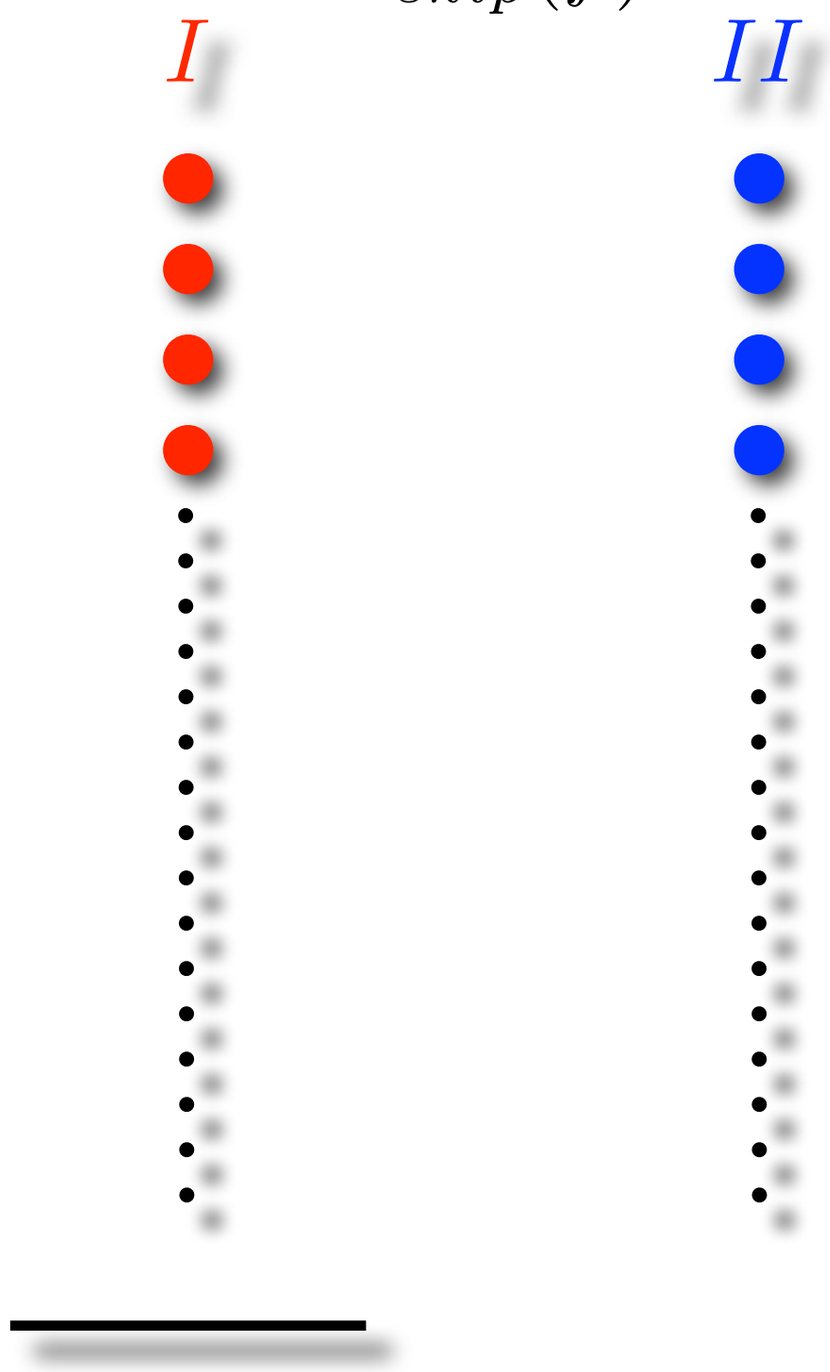
$$\mathbb{G}_{skip}(f)$$



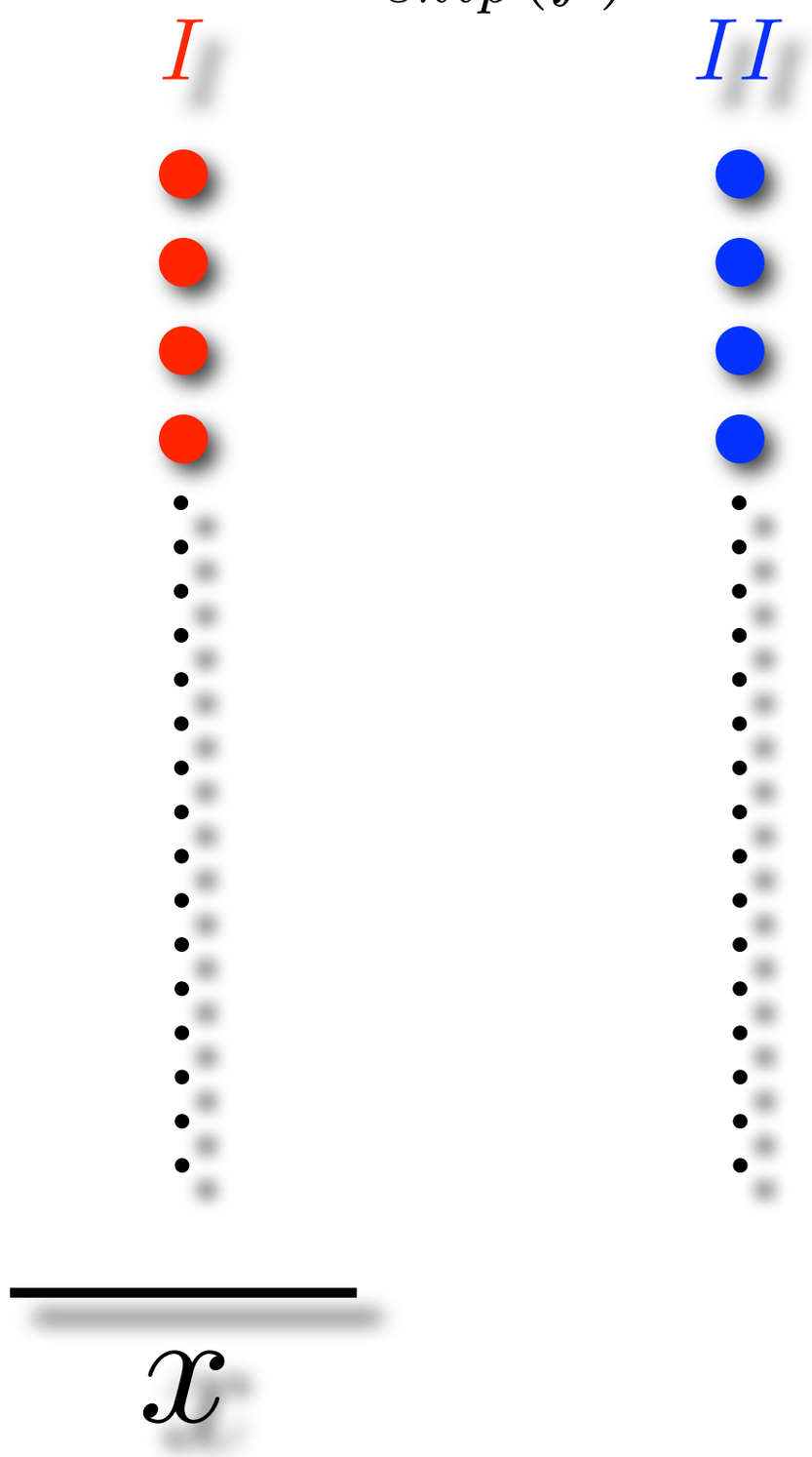
$$\mathbb{G}_{skip}(f)$$



$\mathbb{G}_{skip}(f)$

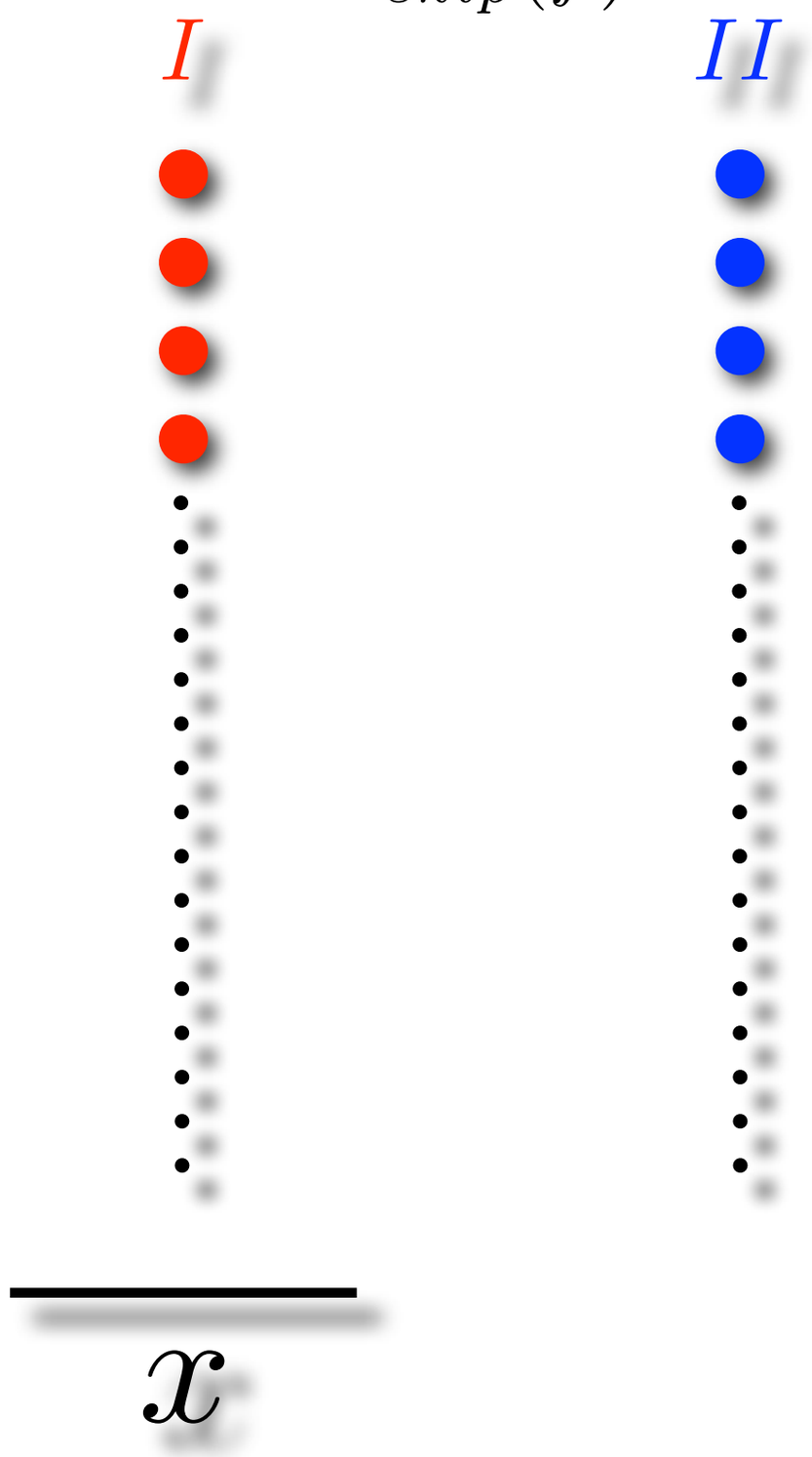


$\mathbb{G}_{skip}(f)$



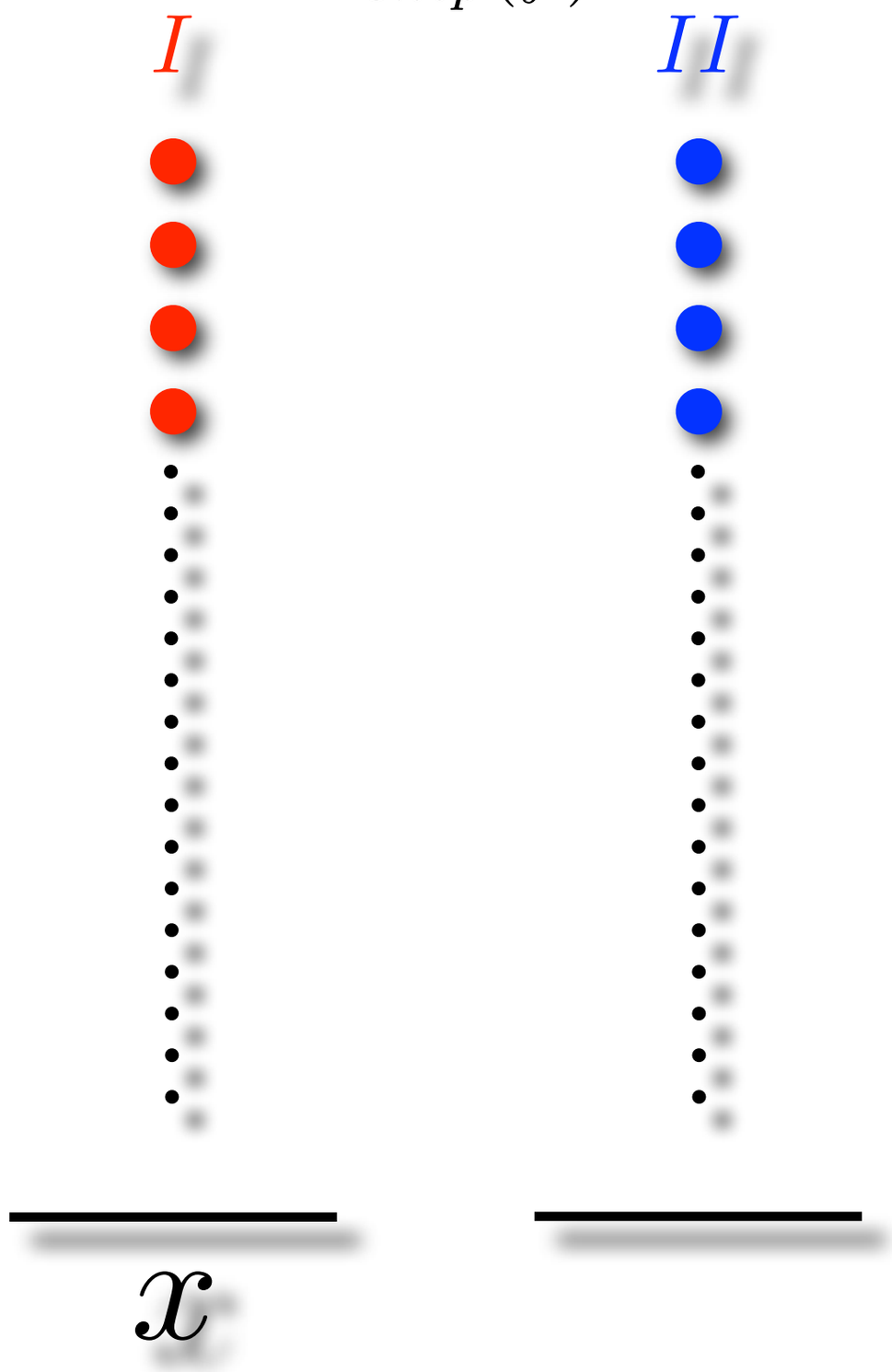
x

$\mathbb{G}_{skip}(f)$

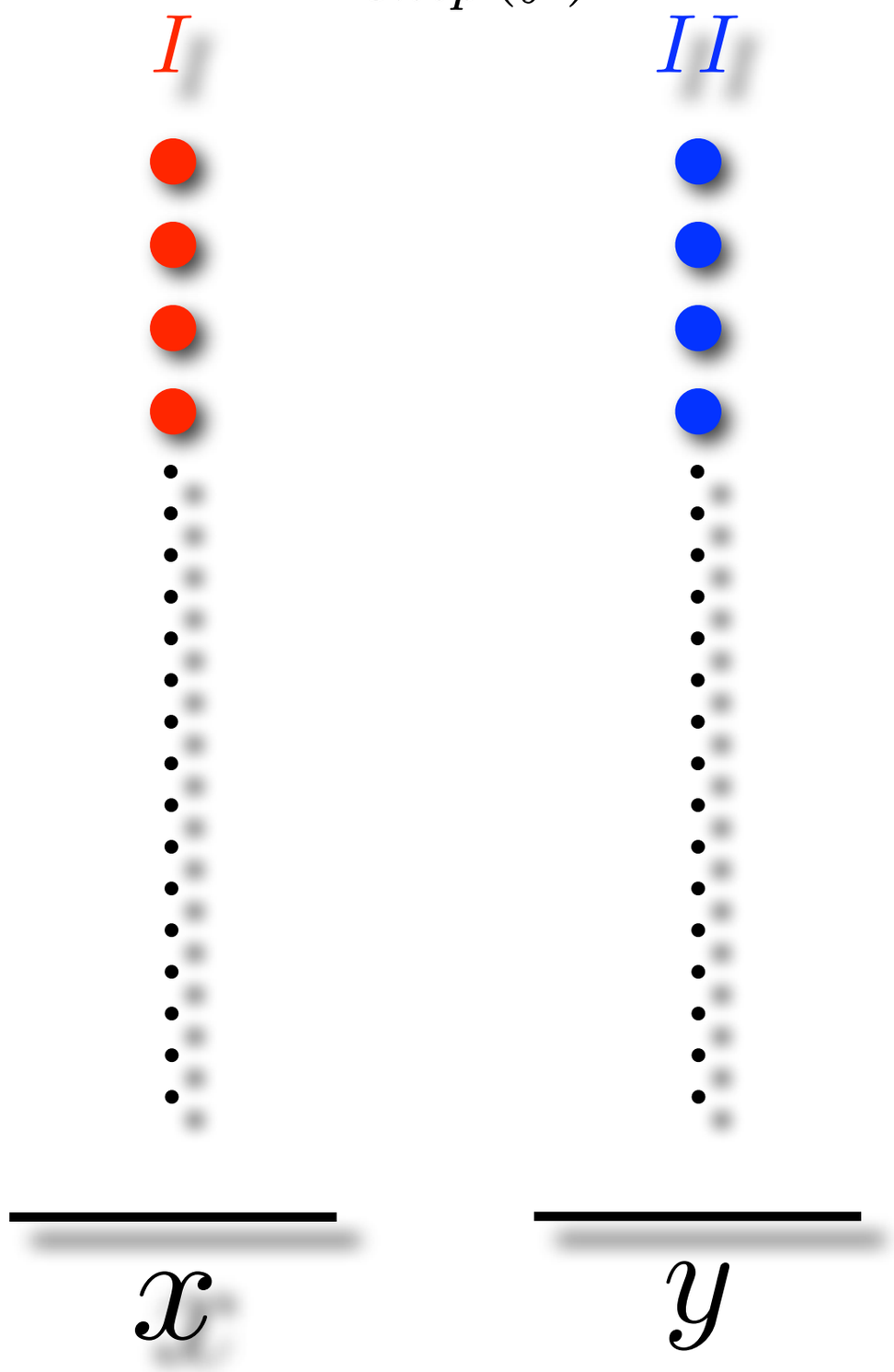


x

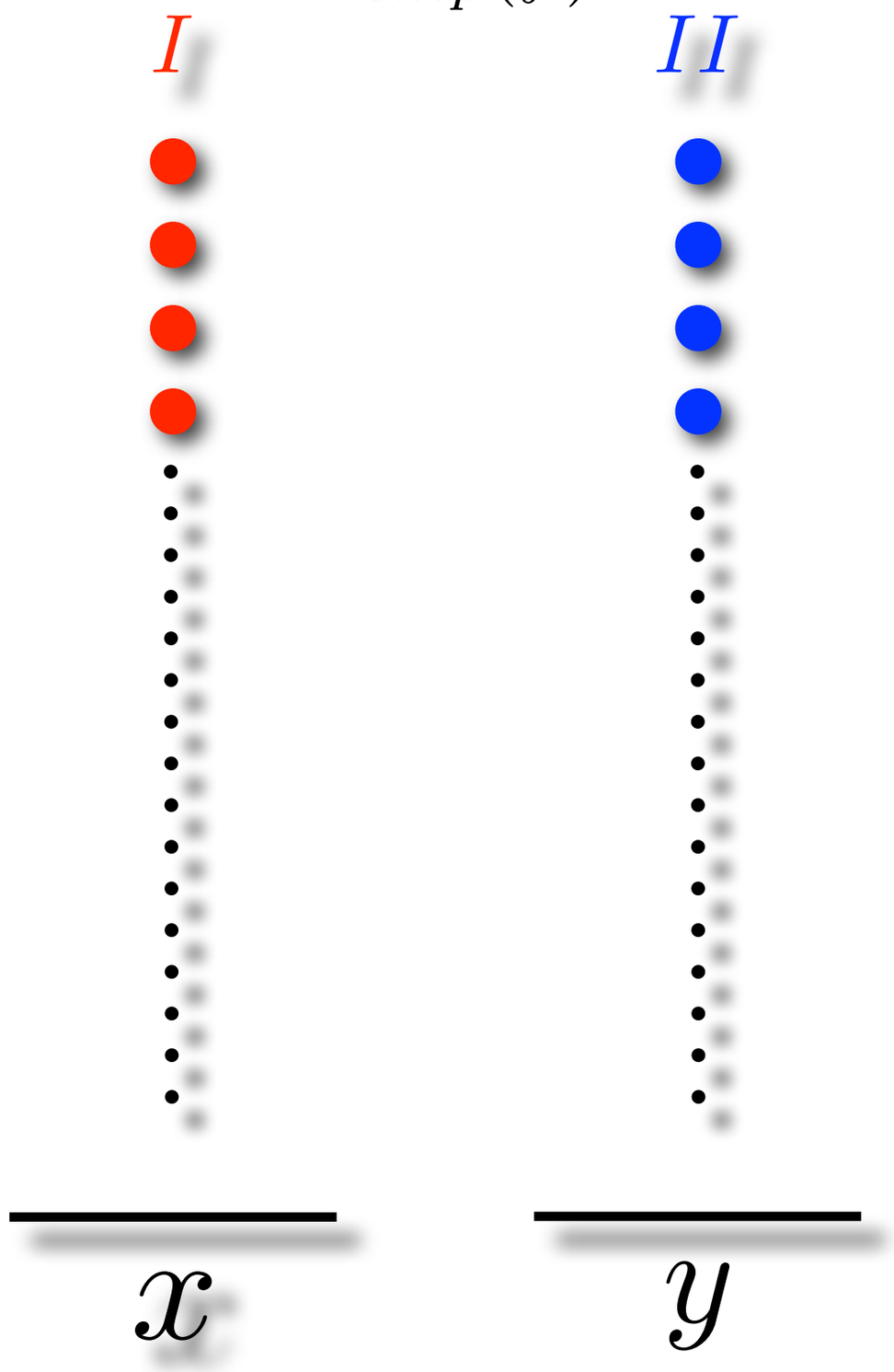
$\mathbb{G}_{skip}(f)$



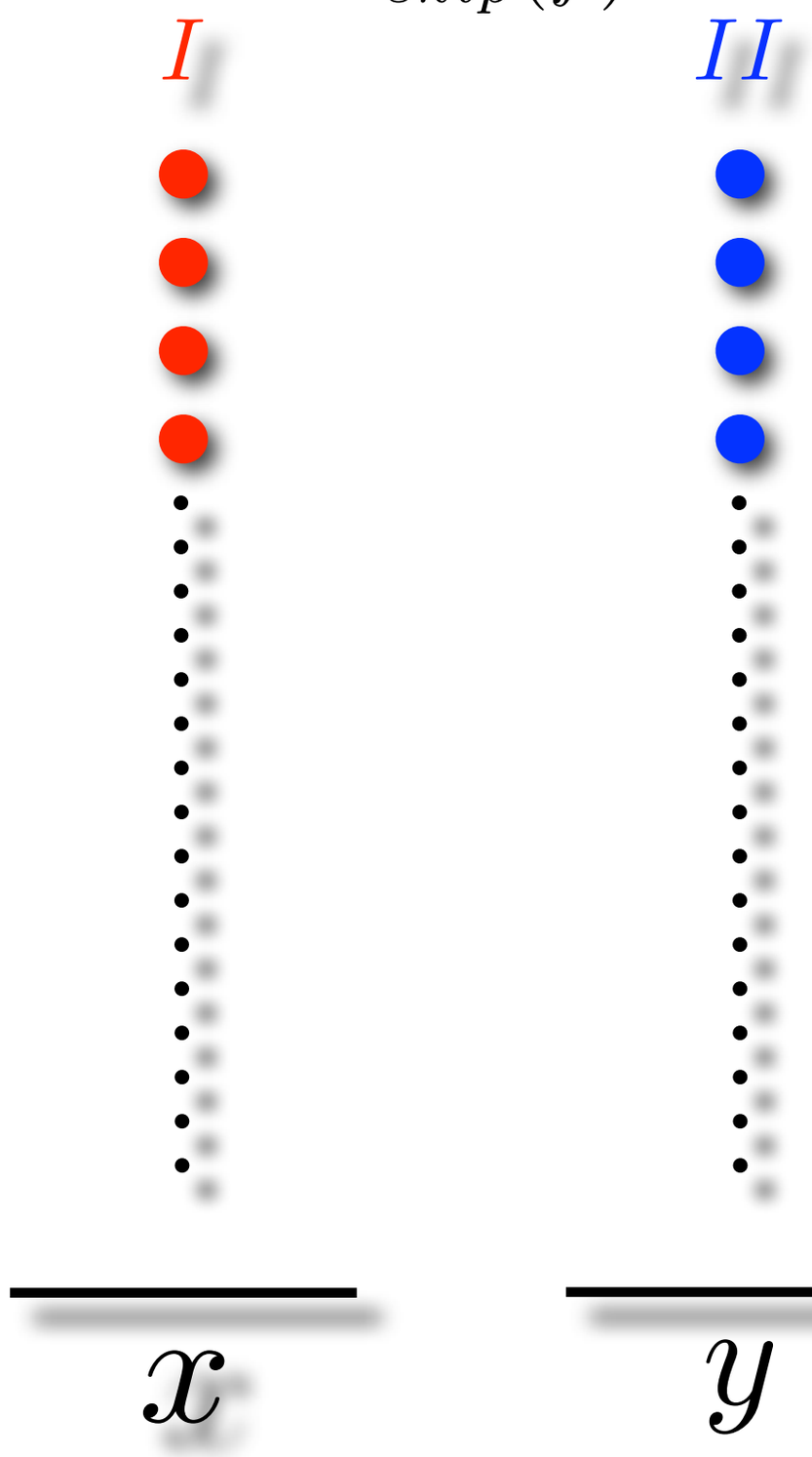
$\mathbb{G}_{skip}(f)$



$\mathbb{G}_{skip}(f)$



$\mathbb{G}_{skip}(f)$



II wins $\iff f(x) = y$

$$\mathbb{G}_{erase}(f)$$

f: any mapping from

infinite words to infinite words

$$x \longmapsto f(x)$$


I
II


I
II


I
II


I
II


I
II

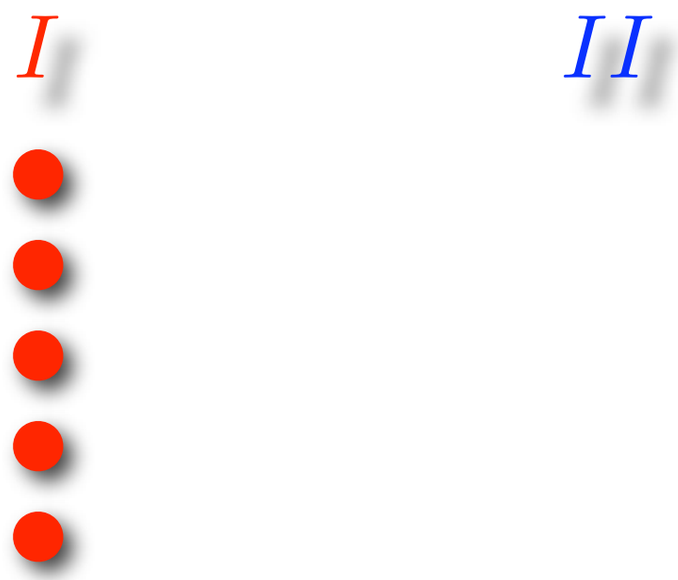

I
II

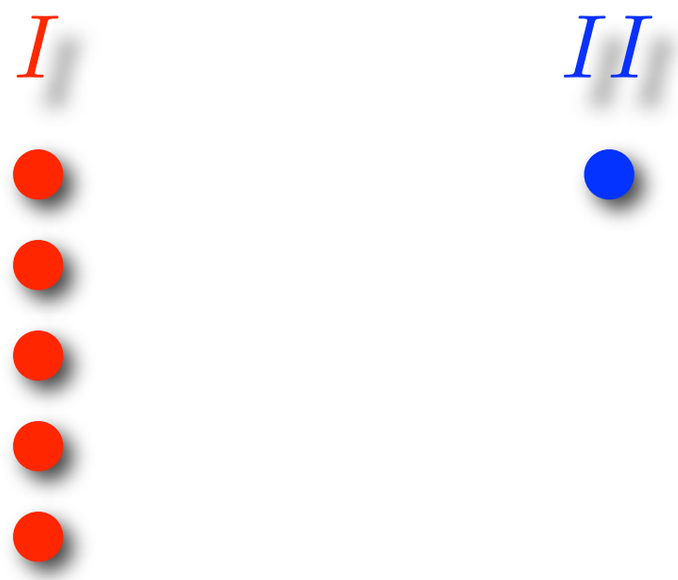

I
II

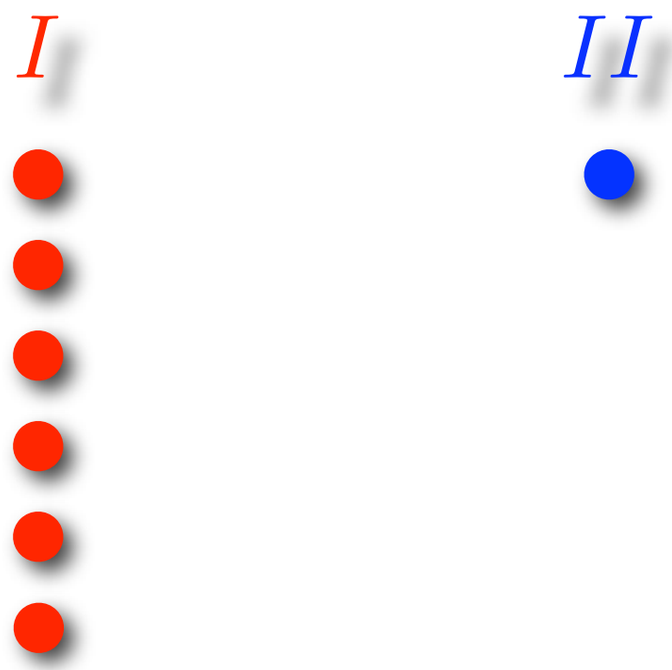

I
II

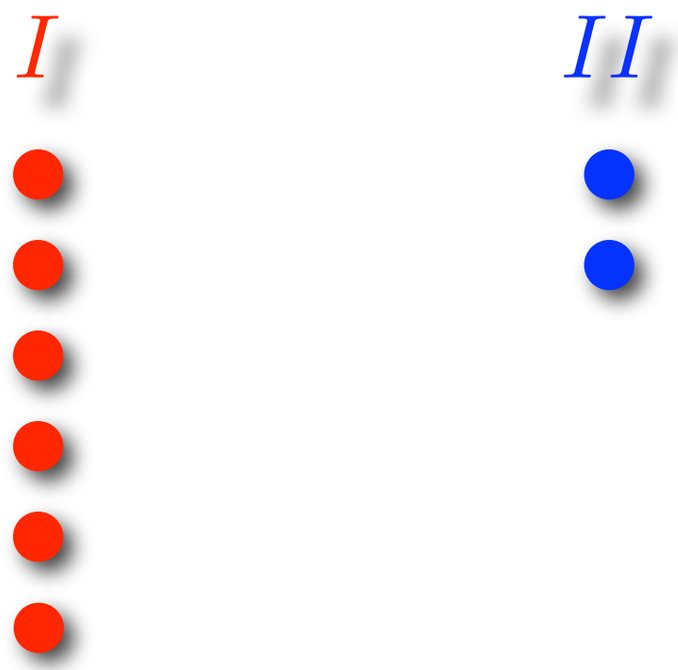

I
II

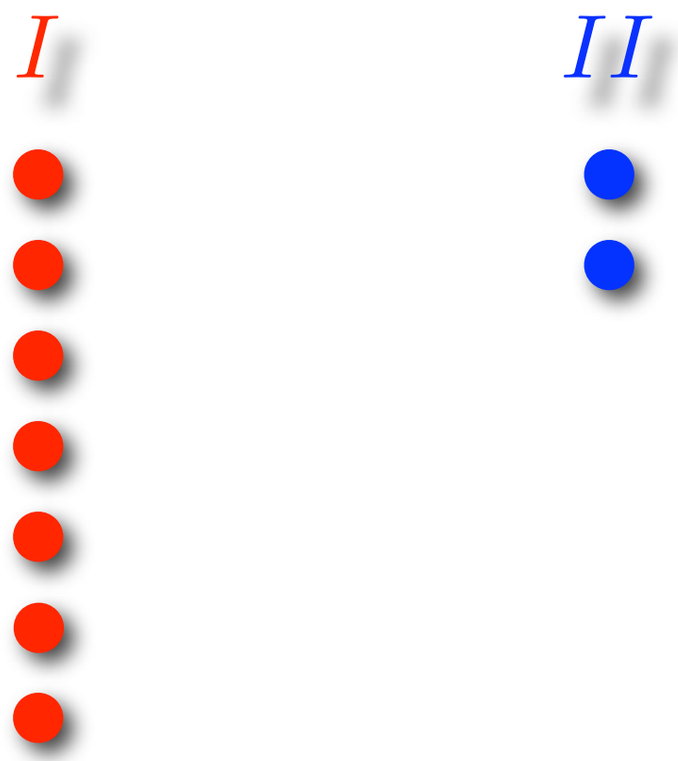

I
II

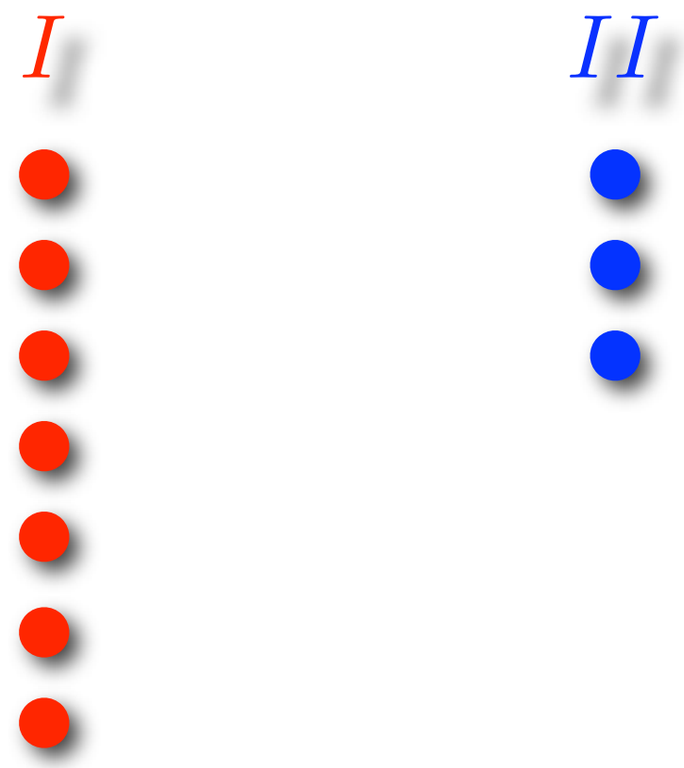

I
II

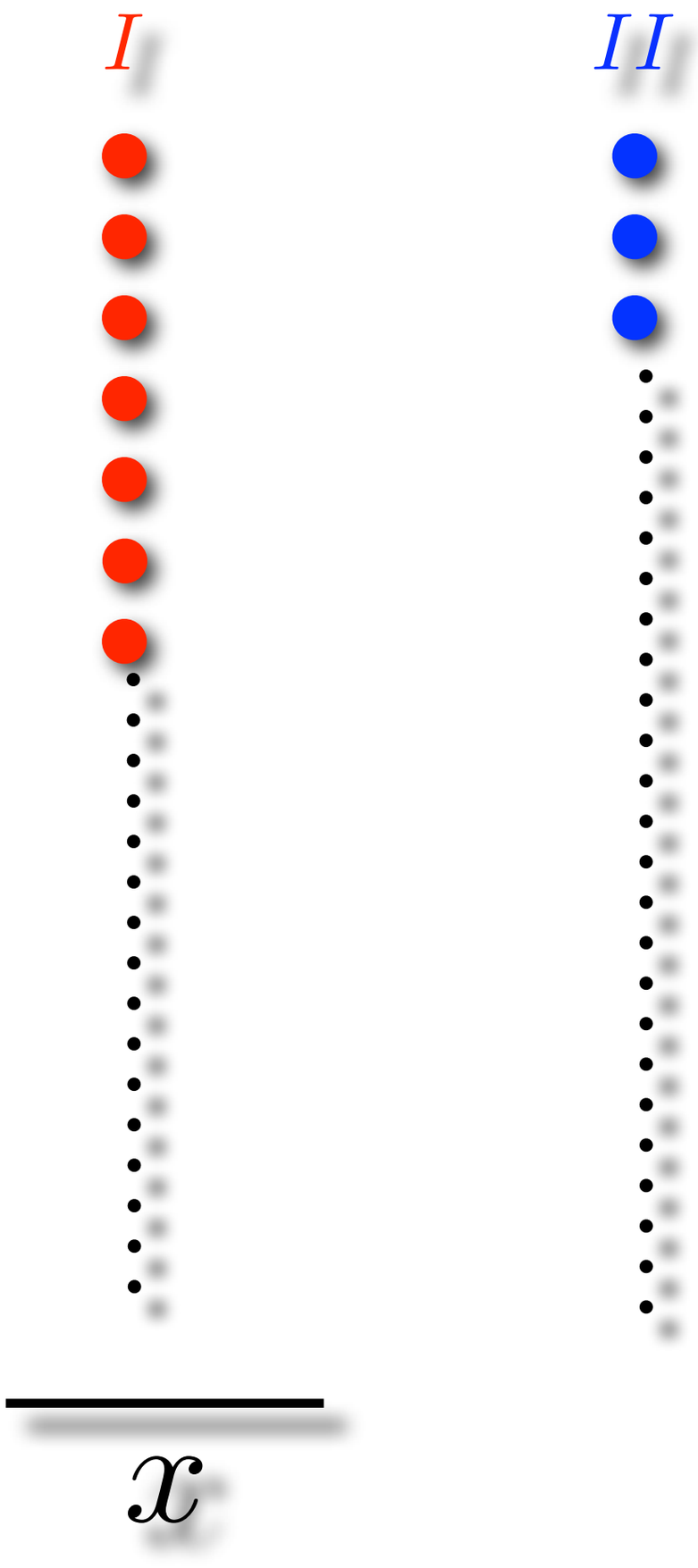

I
II

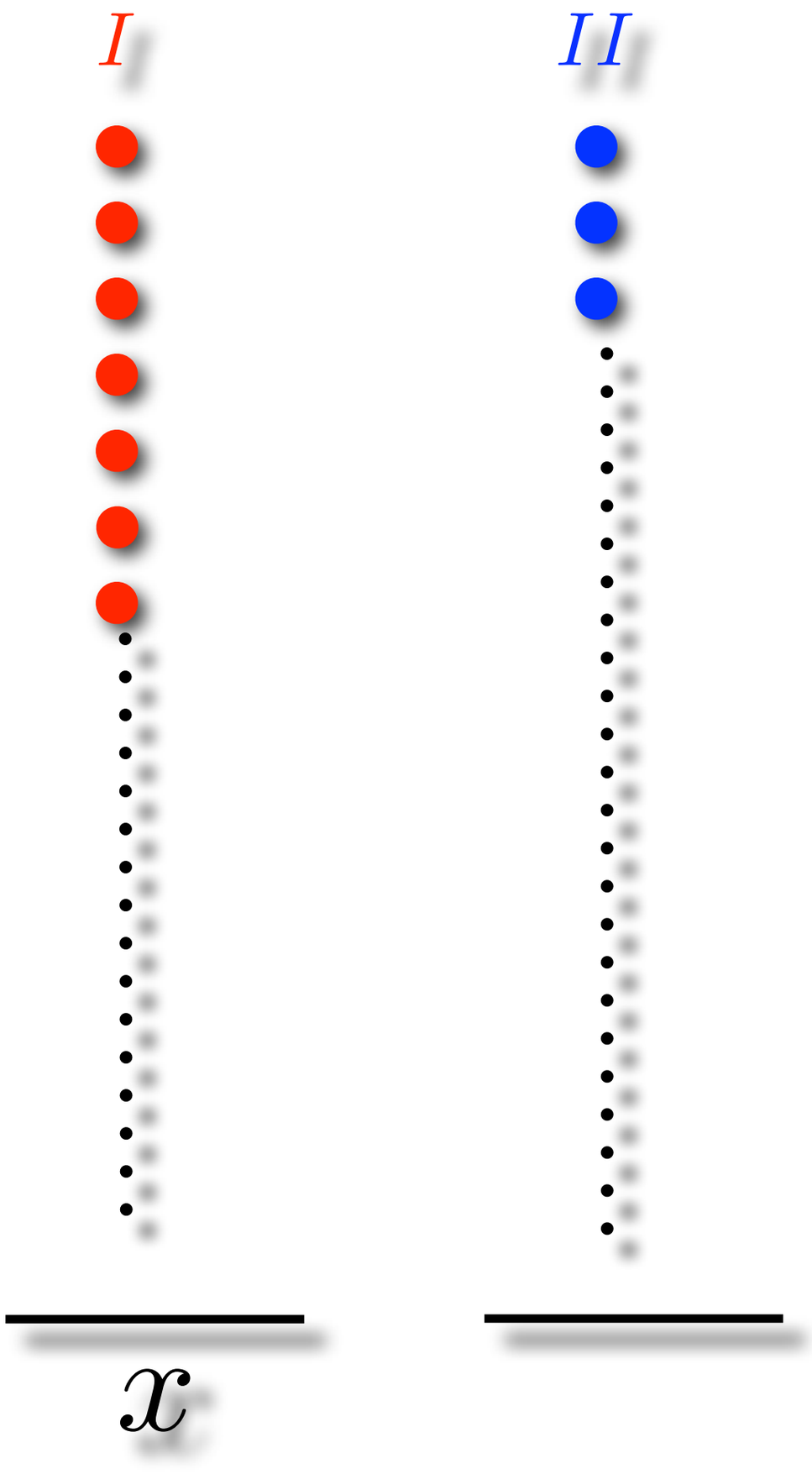

I
II

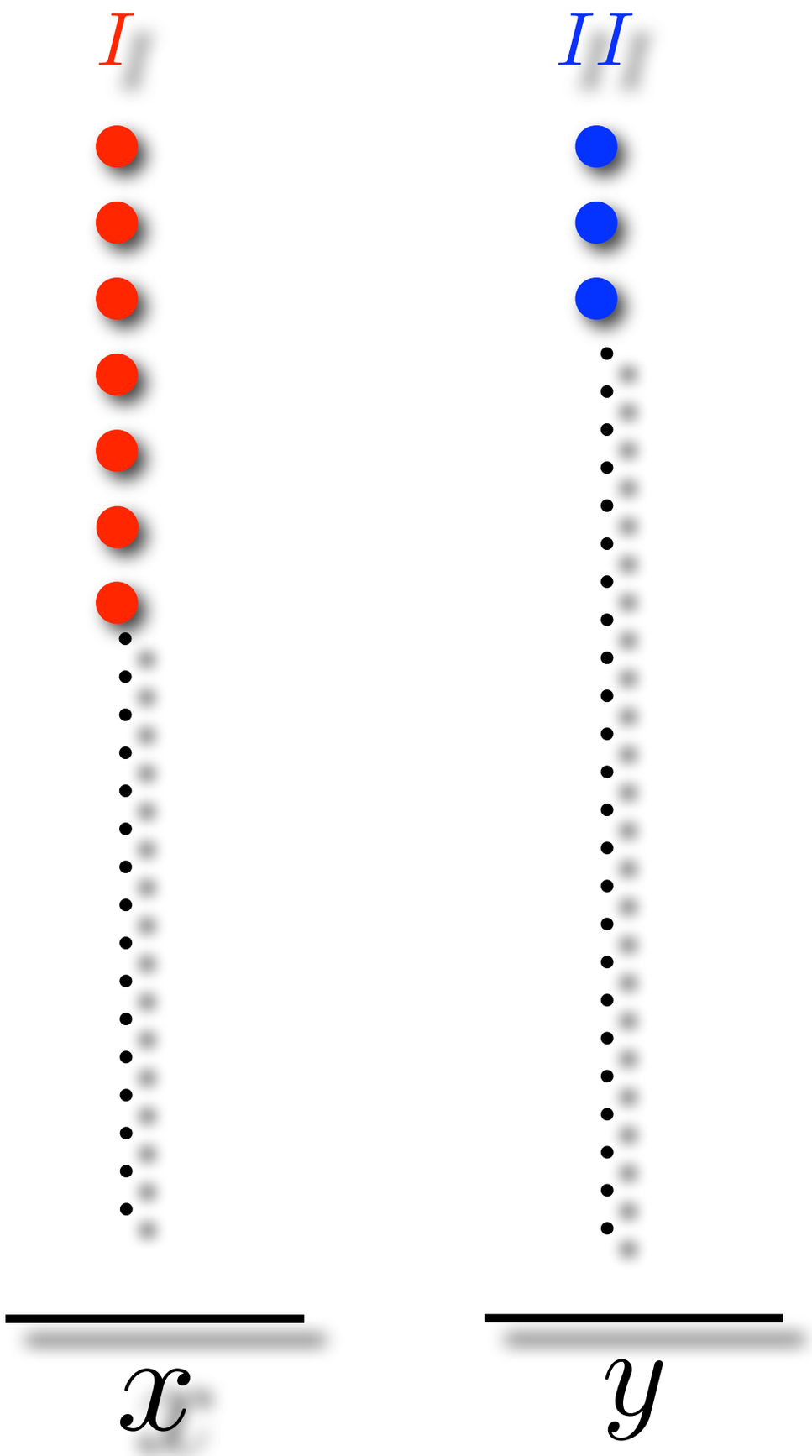

I
II

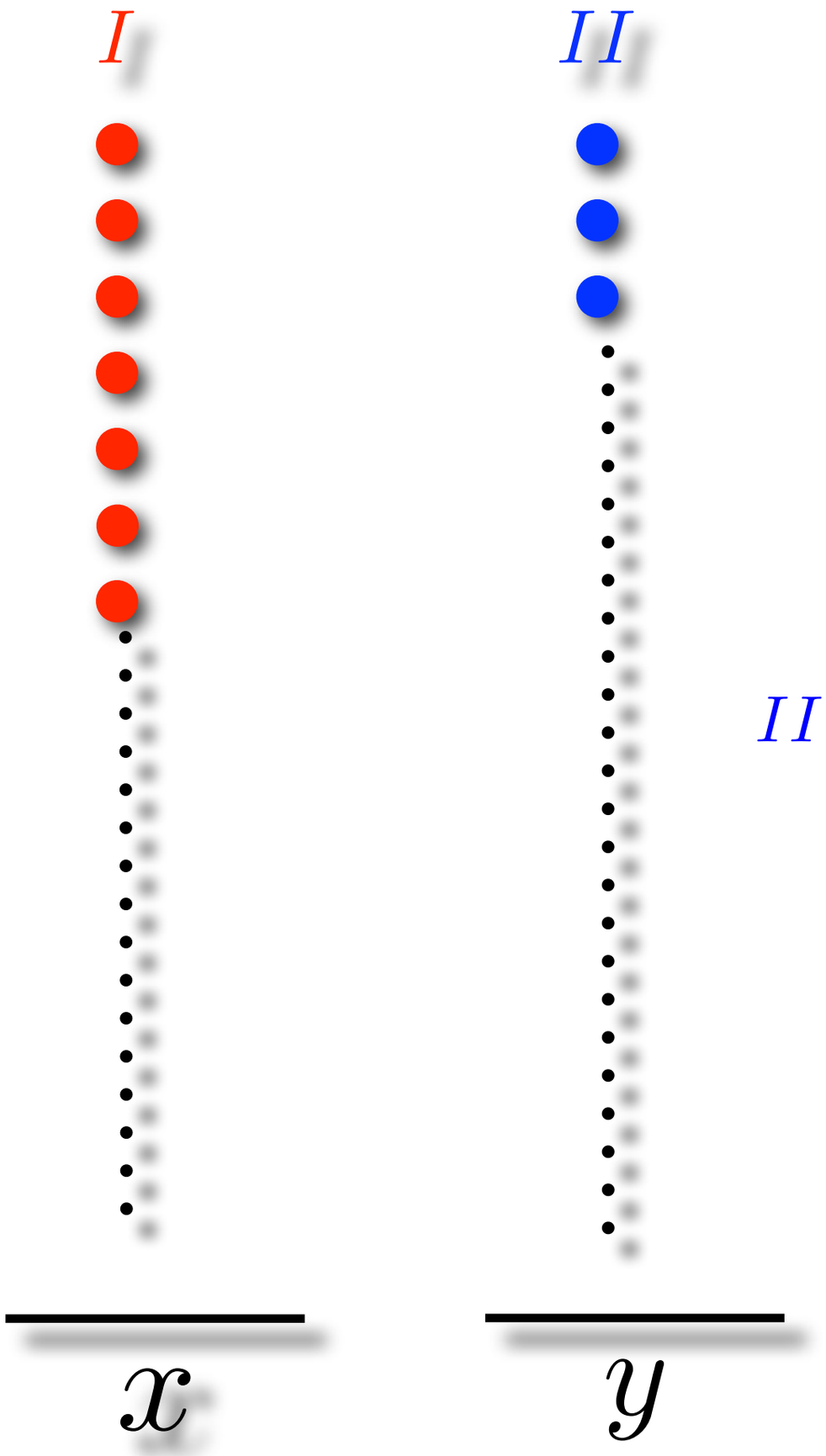

I
II


I
II


I
II
x


I
II
x


I
II
x
y



II wins $\iff f(x) = y$

$$\mathbb{G}_{undo}(f)$$

f: any mapping from

infinite words to infinite words

$$x \longmapsto f(x)$$


I
II


I
II


I
II


I
II


I
II

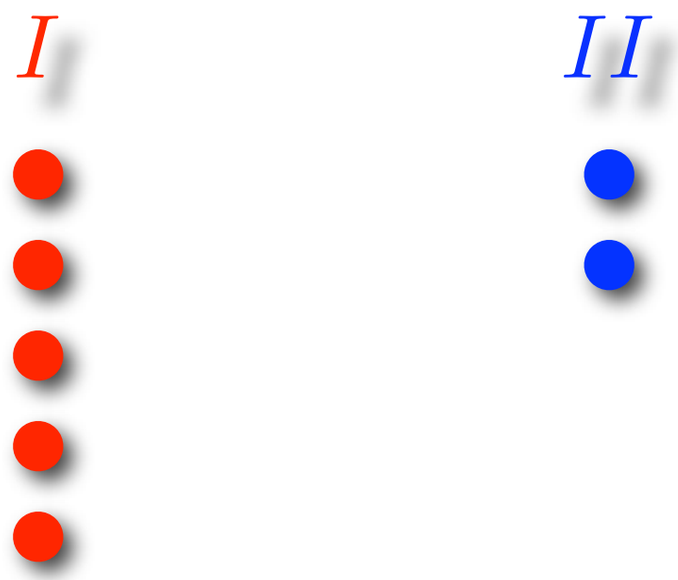

I
II

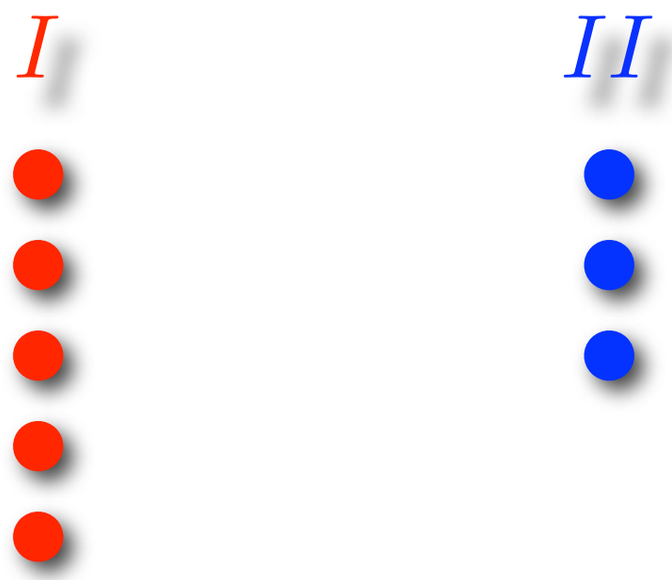

I
II

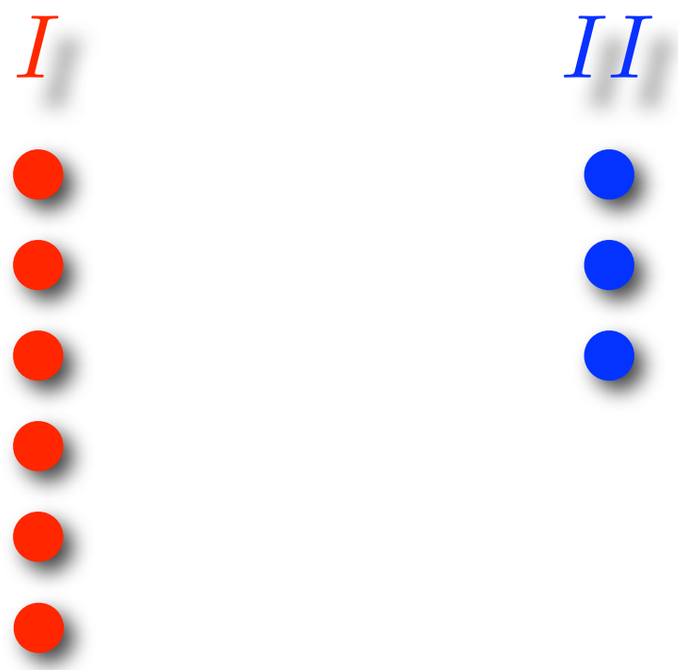

I
II

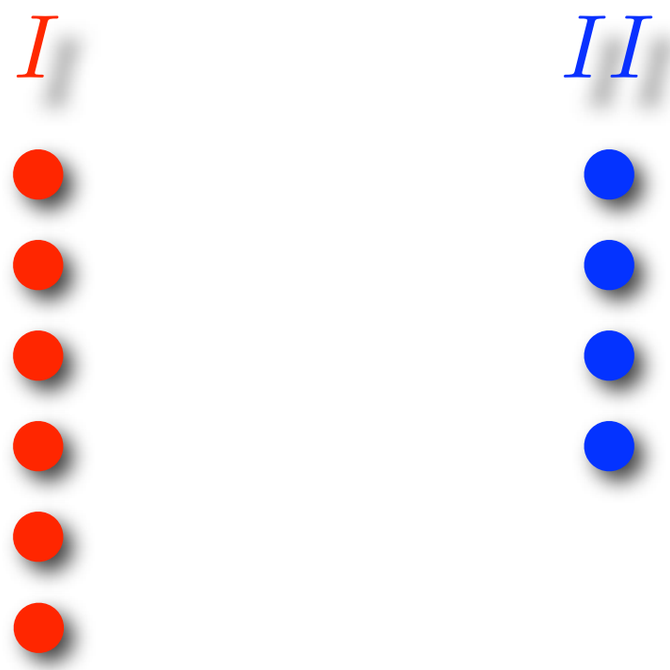

I
II

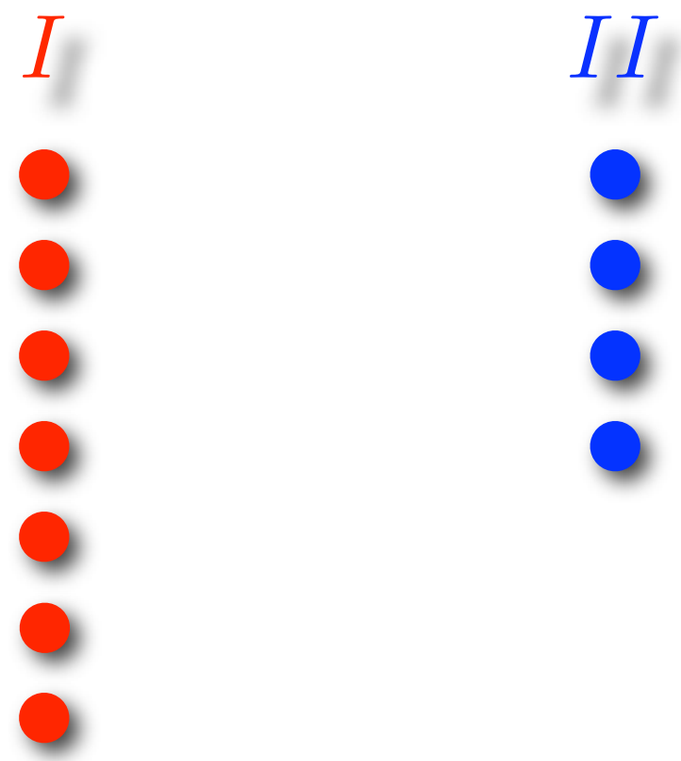

I
II

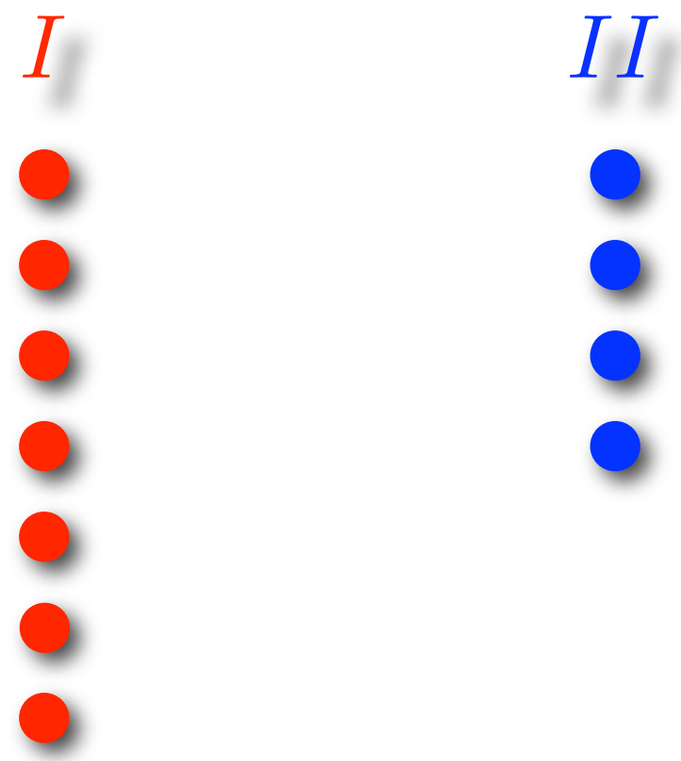

I
II

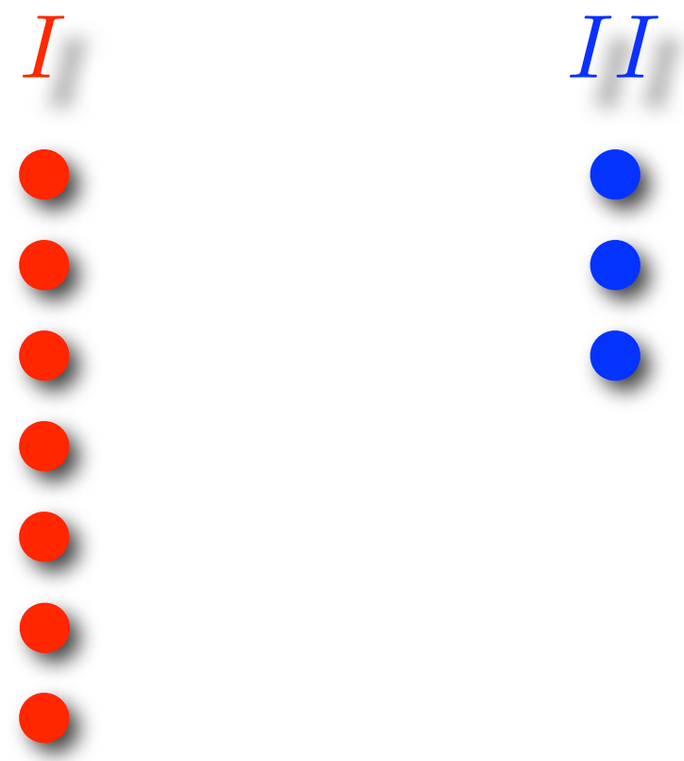

I
II

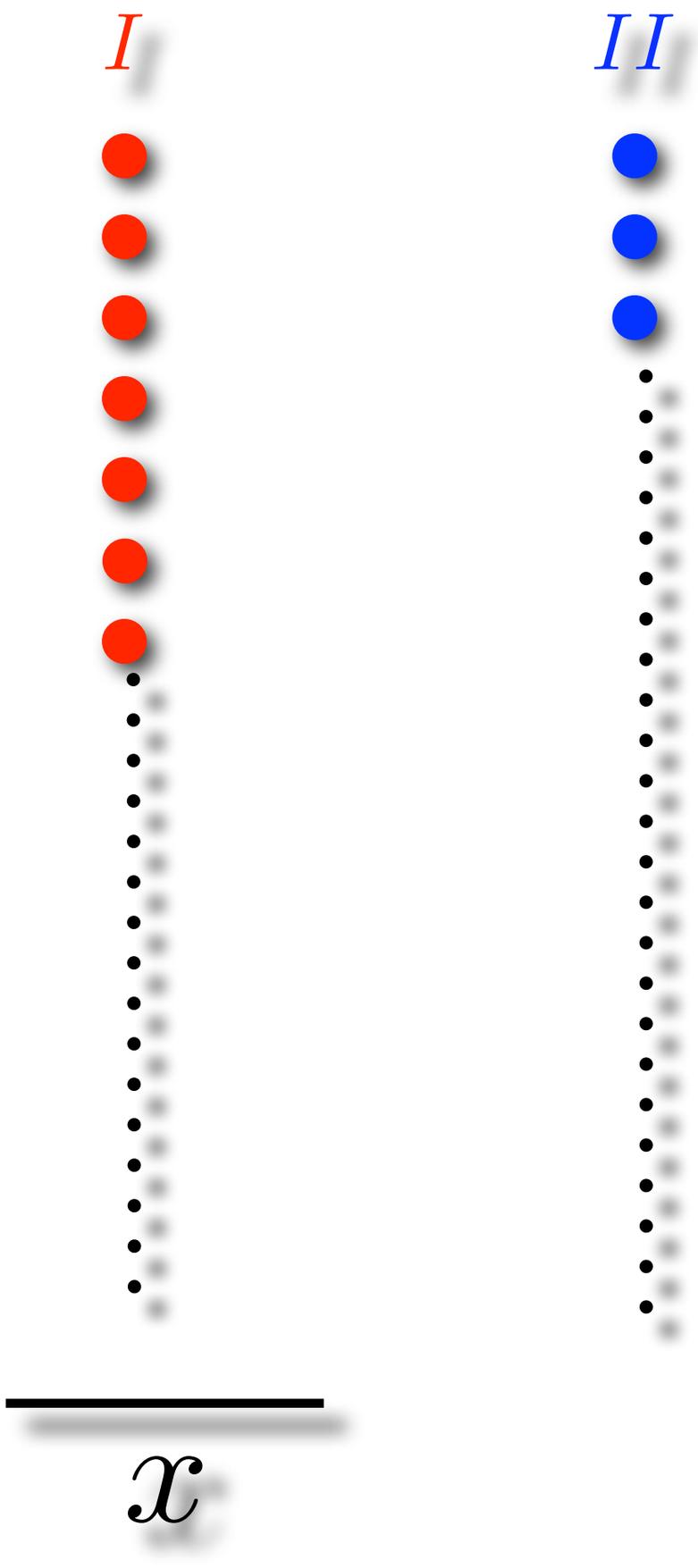

I
II

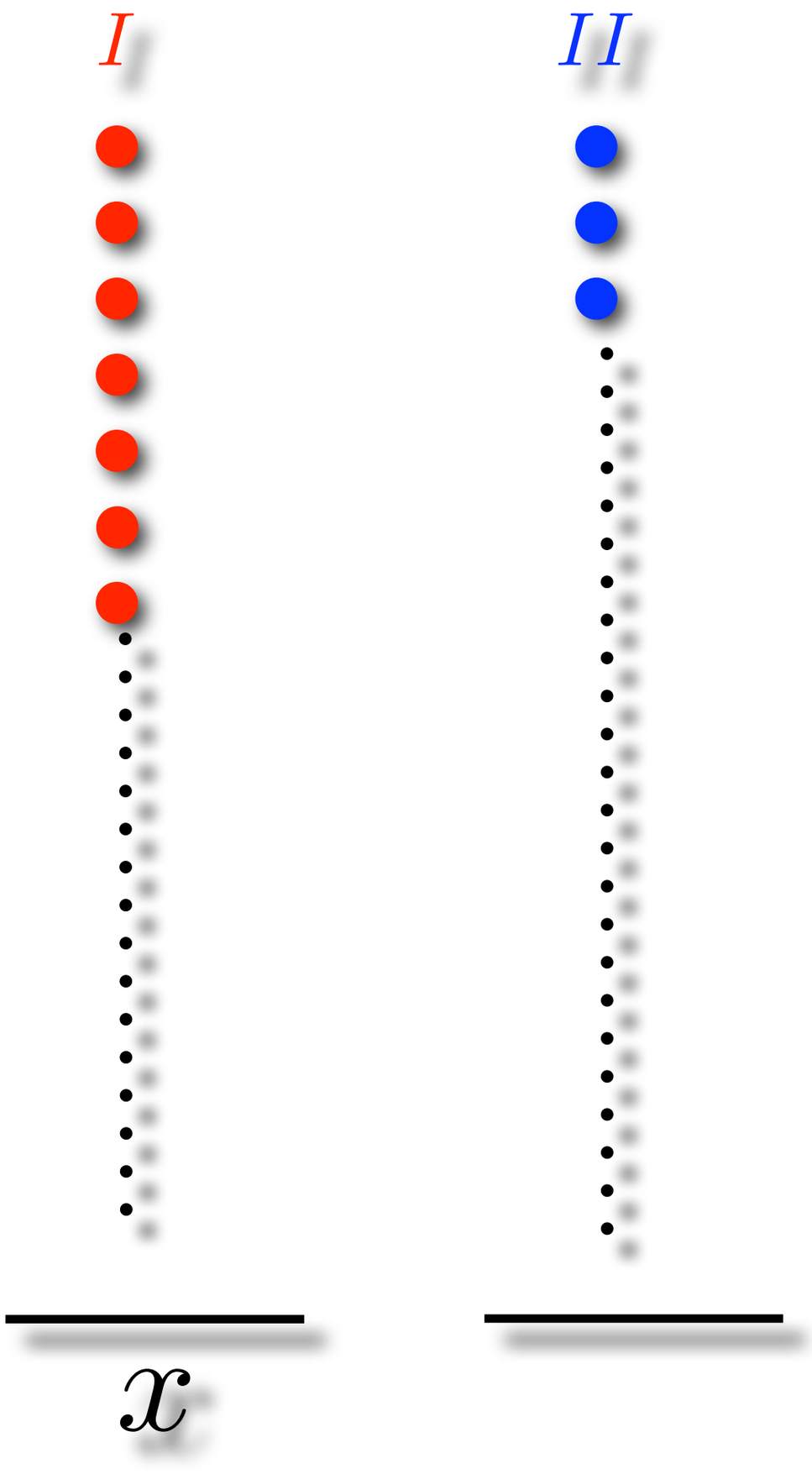

I
II

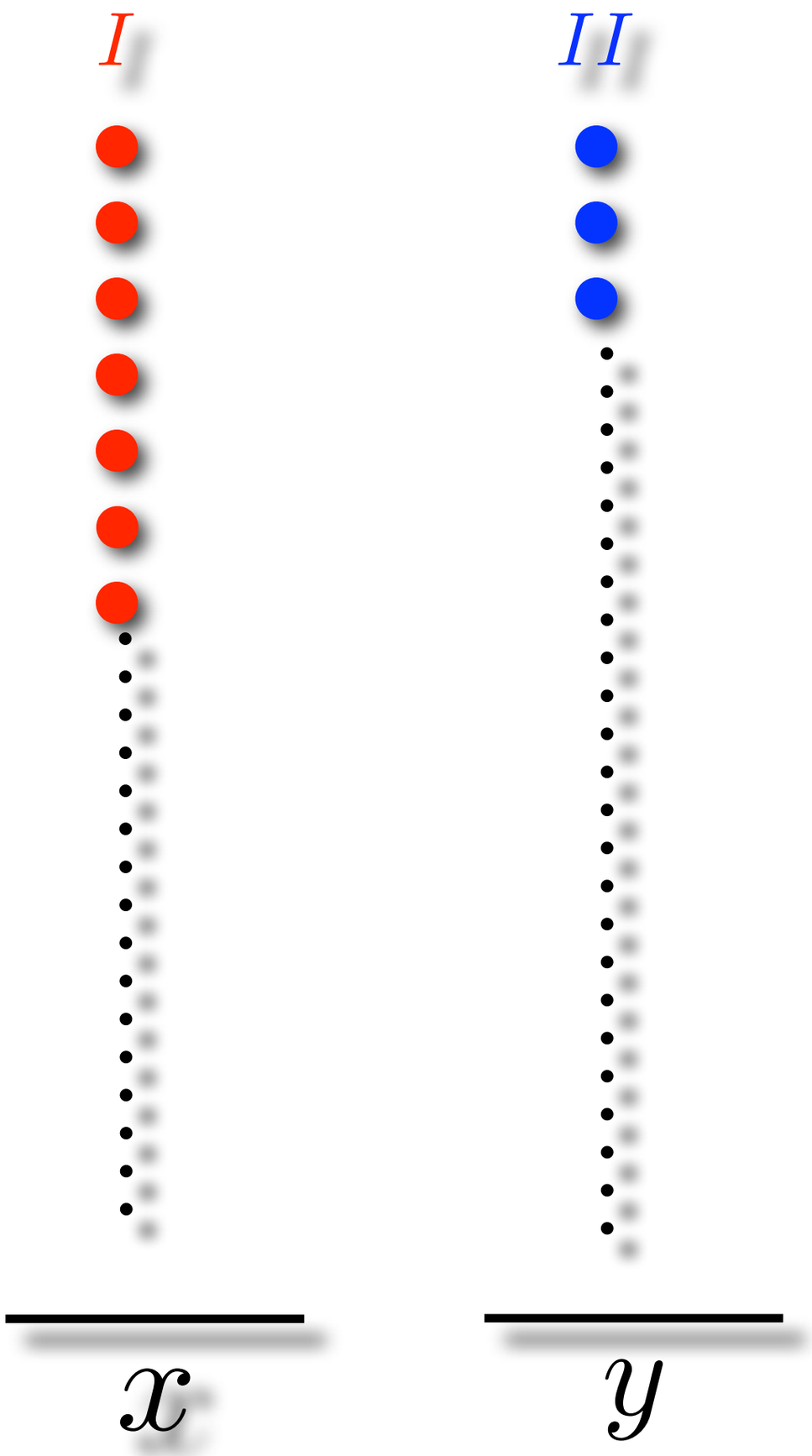

I
II

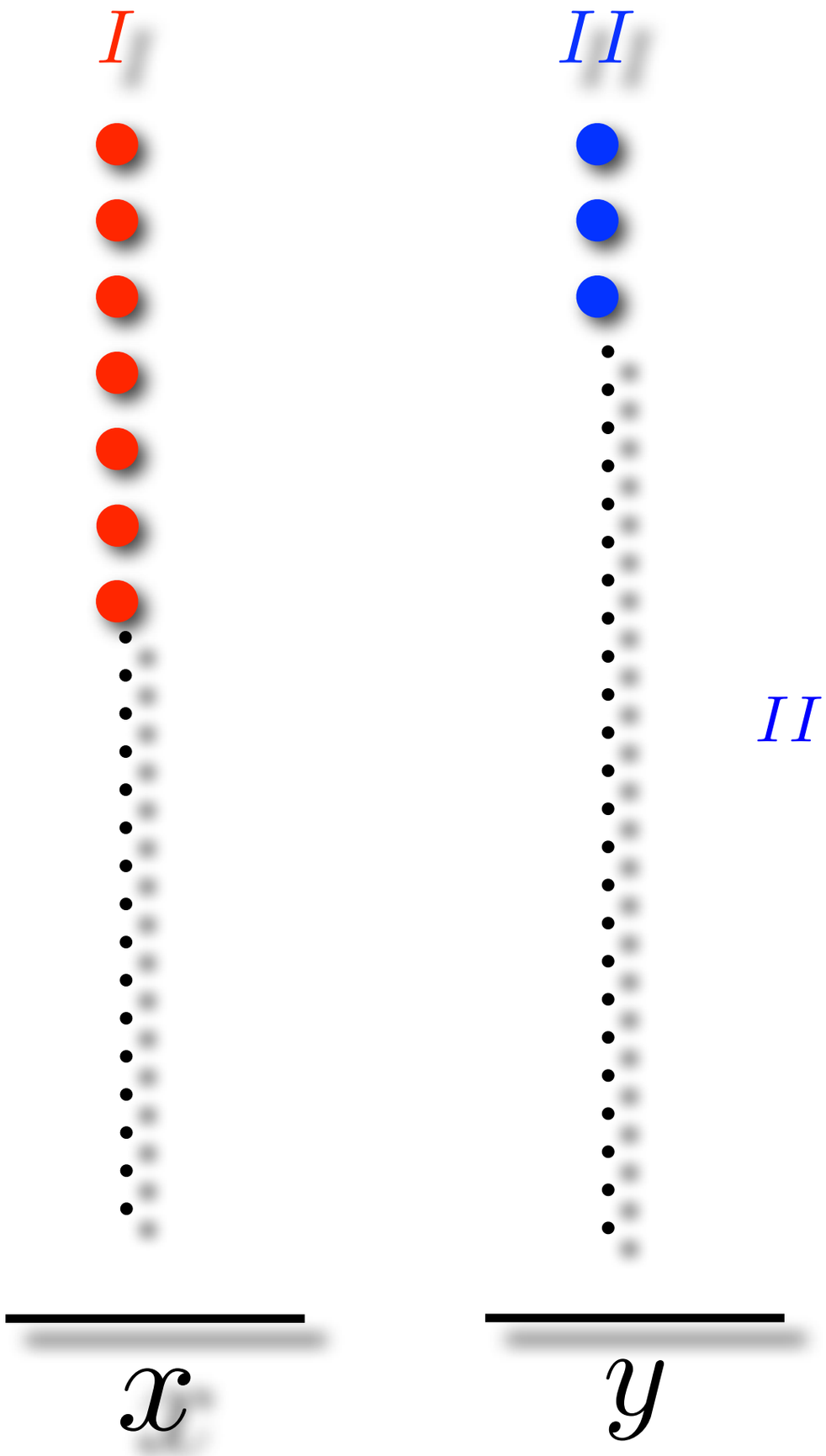

I
II


I
II


I
II
x


I
II
x


I
II
x
y



II wins $\iff f(x) = y$

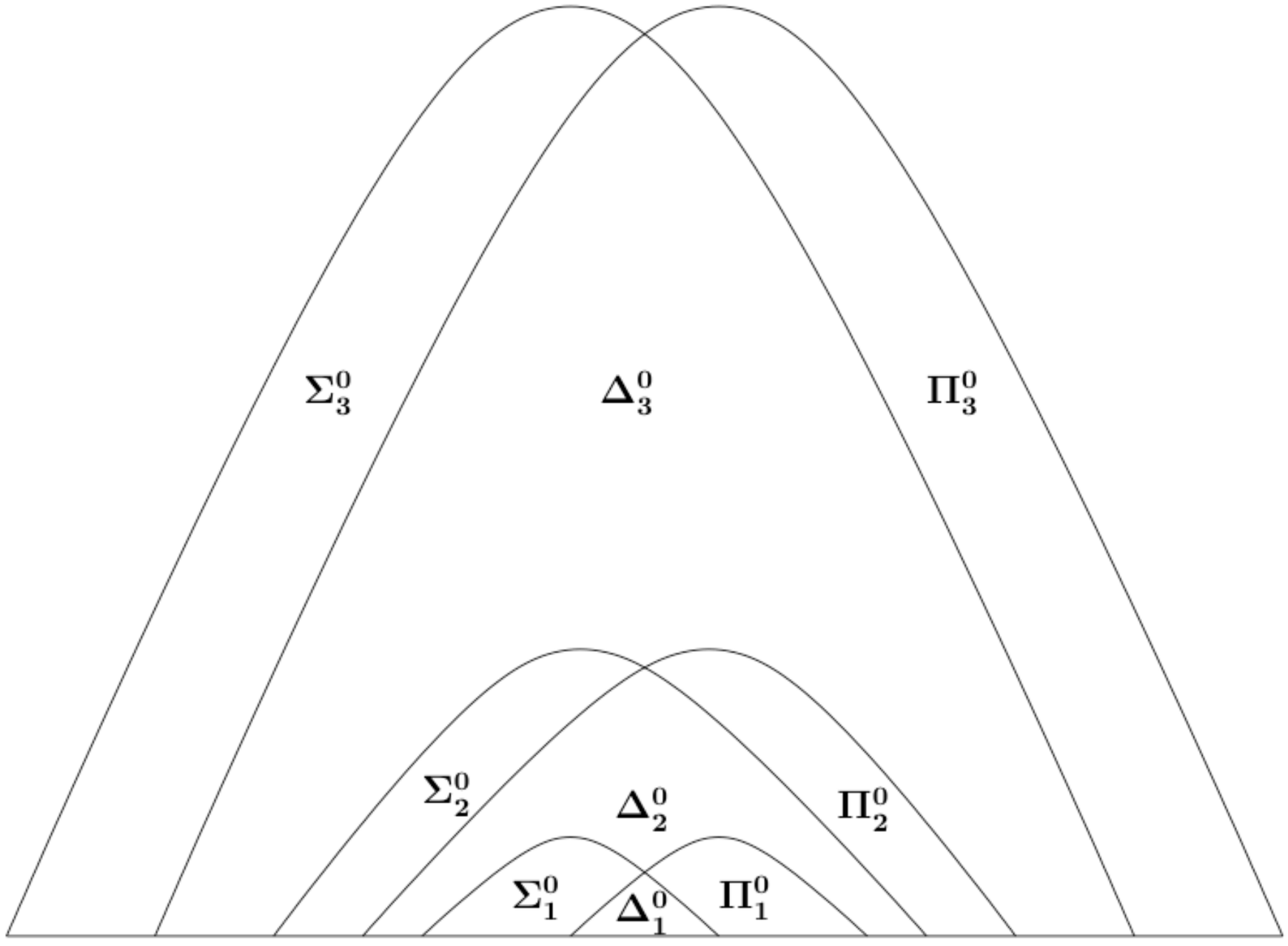
- $II\ has\ a\ w.s.\ in\ \mathbb{G}(f) \iff f\ Lipschiptz$
- $II\ has\ a\ w.s.\ in\ \mathbb{G}_{skip}(f) \iff f\ continuous$
- $II\ has\ a\ w.s.\ in\ \mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- $II\ has\ a\ w.s.\ in\ \mathbb{G}_{undo}(f) \iff f\ Baire\ class\ 1$



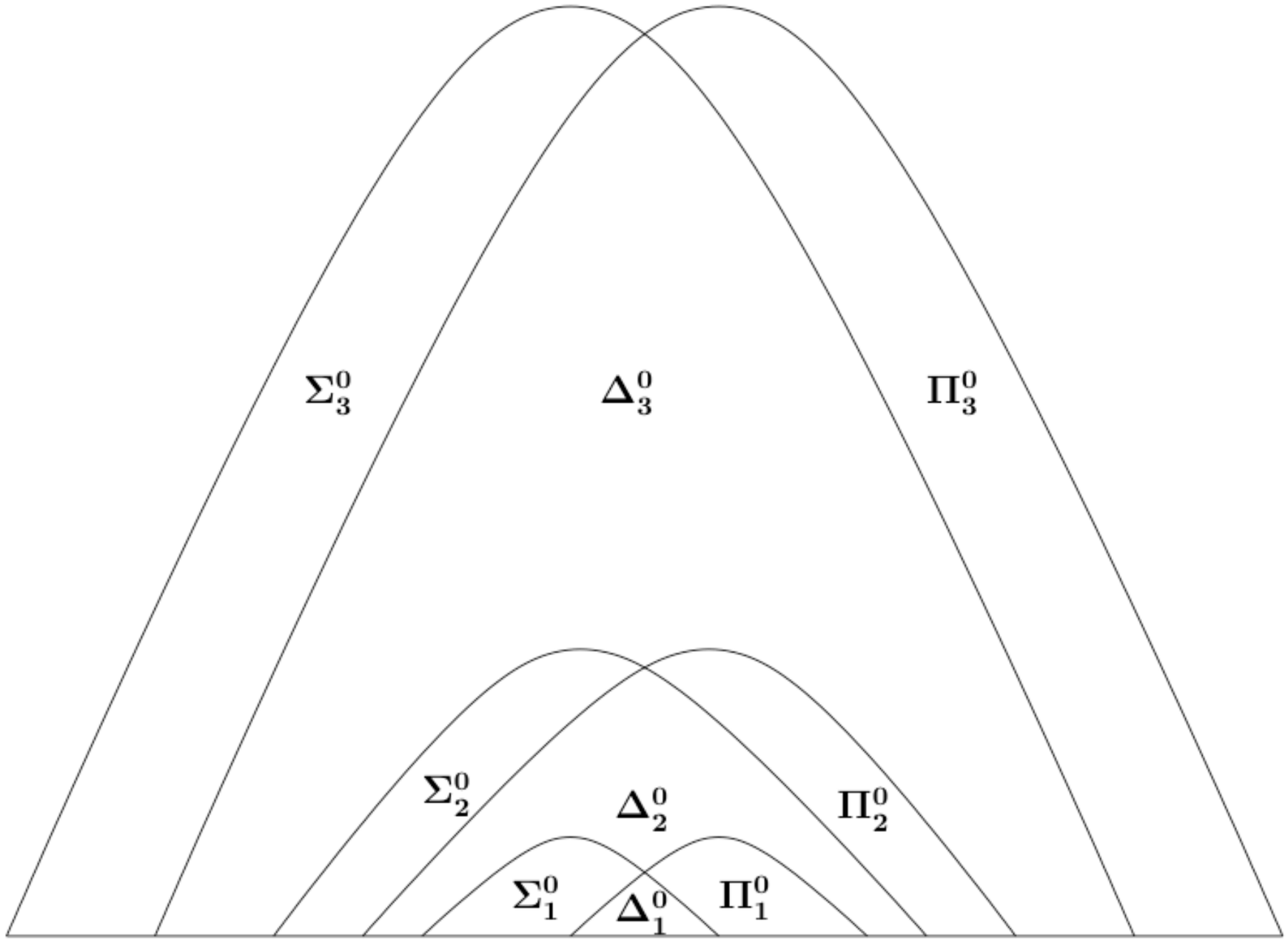
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*

- $II\ has\ a\ w.s.\ in\ \mathbb{G}(f) \iff f\ Lipschiptz$
- $II\ has\ a\ w.s.\ in\ \mathbb{G}_{skip}(f) \iff f\ continuous$
- $II\ has\ a\ w.s.\ in\ \mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- $II\ has\ a\ w.s.\ in\ \mathbb{G}_{undo}(f) \iff f\ Baire\ class\ 1$

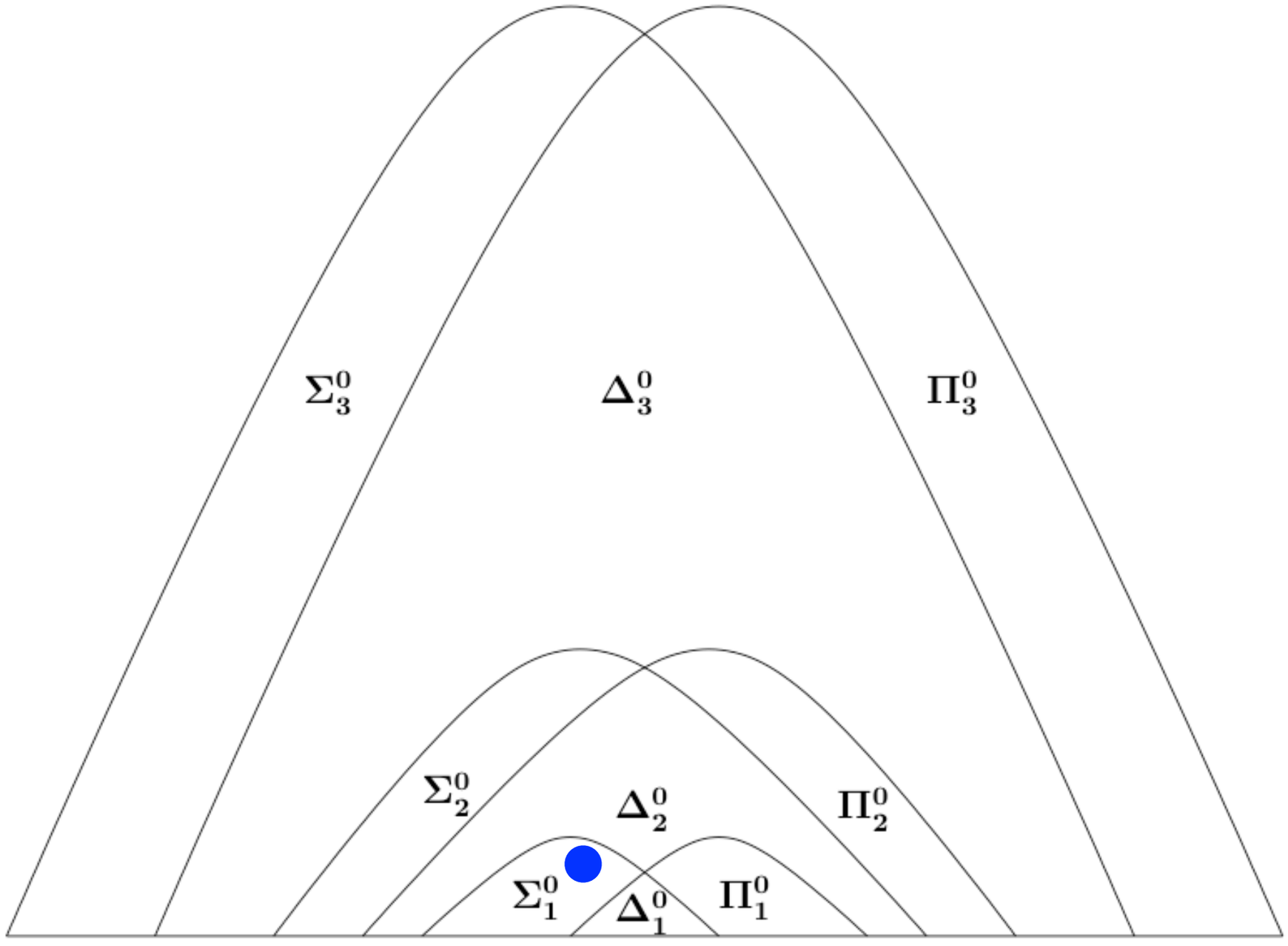
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



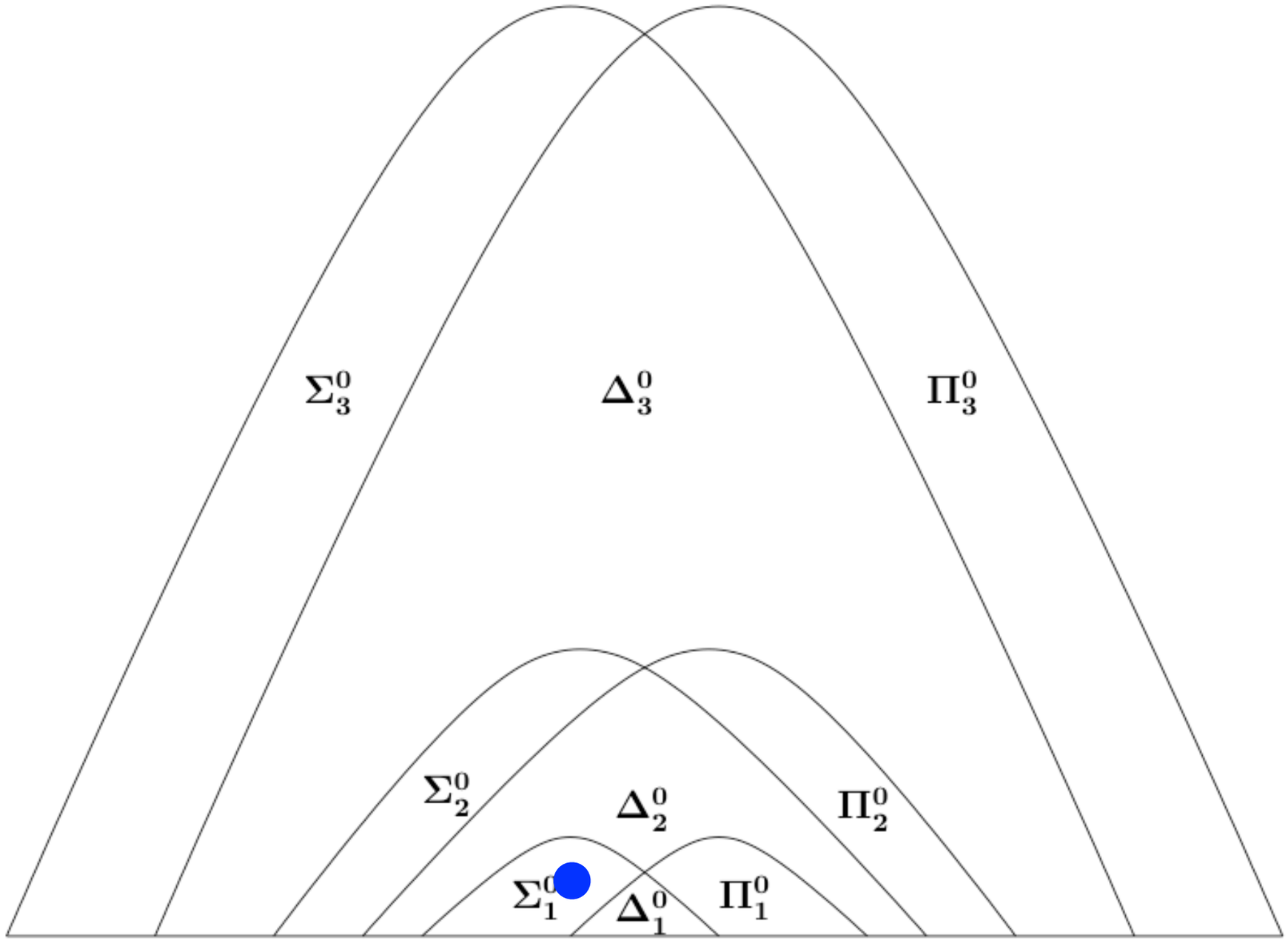
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



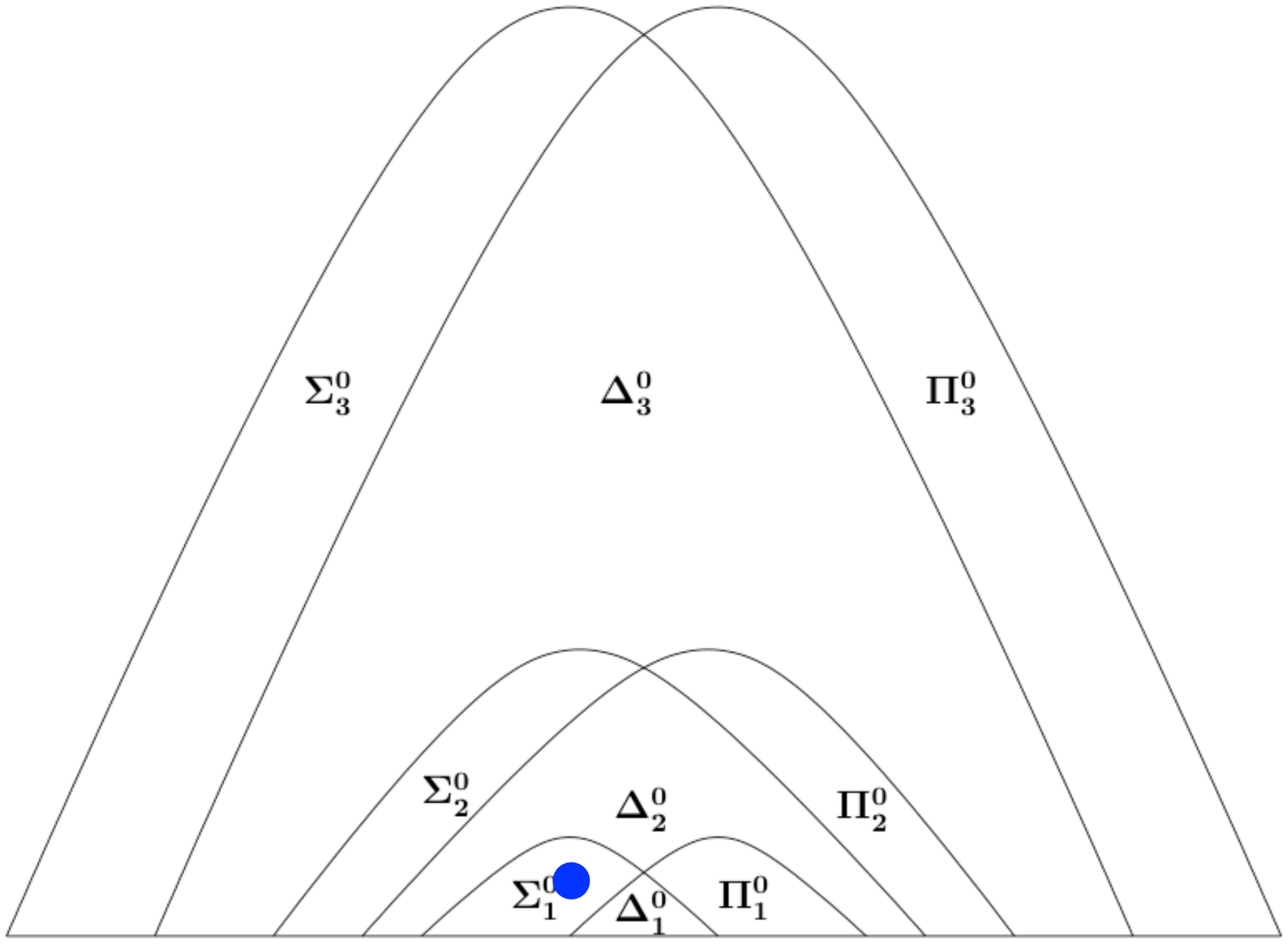
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



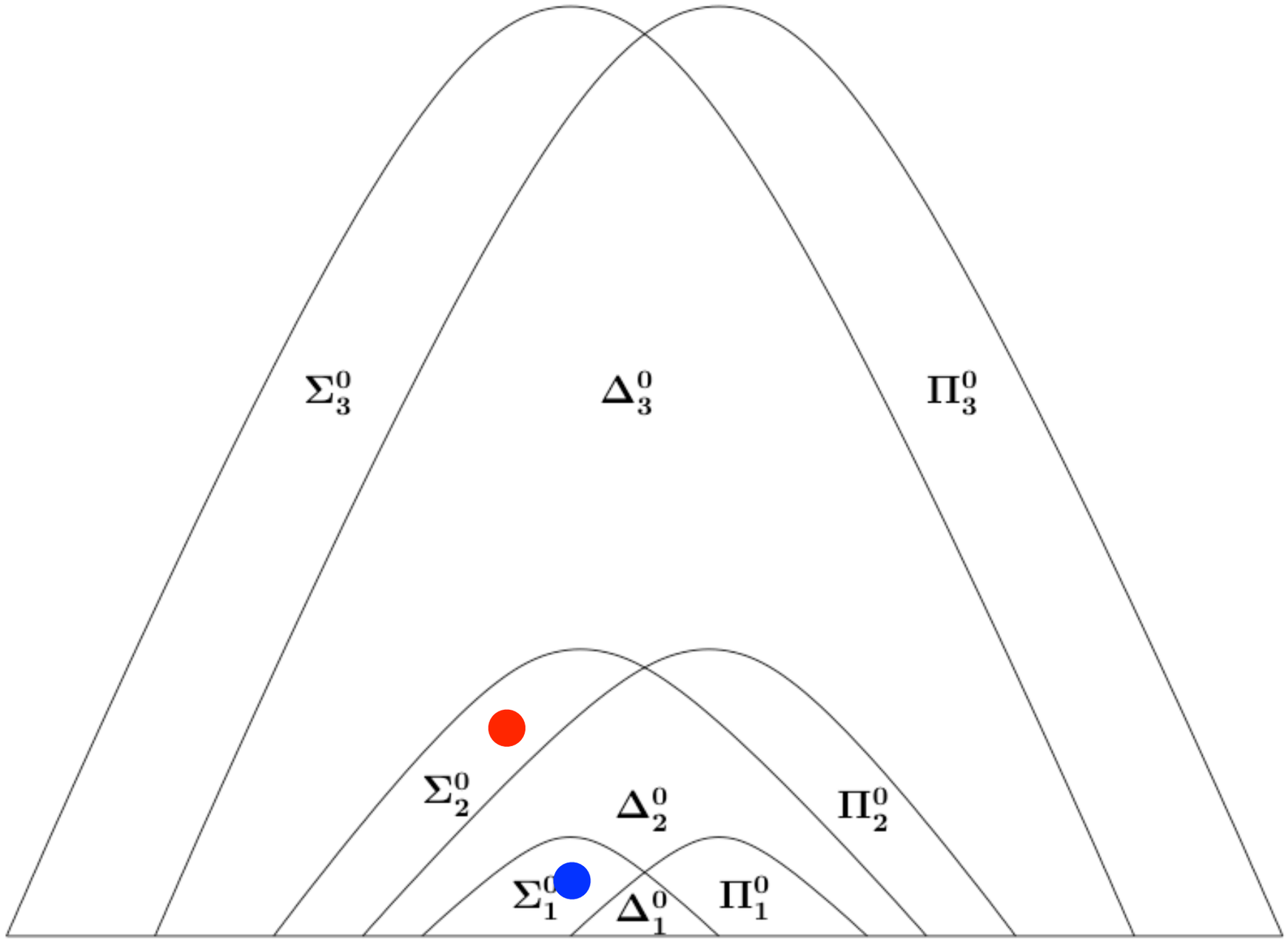
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



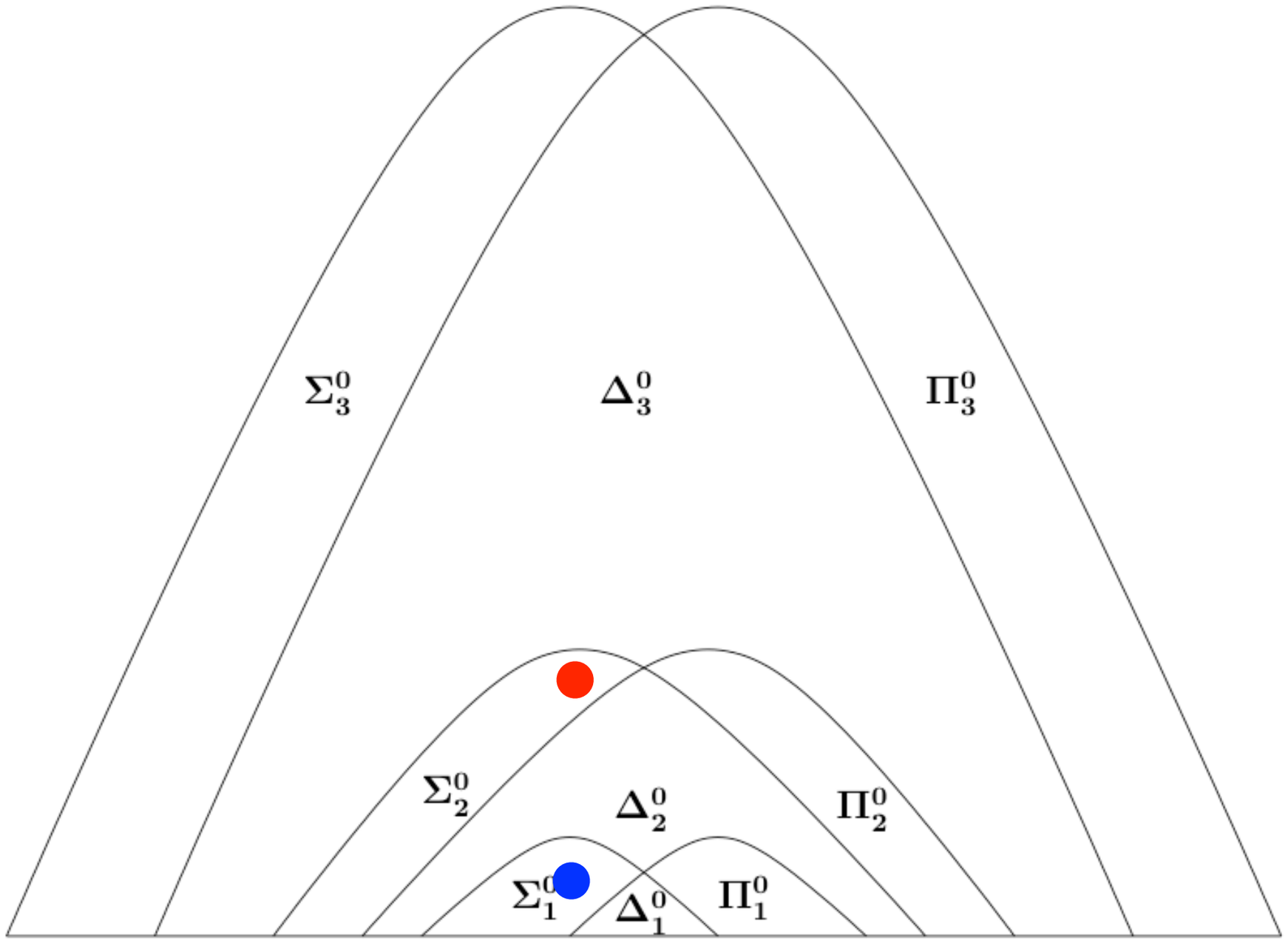
- *II has a w.s. in* $\mathbb{G}(f)$ $\iff$ f *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f)$ $\iff$ f *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f)$ $\iff$ f Δ_2^0
- *II has a w.s. in* $\mathbb{G}_{undo}(f)$ $\iff$ f *Baire class 1*



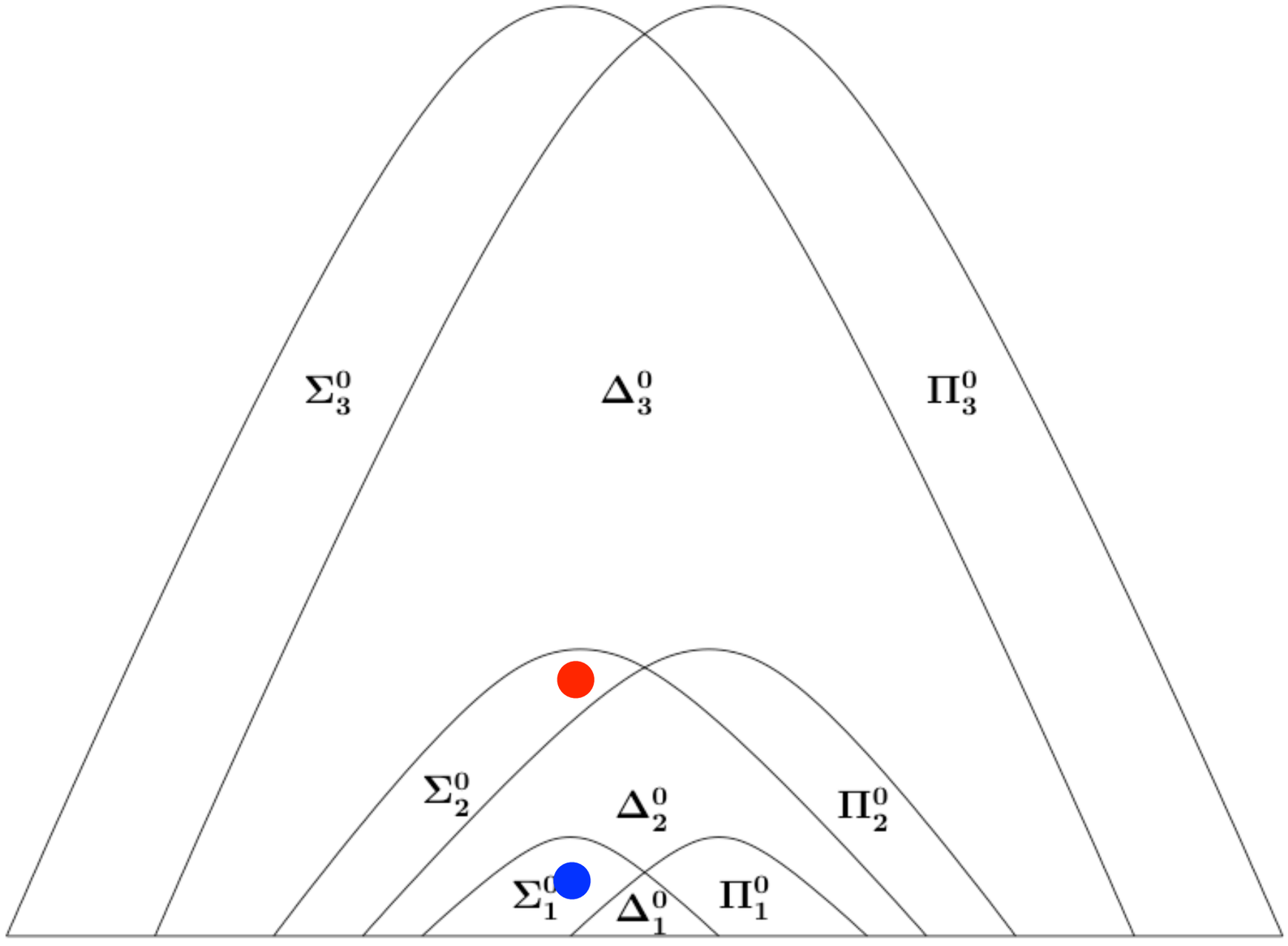
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



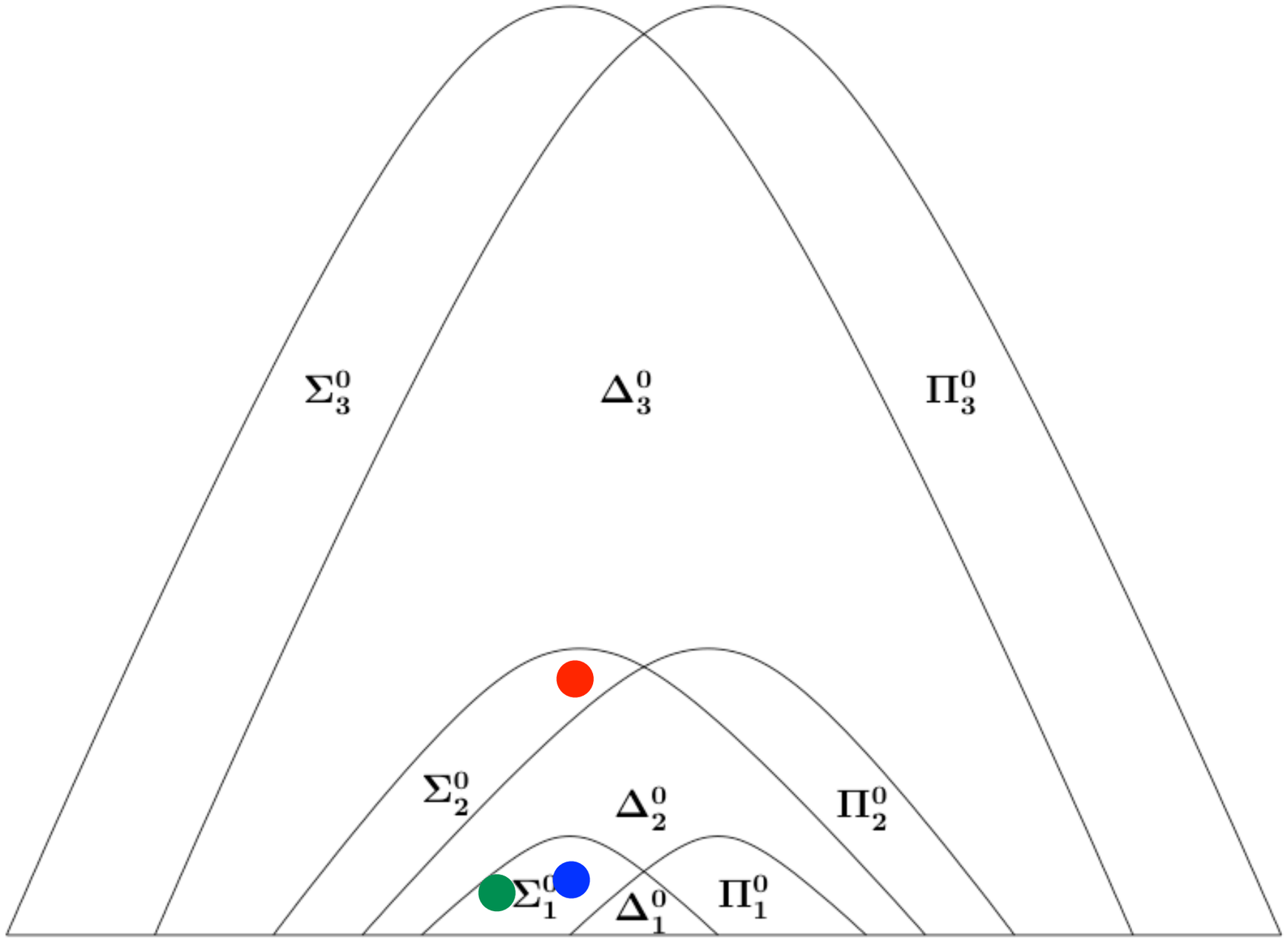
- *II has a w.s. in* $\mathbb{G}(f)$ $\iff$ f *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f)$ $\iff$ f *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f)$ $\iff$ f Δ_2^0
- *II has a w.s. in* $\mathbb{G}_{undo}(f)$ $\iff$ f *Baire class 1*



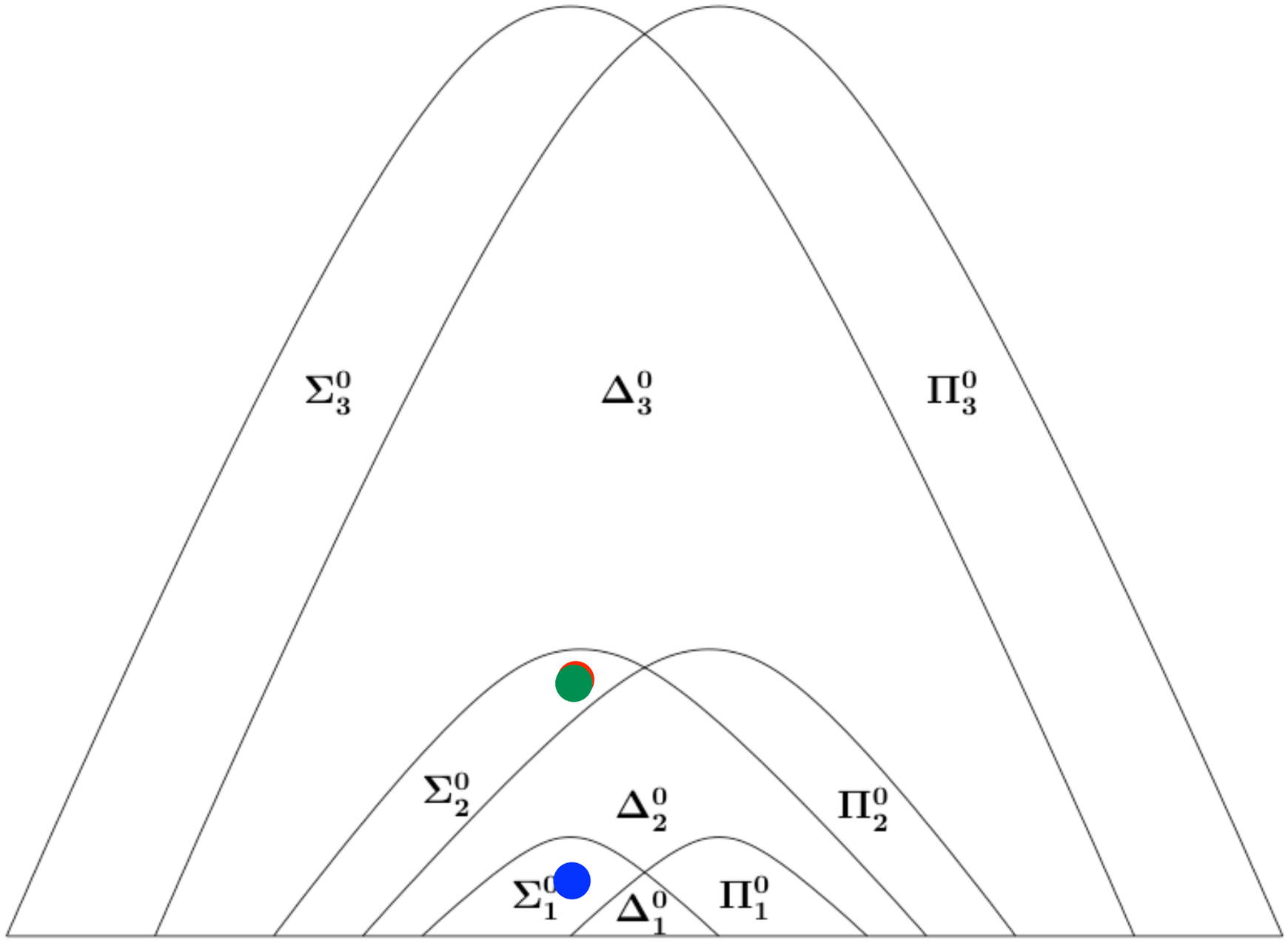
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



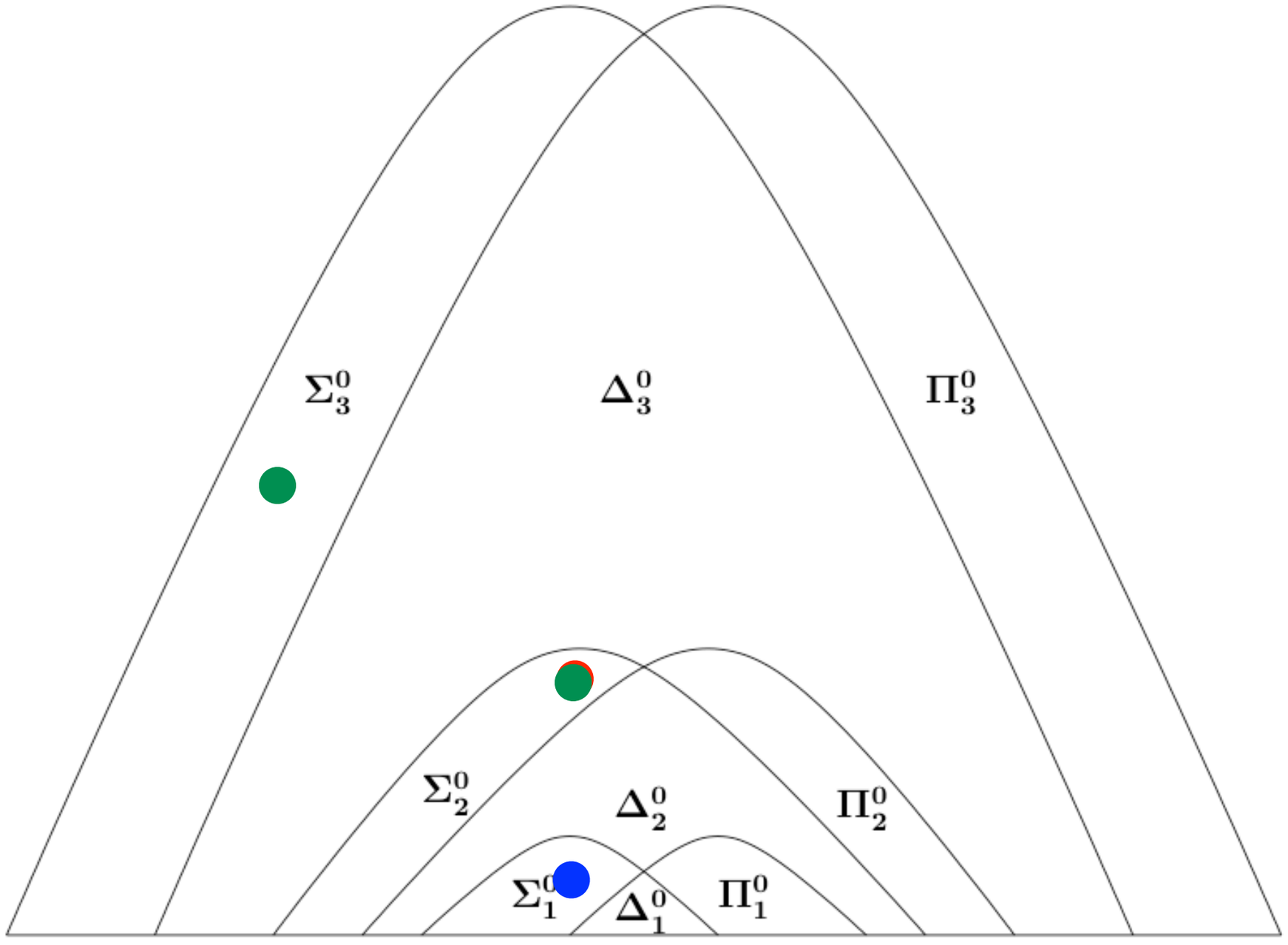
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*



- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*

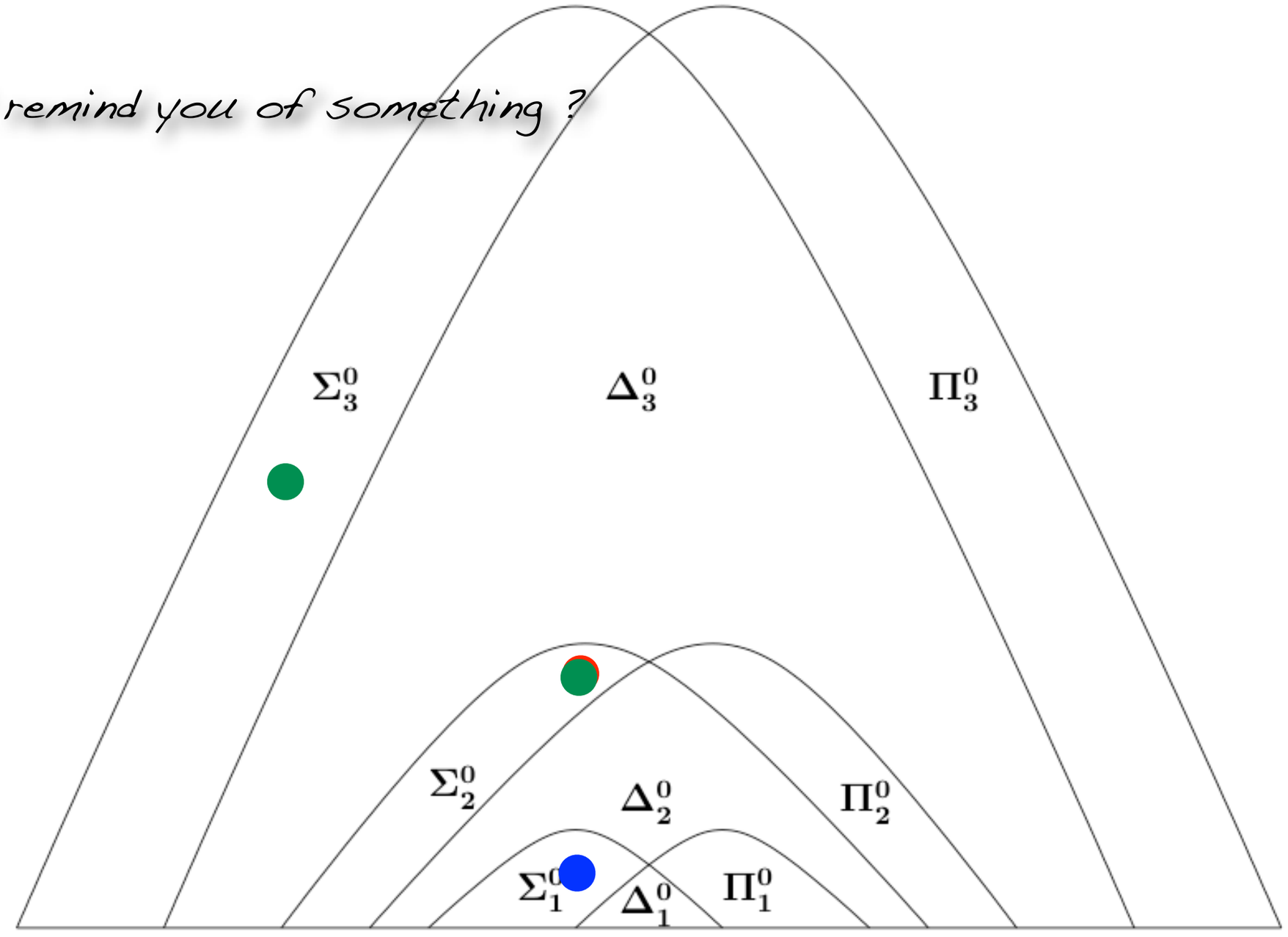


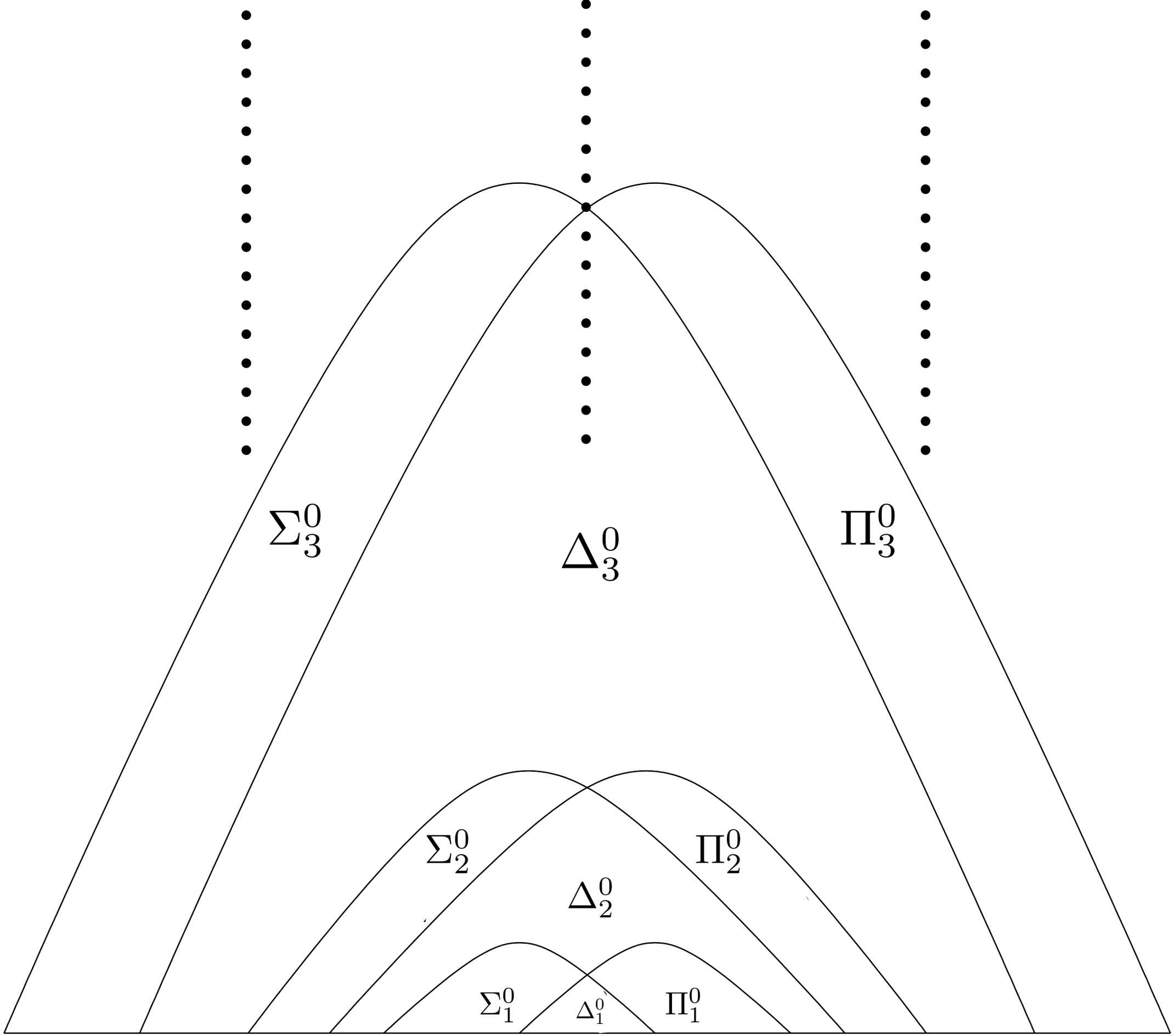
- *II has a w.s. in* $\mathbb{G}(f) \iff f$ *Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f) \iff f$ *continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f) \iff f\ \Delta_2^0$
- *II has a w.s. in* $\mathbb{G}_{undo}(f) \iff f$ *Baire class 1*

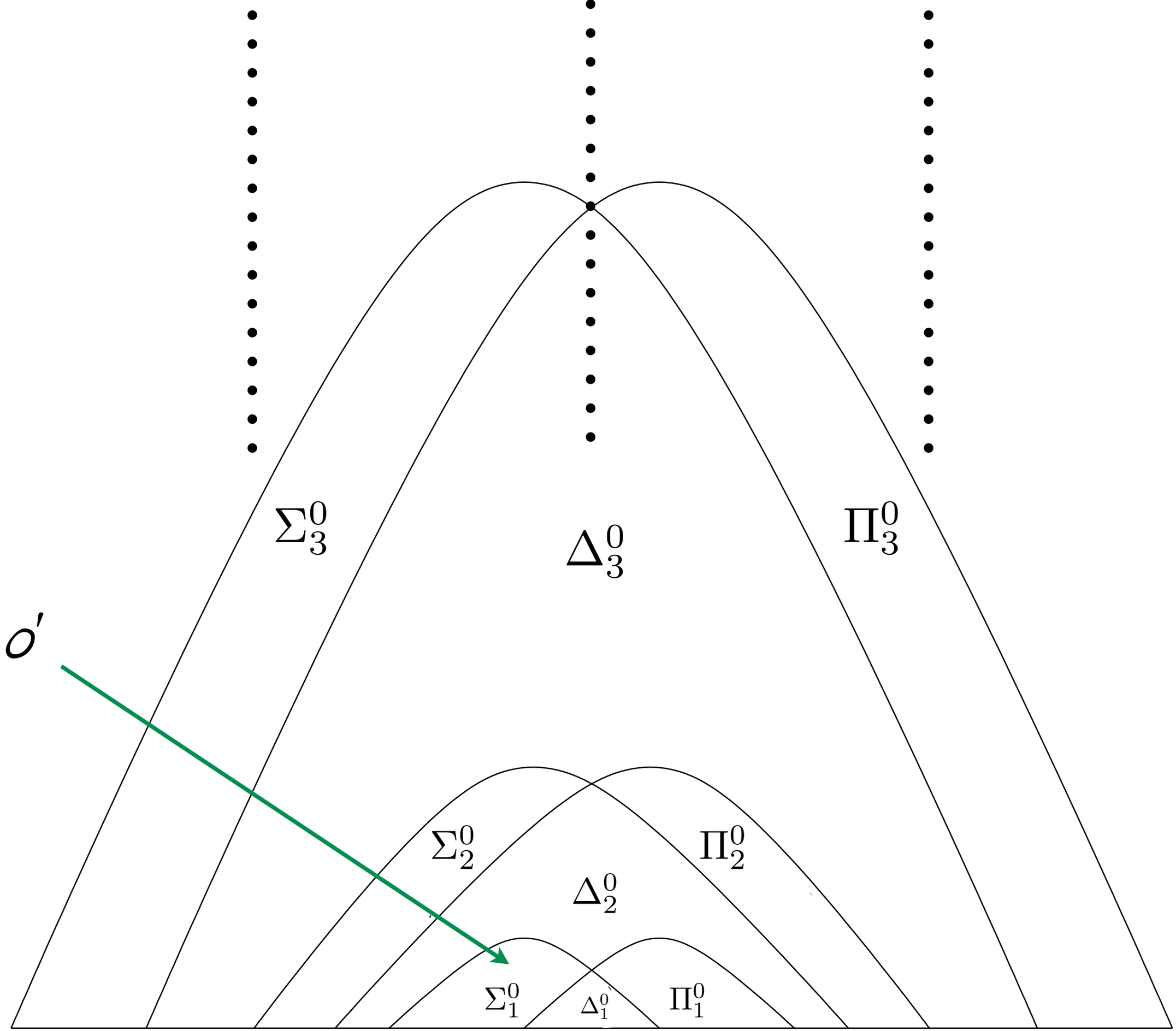


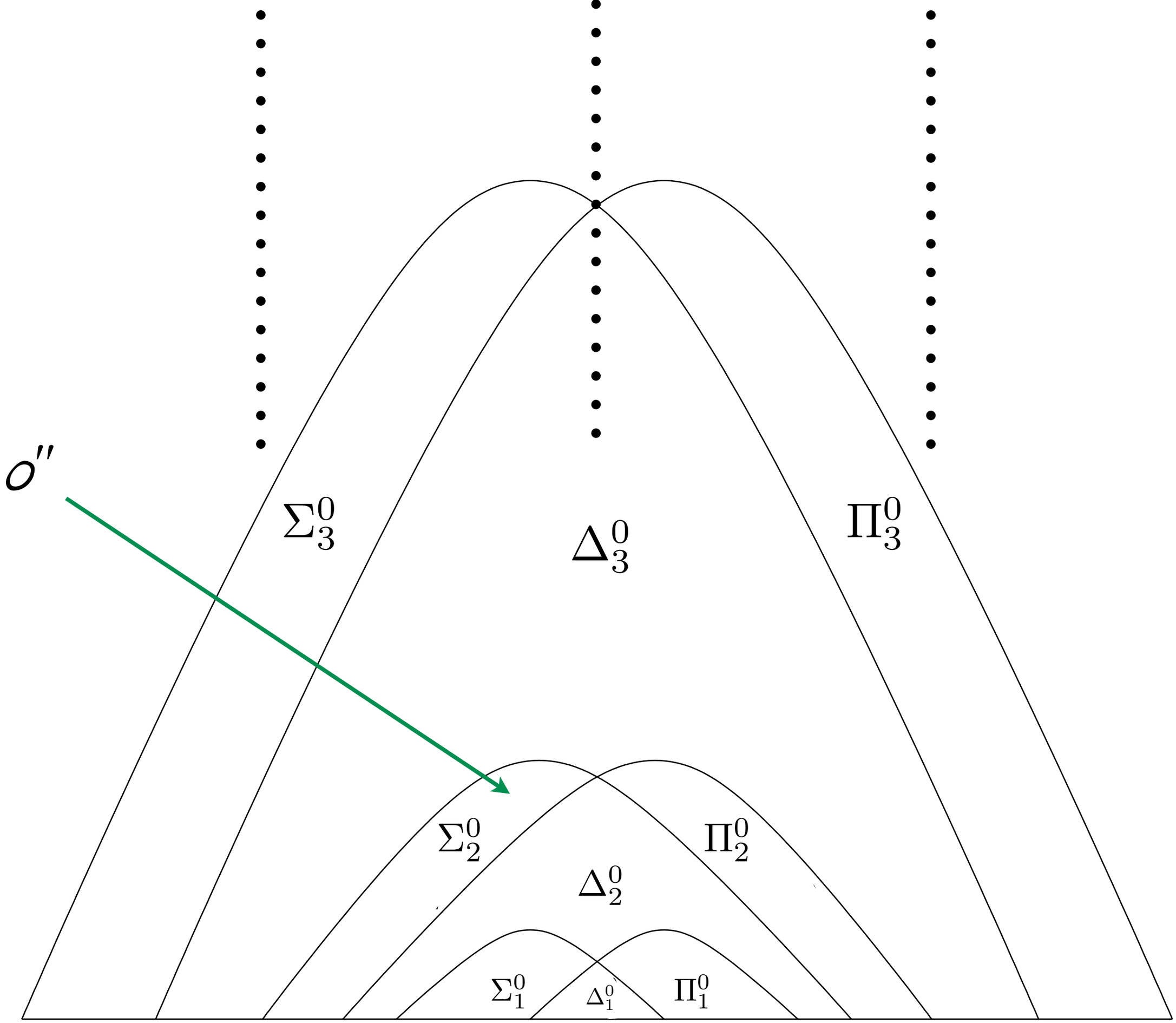
- *II has a w.s. in* $\mathbb{G}(f)$ $\iff$ *f Lipschiptz*
- *II has a w.s. in* $\mathbb{G}_{skip}(f)$ $\iff$ *f continuous*
- *II has a w.s. in* $\mathbb{G}_{erase}(f)$ $\iff$ *f* Δ_2^0
- *II has a w.s. in* $\mathbb{G}_{undo}(f)$ $\iff$ *f Baire class 1*

Does it remind you of something ?




Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


Σ_3^0
Δ_3^0
Π_3^0
O'
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


o''
Σ_3^0
Δ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0

O'''

Σ_3^0

Δ_3^0

Π_3^0

Σ_2^0

Δ_2^0

Π_2^0

Σ_1^0

Δ_1^0

Π_1^0

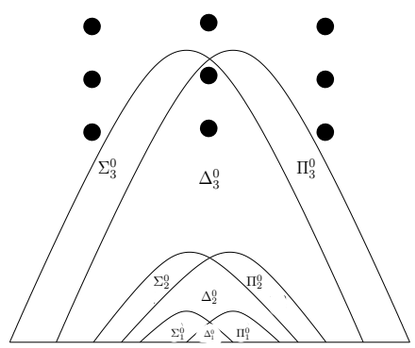


Games that construct mappings

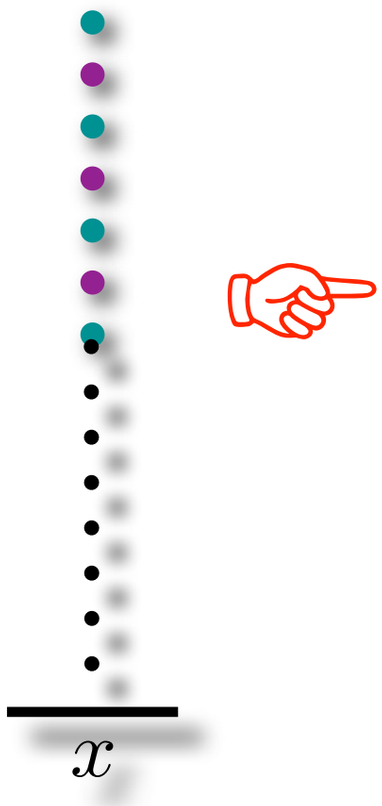


An example: evaluation game for 1st order logic

non determined game

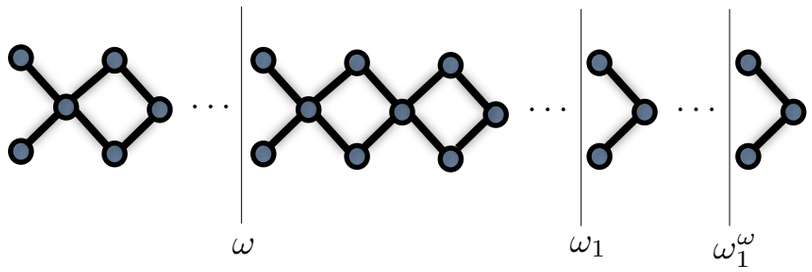


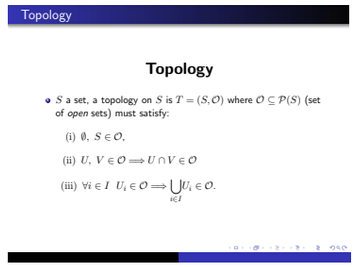
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

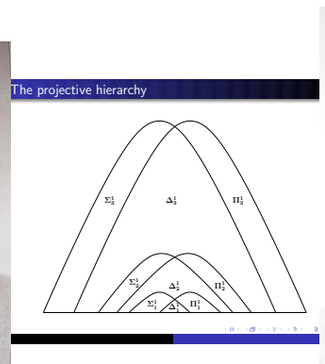
Games that construct mappings



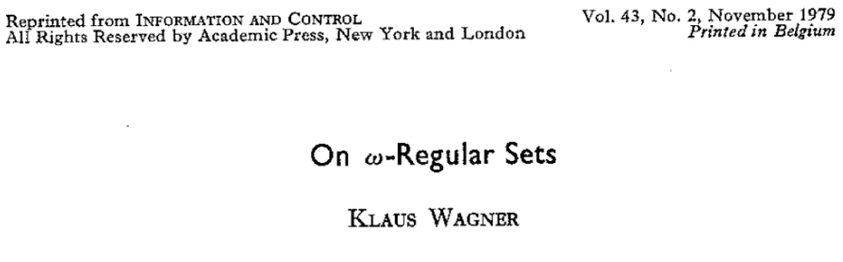










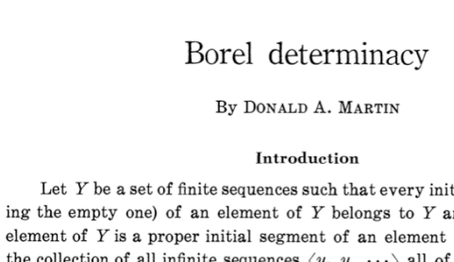


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

Reduction and Games



Reduction and Games





Reduction

Definition

- $X \leq Y \iff \exists f \textit{ simple } (x \in X \longleftrightarrow f(x) \in Y)$
- Y is $\mathcal{C}-$**complete** $\iff \begin{cases} Y \in \mathcal{C} \\ X \leq Y \text{ , any } X \in \mathcal{C} \end{cases}$

- X is less complicated than Y



Reduction

Definition

- $X \leq Y \iff \exists f \text{ simple } (x \in X \longleftrightarrow f(x) \in Y)$

- *simple* w.r. to topological complexity means **continuous**

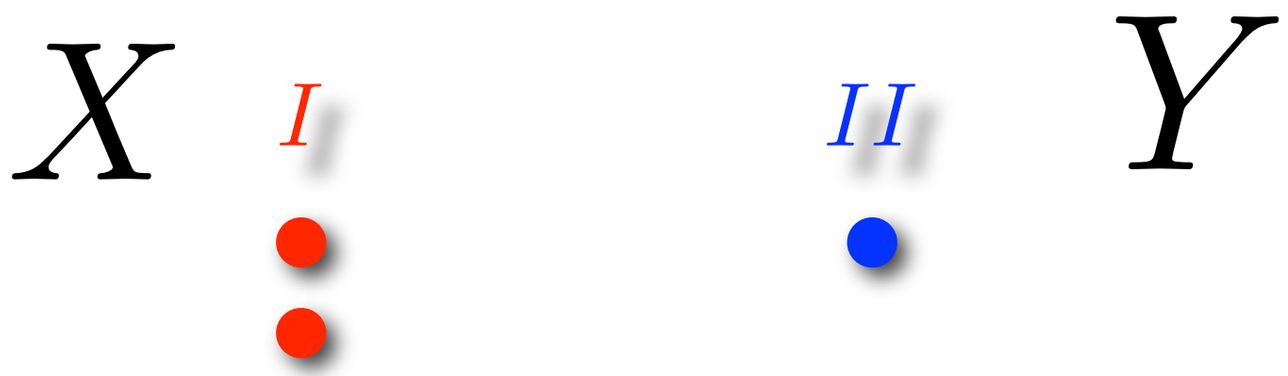
$$\mathbb{W}(X, Y)$$

X and Y are two sets of infinite words

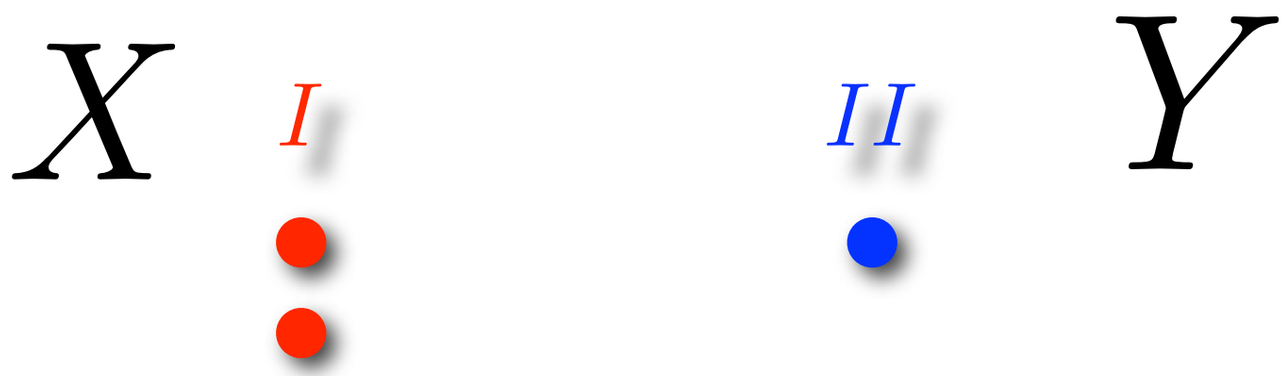
X I II Y


X
I
II
Y


X
I
II
Y

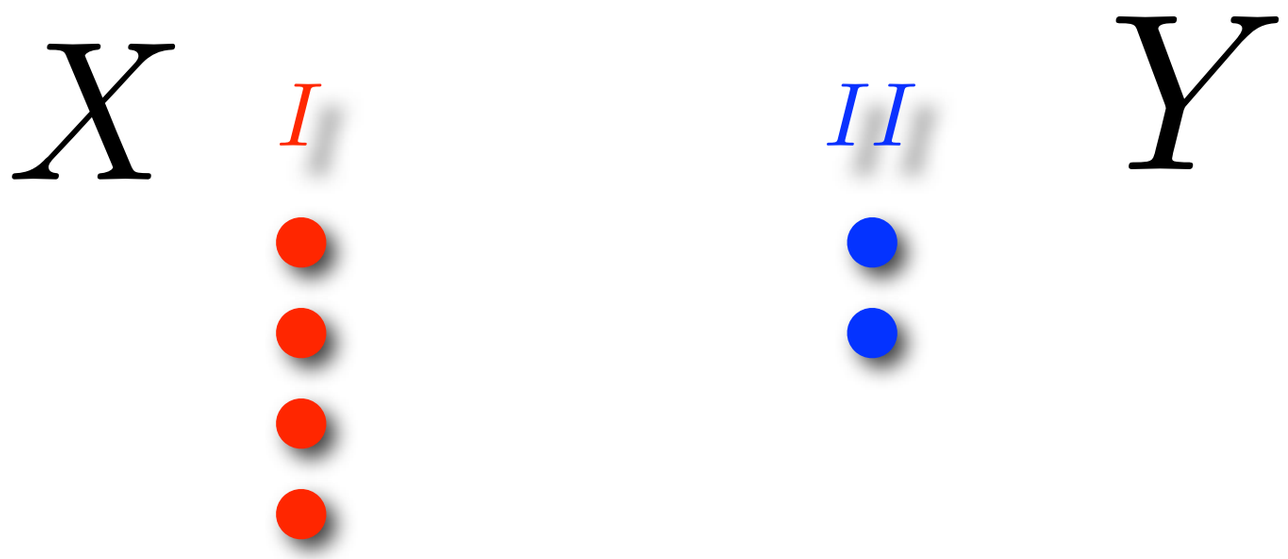

X
I
II
Y

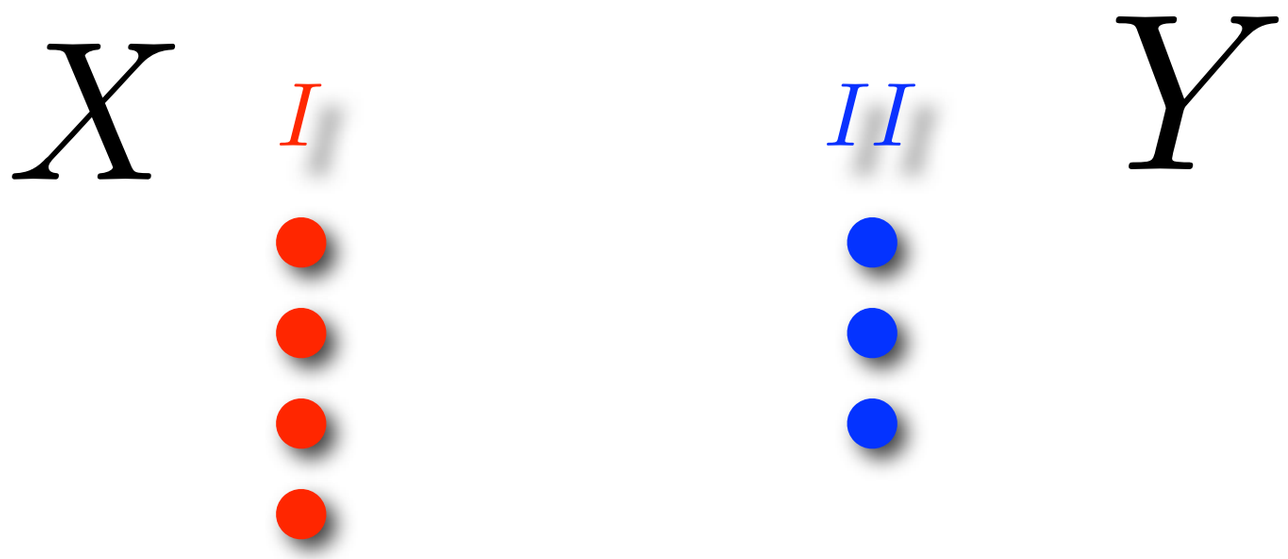

X
I
II
Y
$skip!$

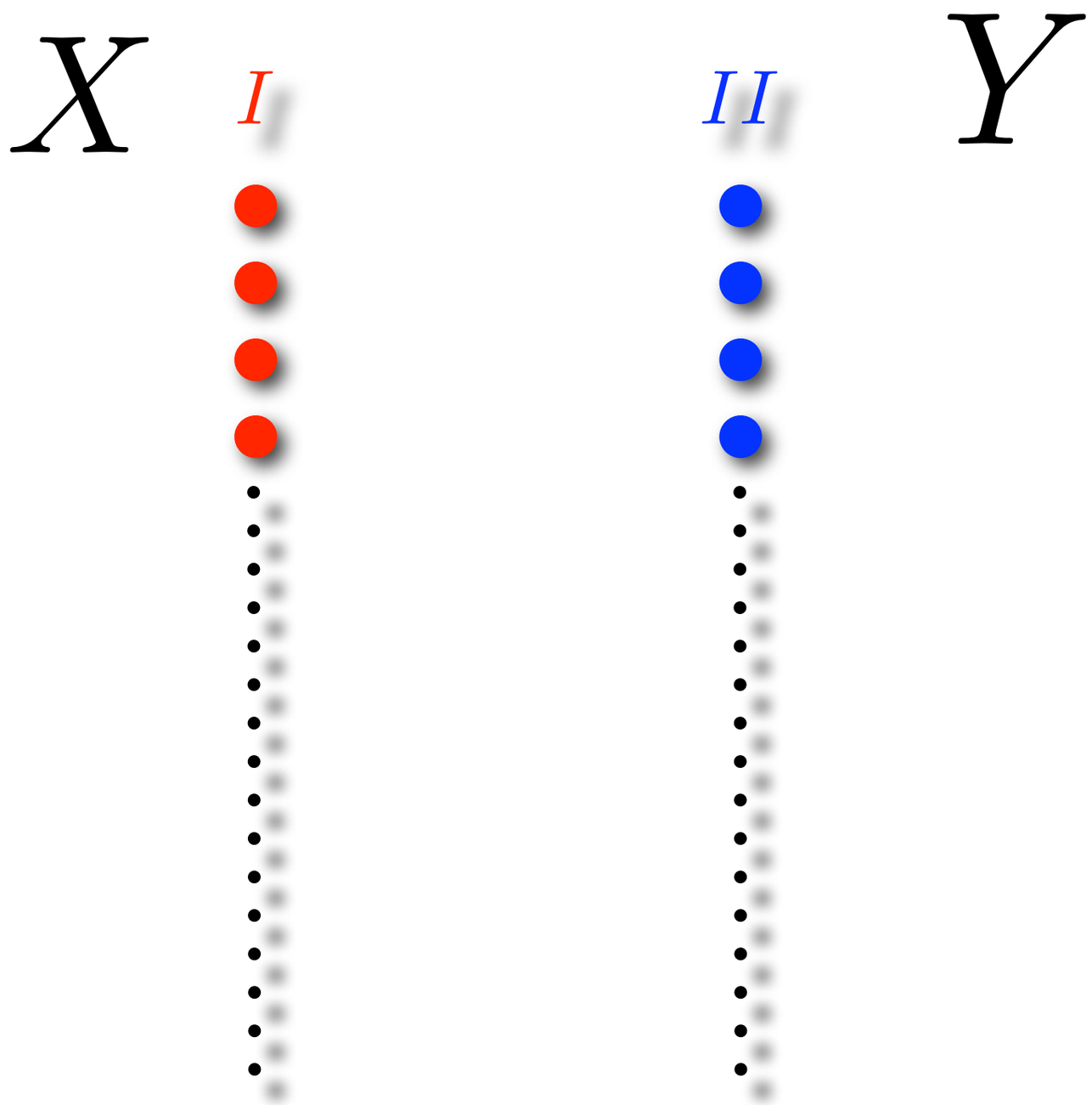

X
I
II
Y

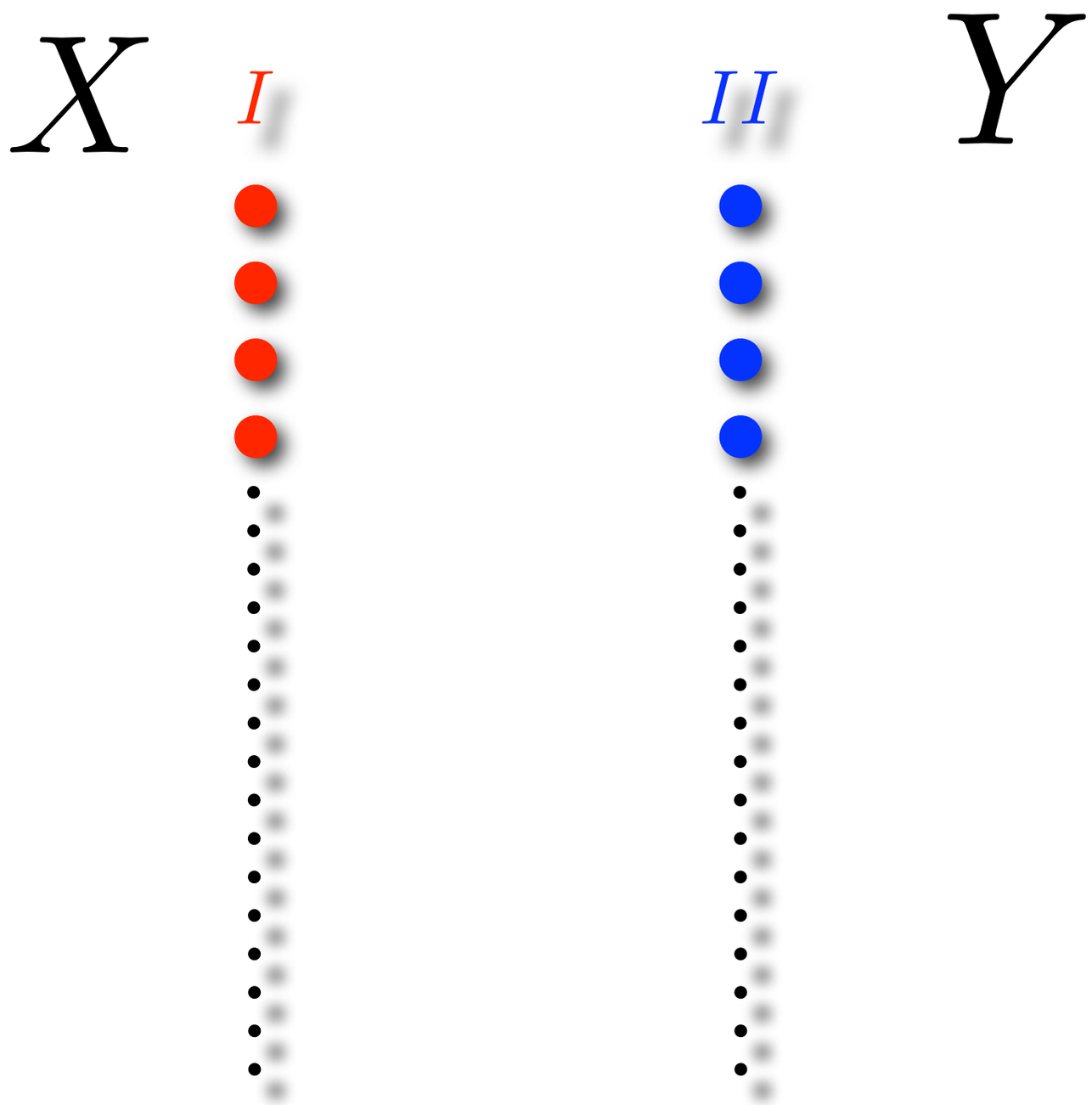

X
I
II
Y

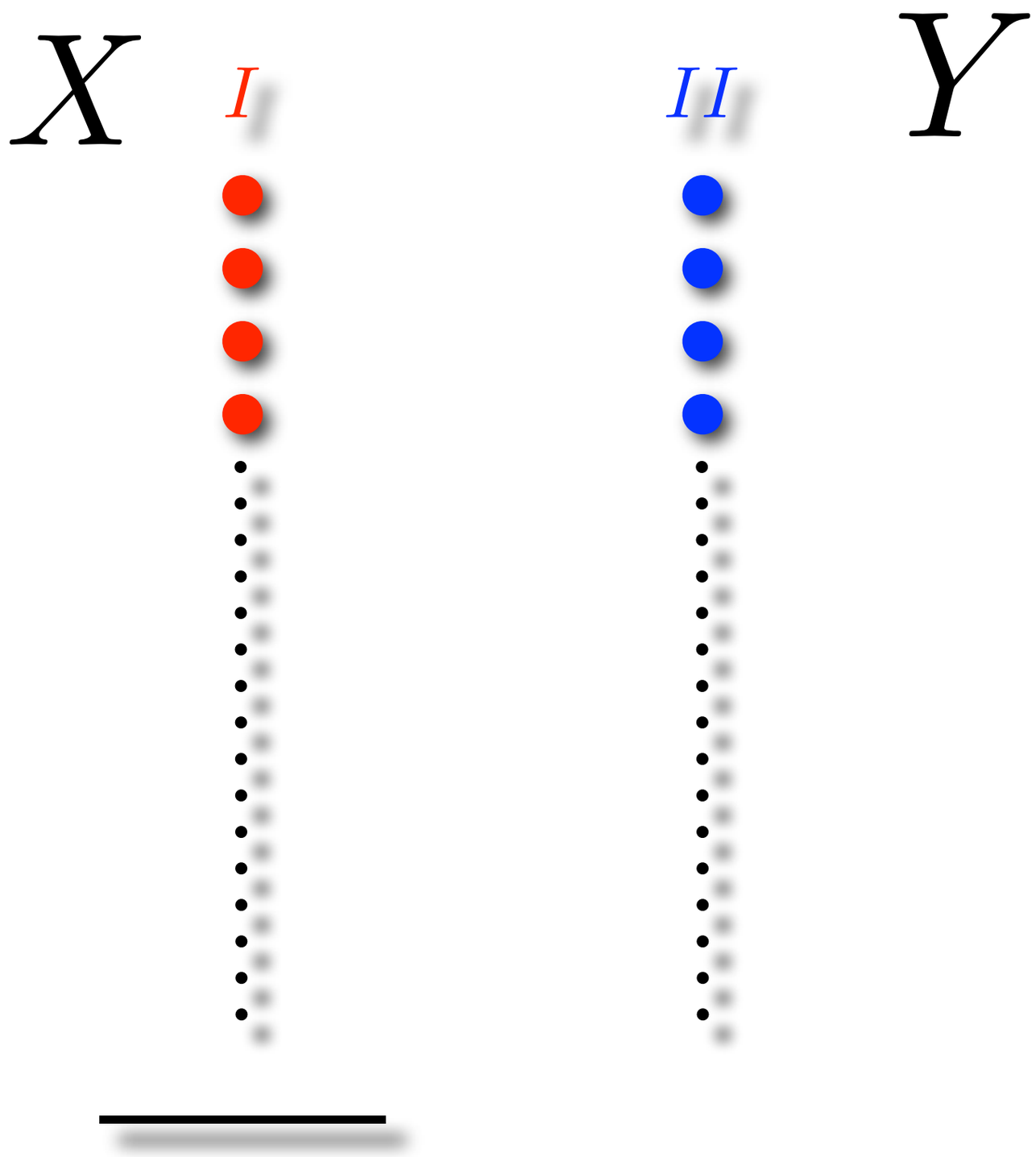

X
I
II
Y

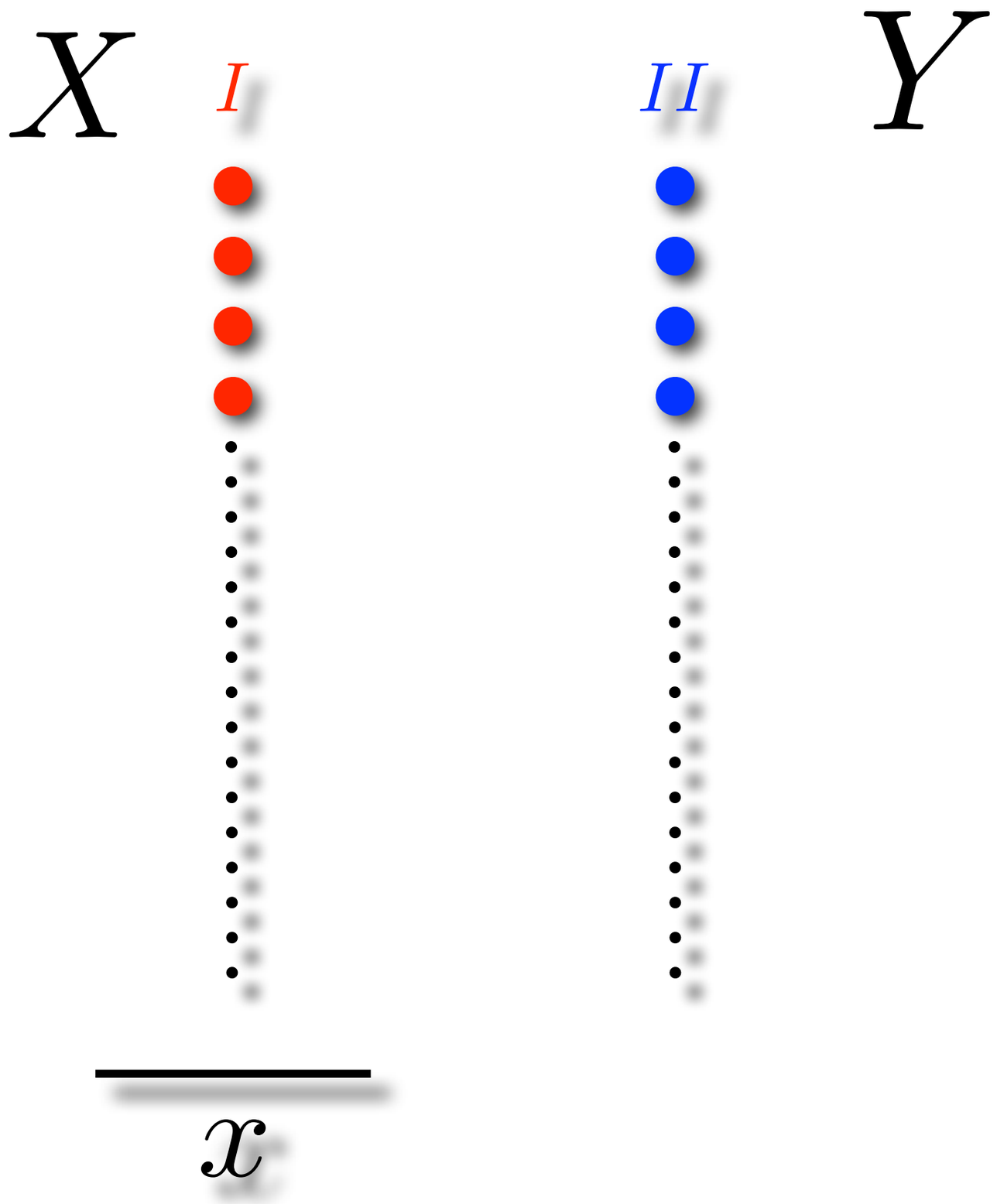

X
I
II
Y

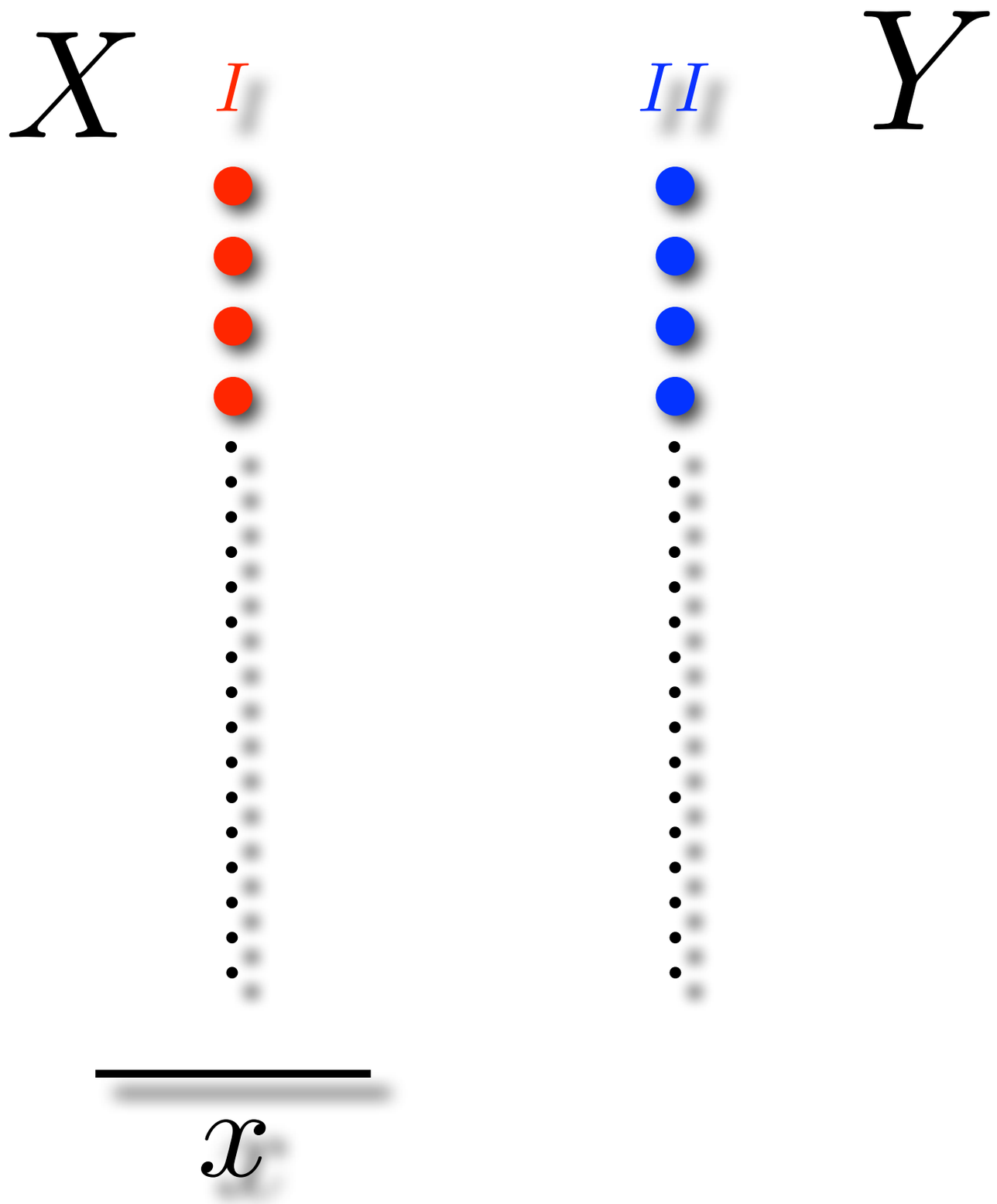

X
I
II
Y

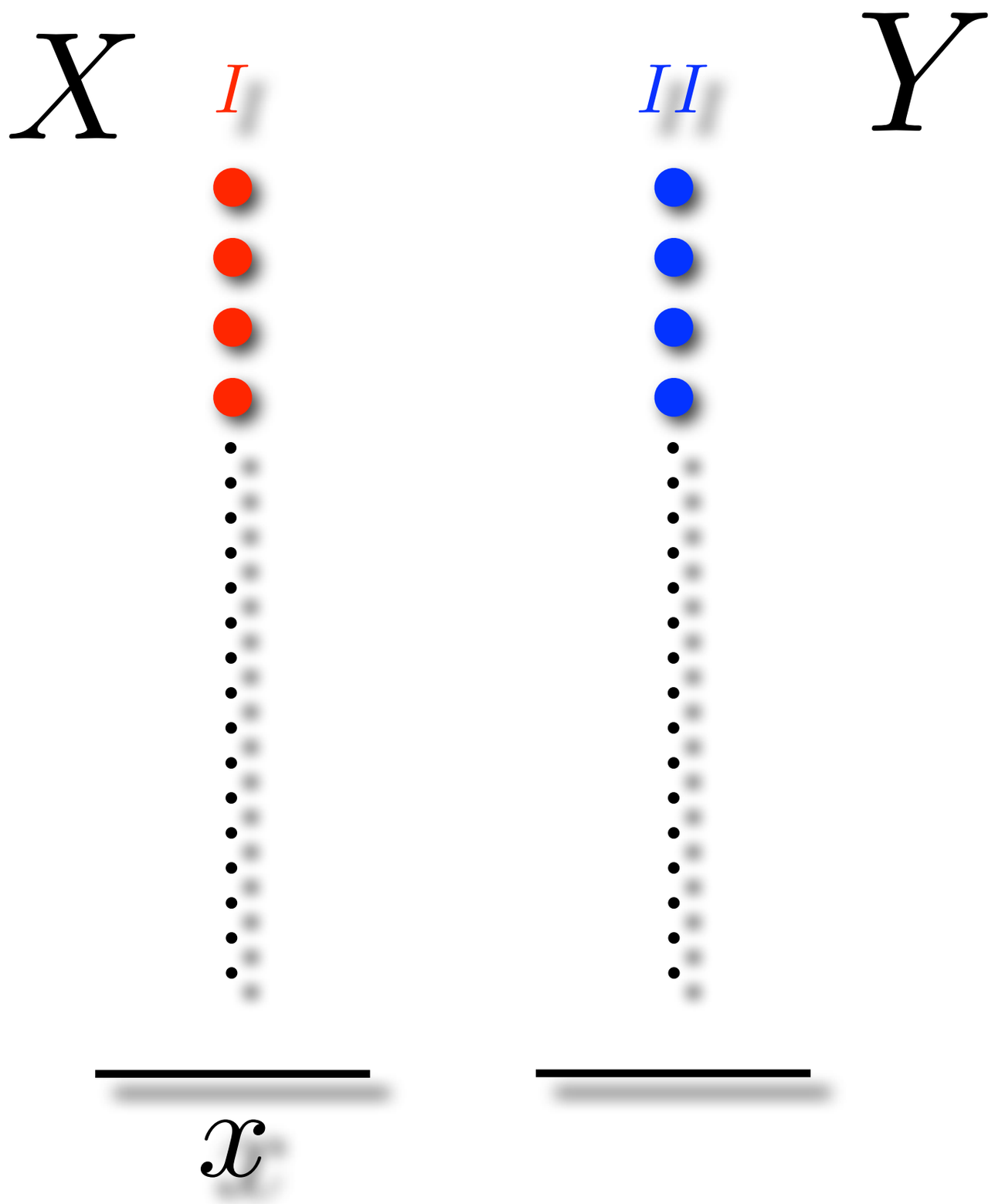

X
I
II
Y

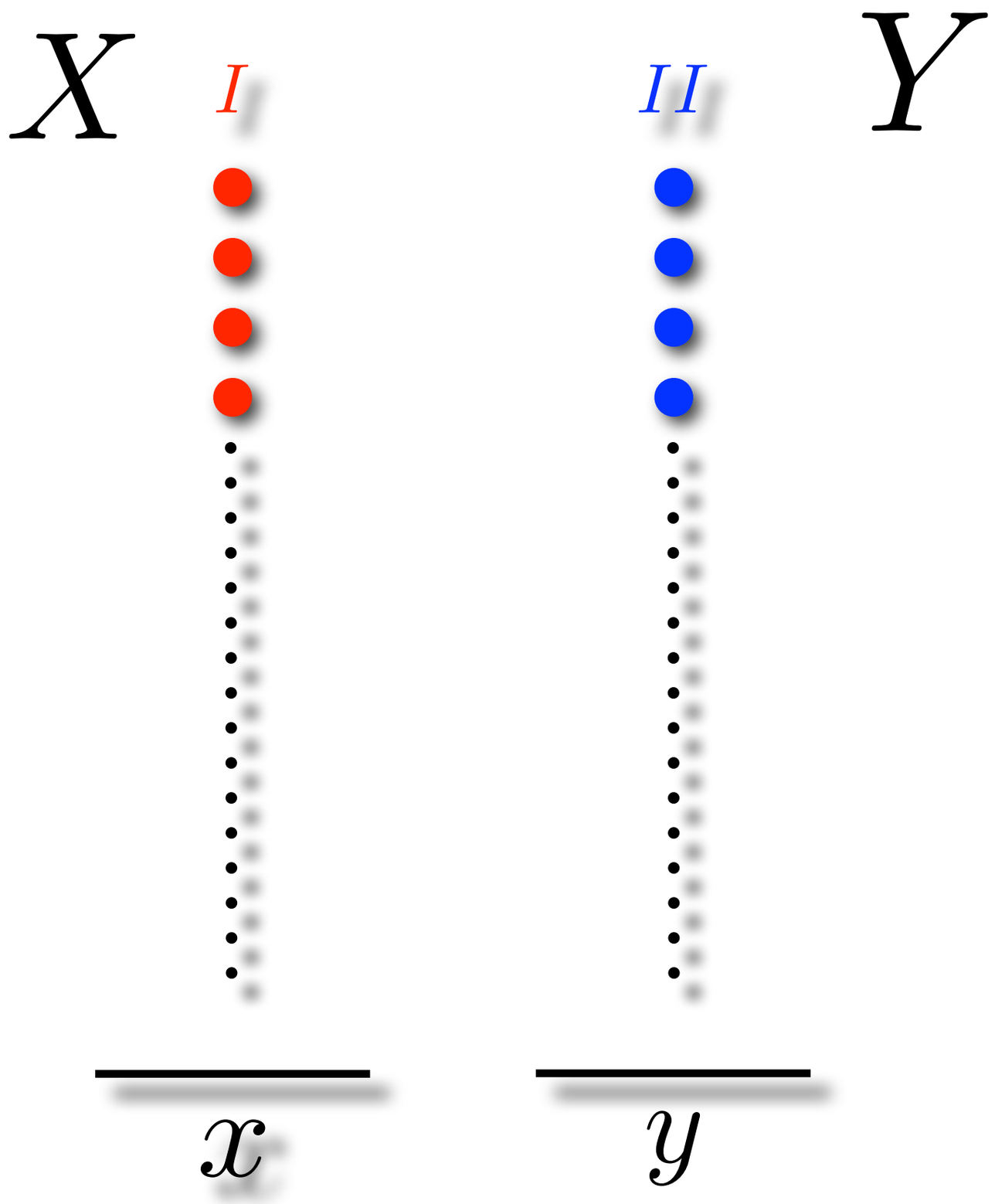

X
I
II
Y

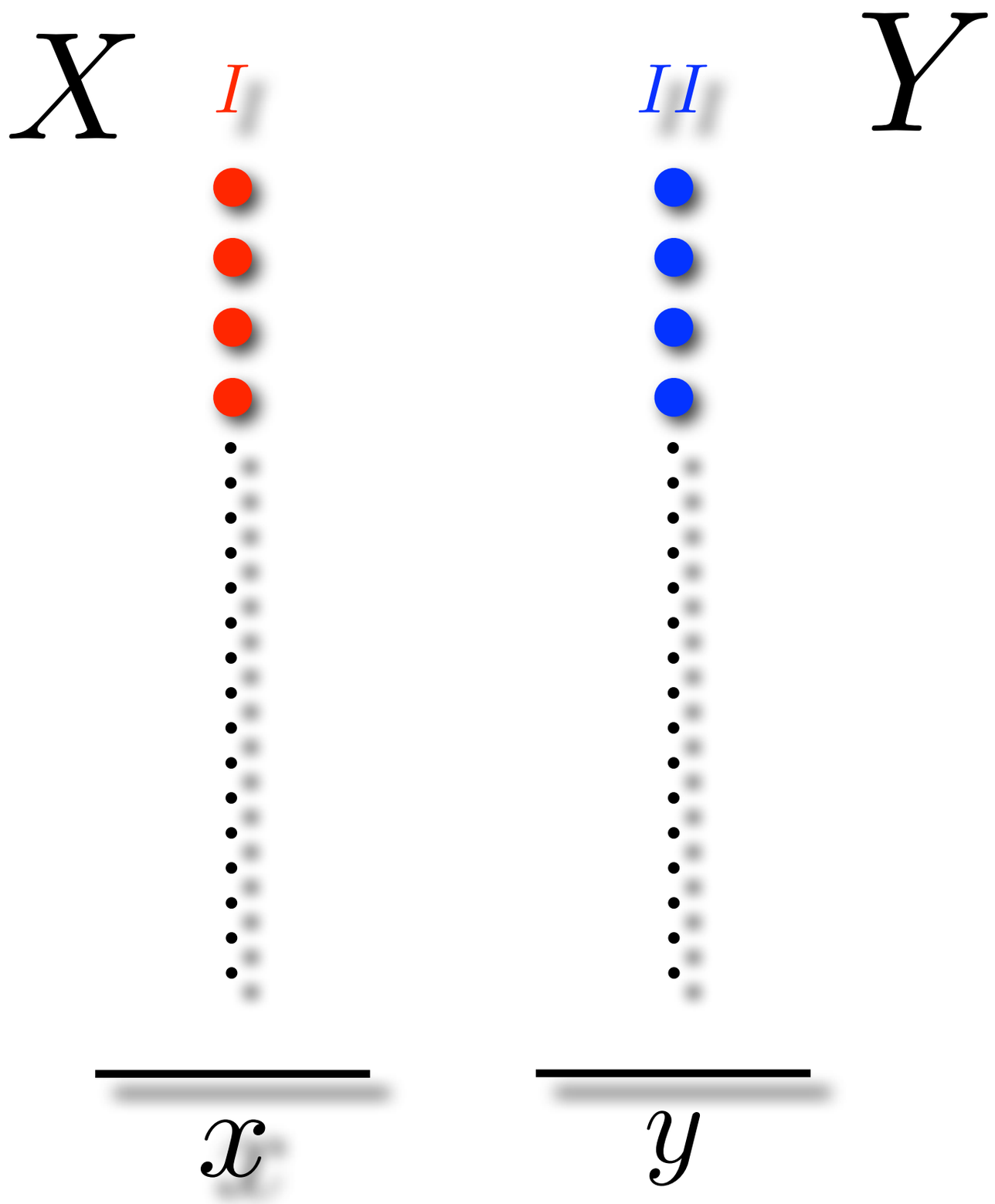

X
I
II
Y


X
I
II
Y
x

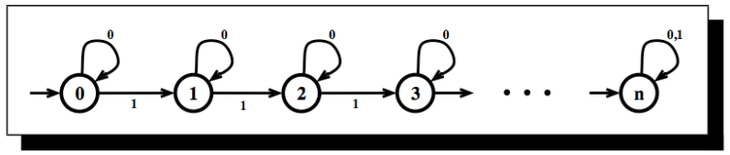

X
I
II
Y
x

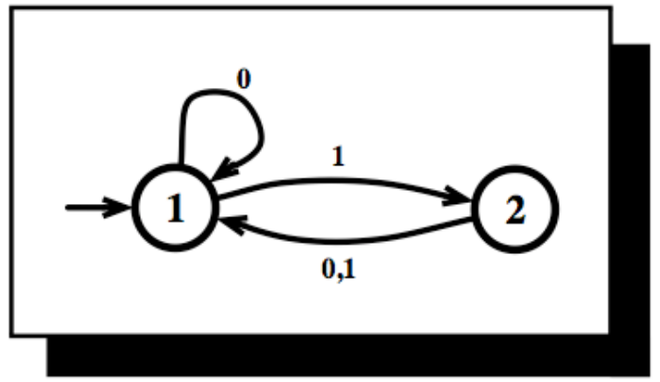

X
I
II
Y
x

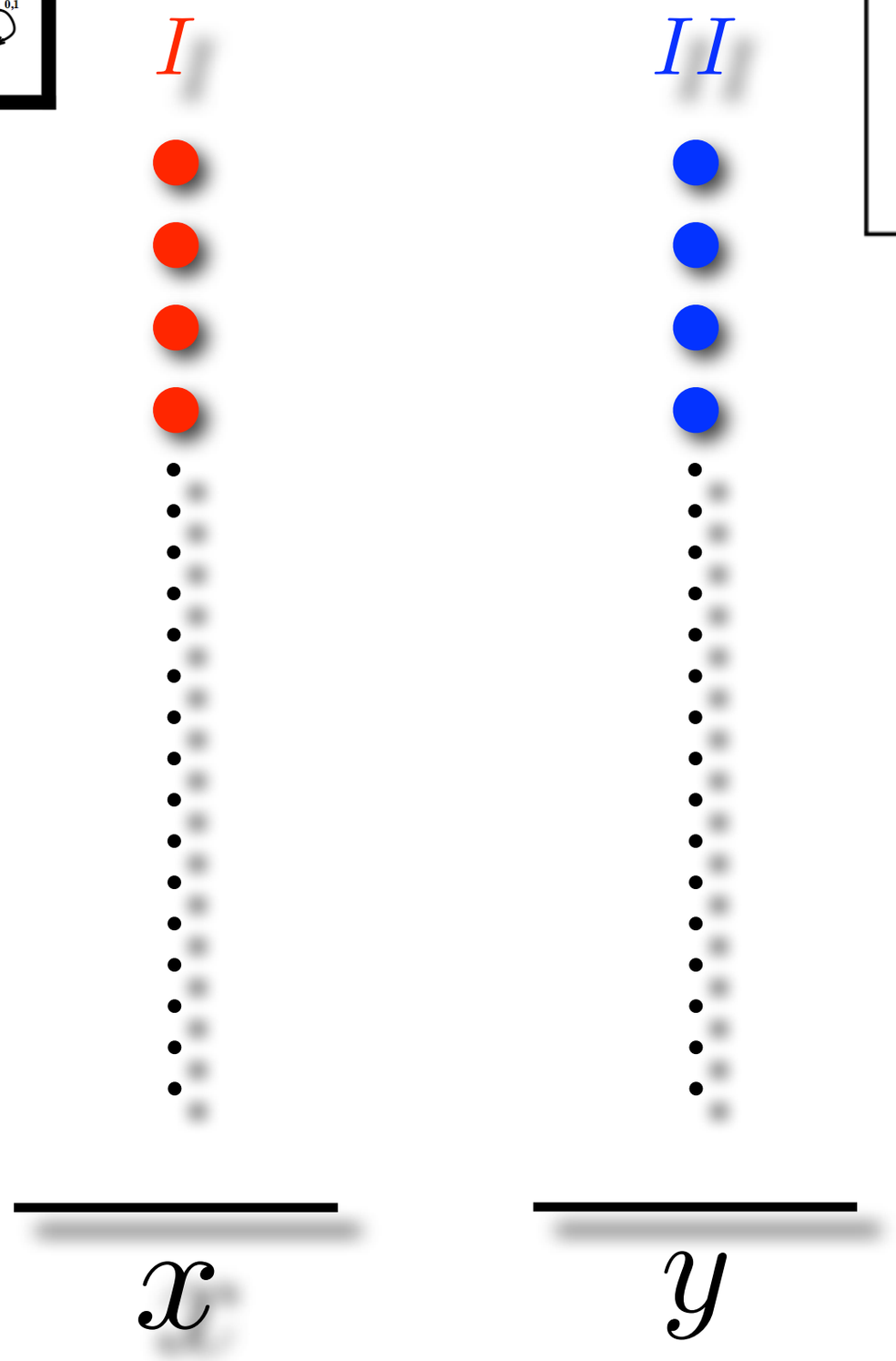

X
I
II
Y
x
y



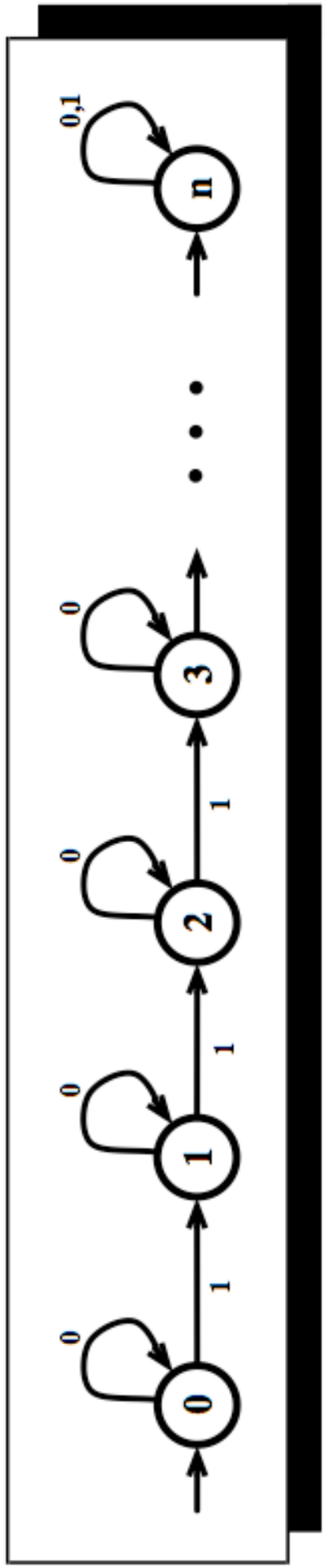
$$II \text{ wins} \iff (x \in X \longleftrightarrow y \in Y)$$





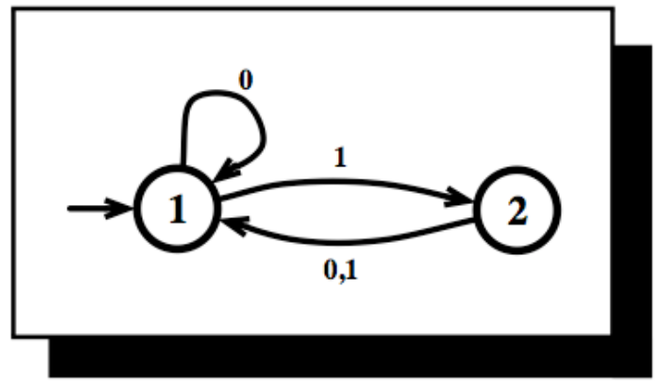


$$II \text{ wins} \iff (x \in X \longleftrightarrow y \in Y)$$



I

II



$$\frac{}{x} \qquad \frac{}{y}$$

II wins $\iff (x \in X \longleftrightarrow y \in Y)$

- $Determinacy\ \mathcal{G} \Longrightarrow Determinacy\ \mathbb{W}$
- $Determinacy \Longrightarrow \leq\ is\ almost\ a\ well\text{-}ordering$
  - $X < Y \iff X \leq Y \not\leq X$
  - $X \equiv Y \iff X \leq Y \leq X$

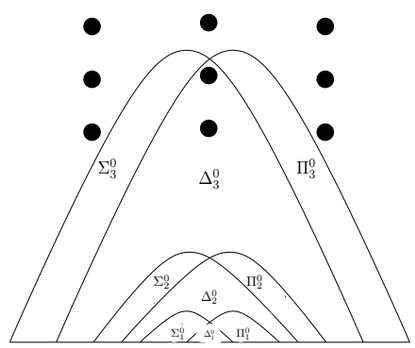
Reduction and Games



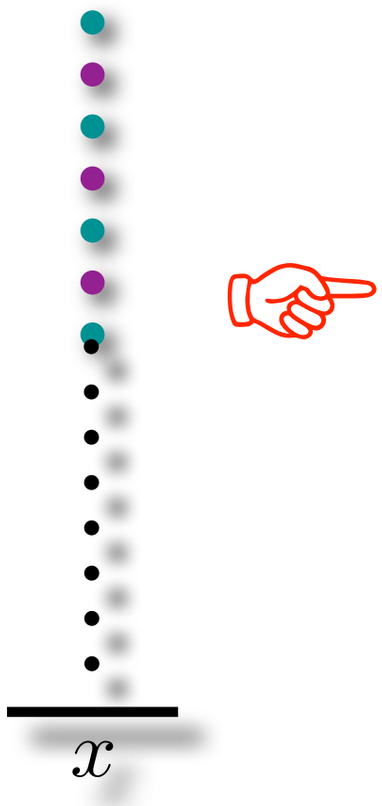


An example: evaluation game for 1st order logic

non determined game

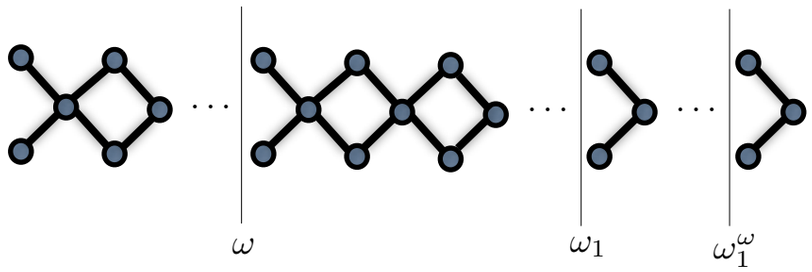


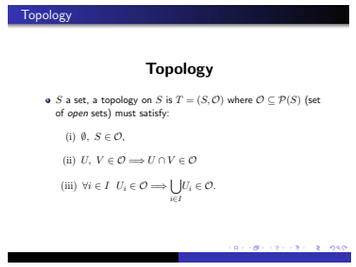
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

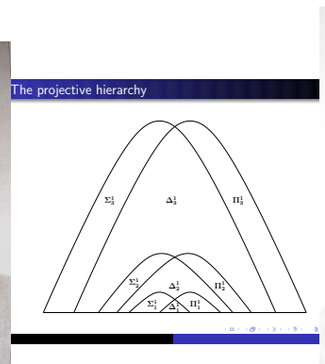
Games that construct mappings



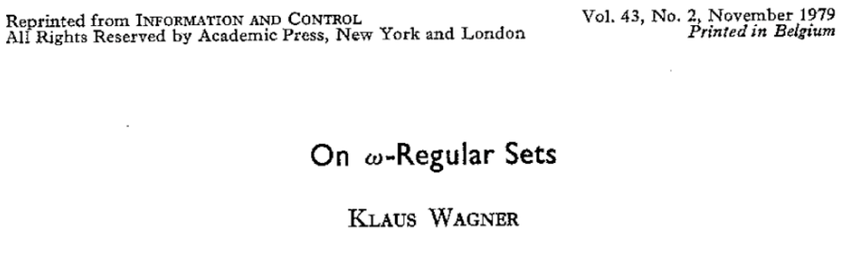










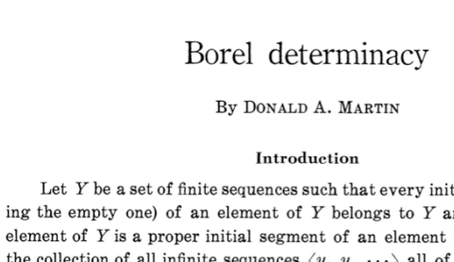


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



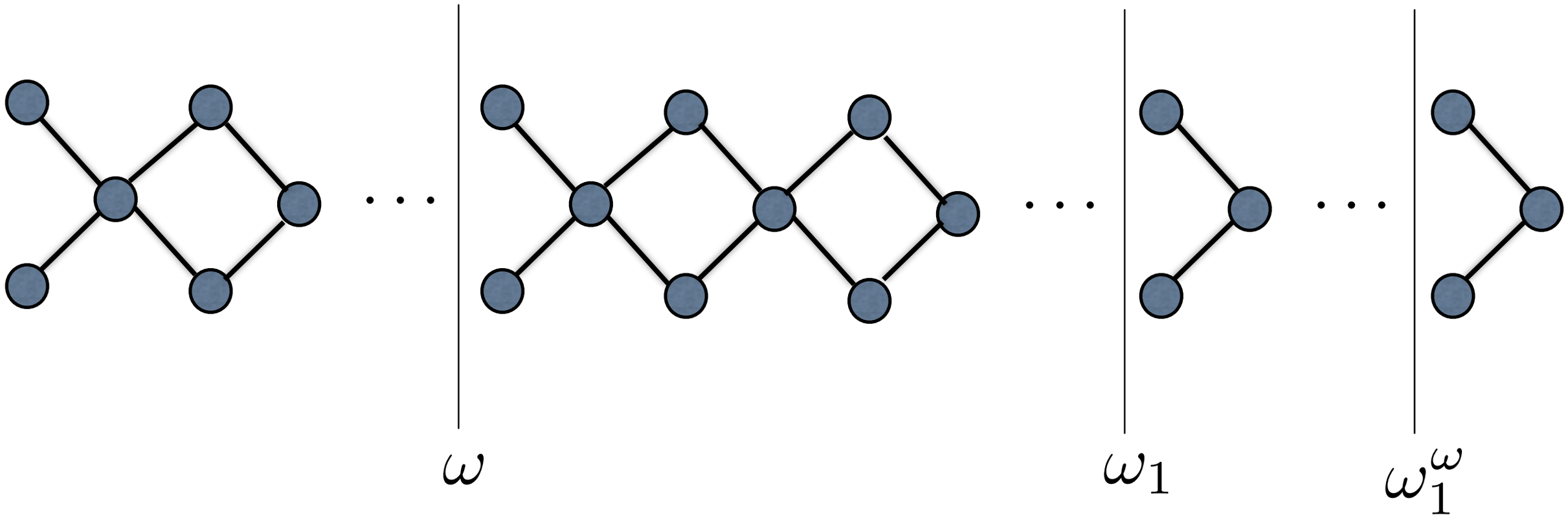
Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

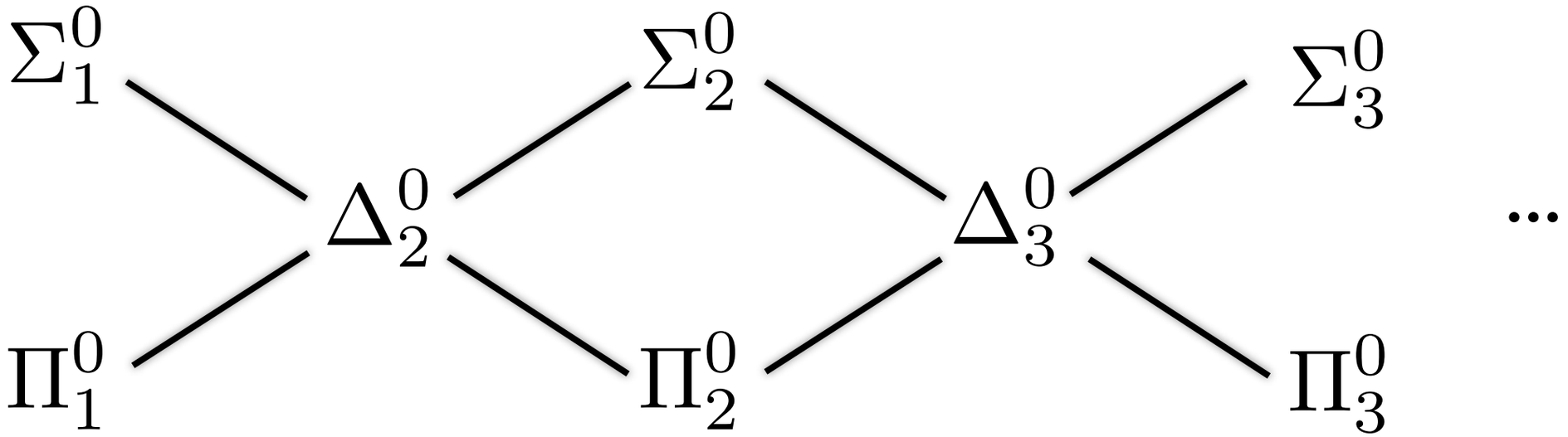
By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of


ω
ω_1
ω_1^ω

The Borel Hierarchy

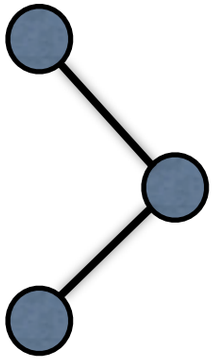


The Wadge Hierarchy

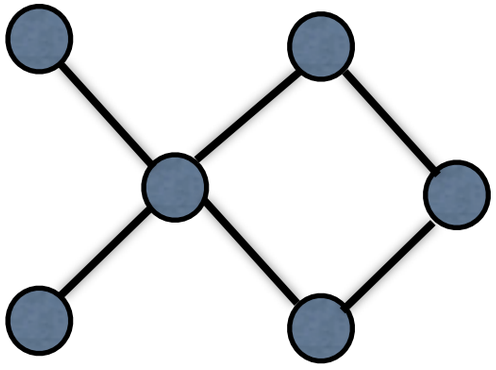
The Wadge Hierarchy



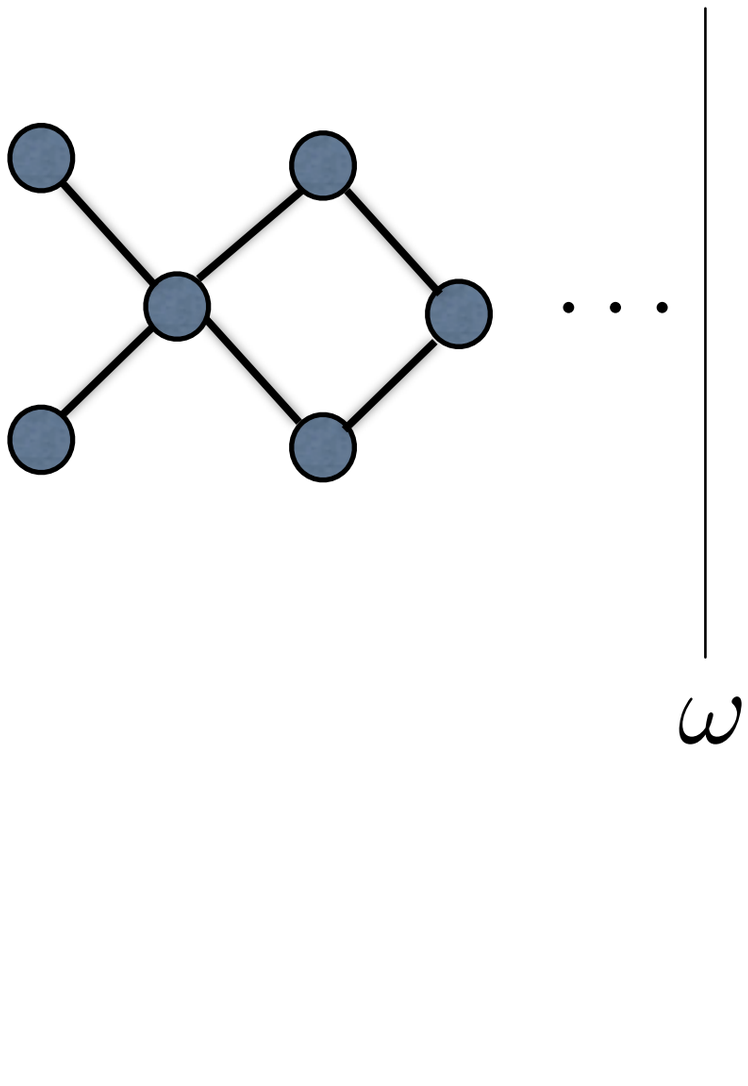
The Wadge Hierarchy



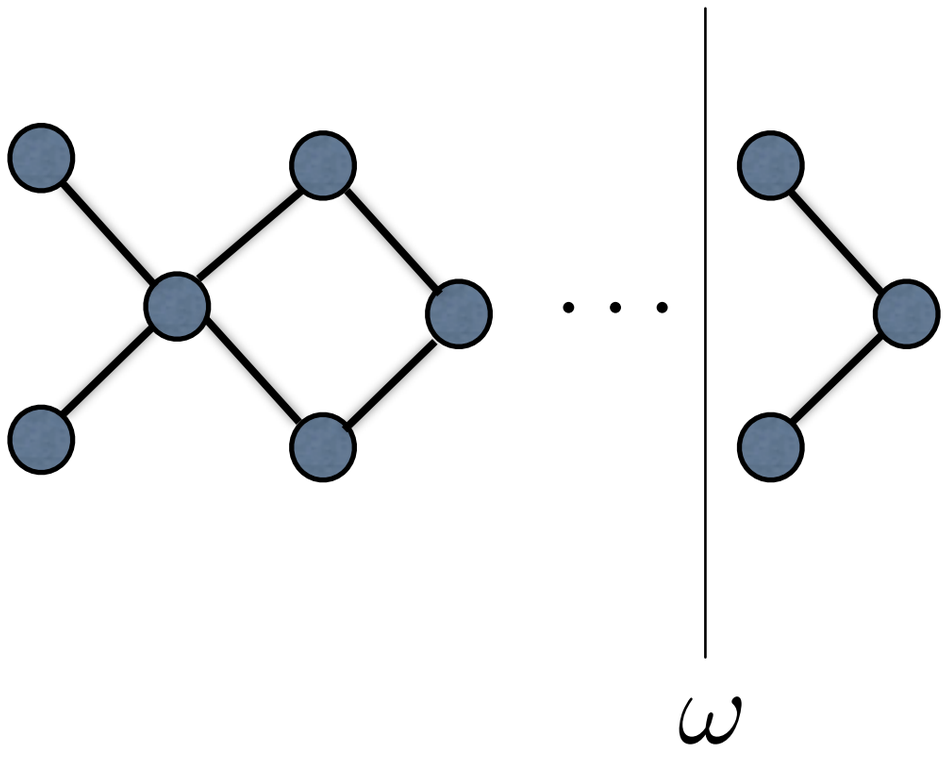
The Wadge Hierarchy



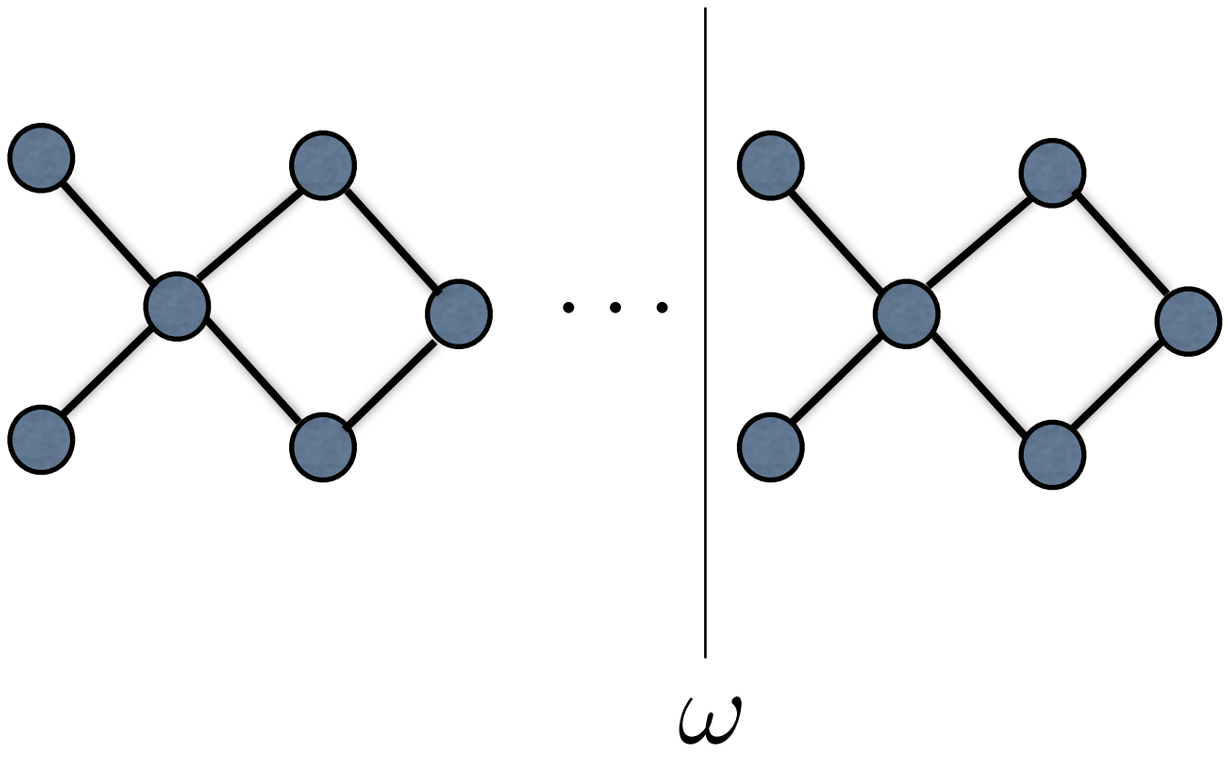
The Wadge Hierarchy



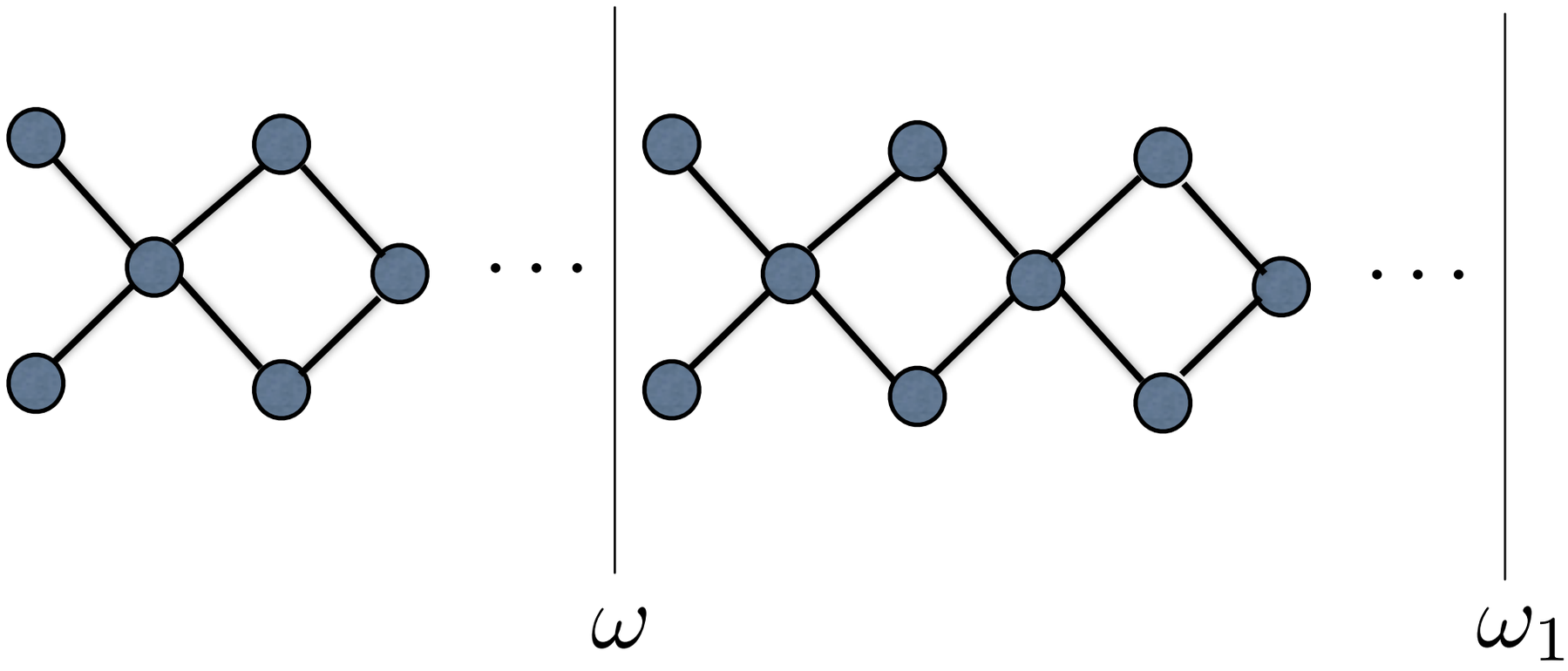
The Wadge Hierarchy



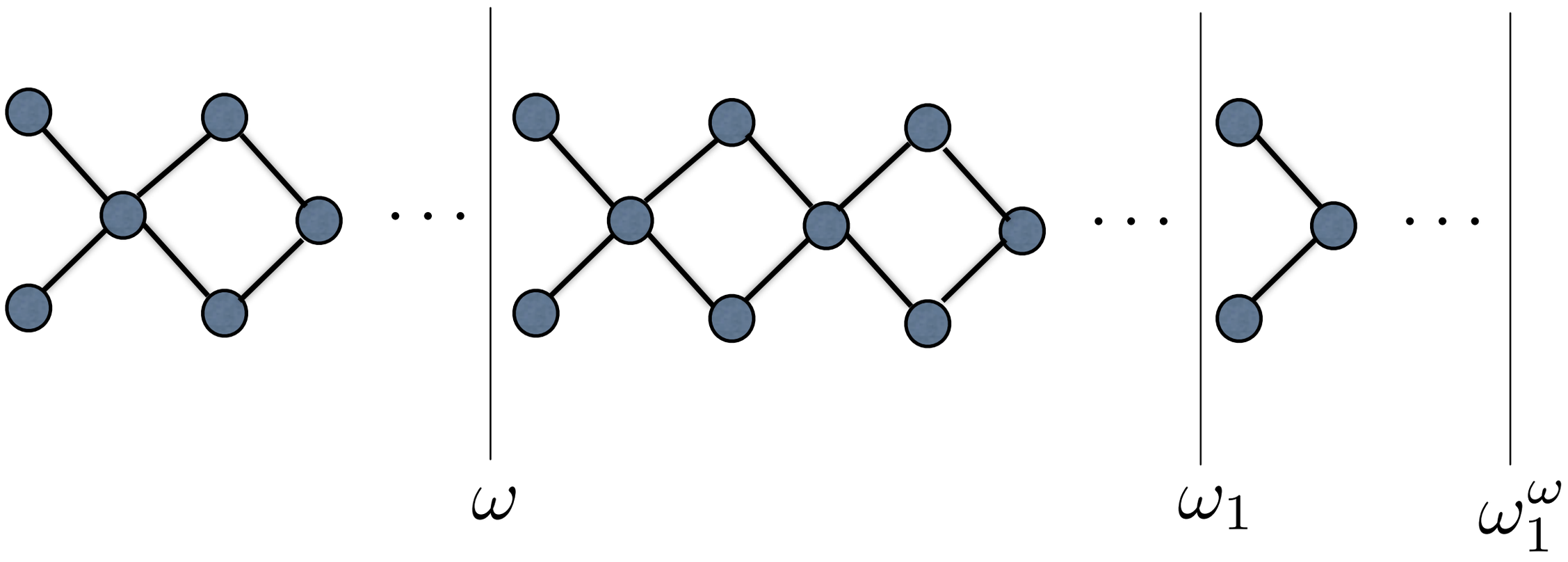
The Wadge Hierarchy



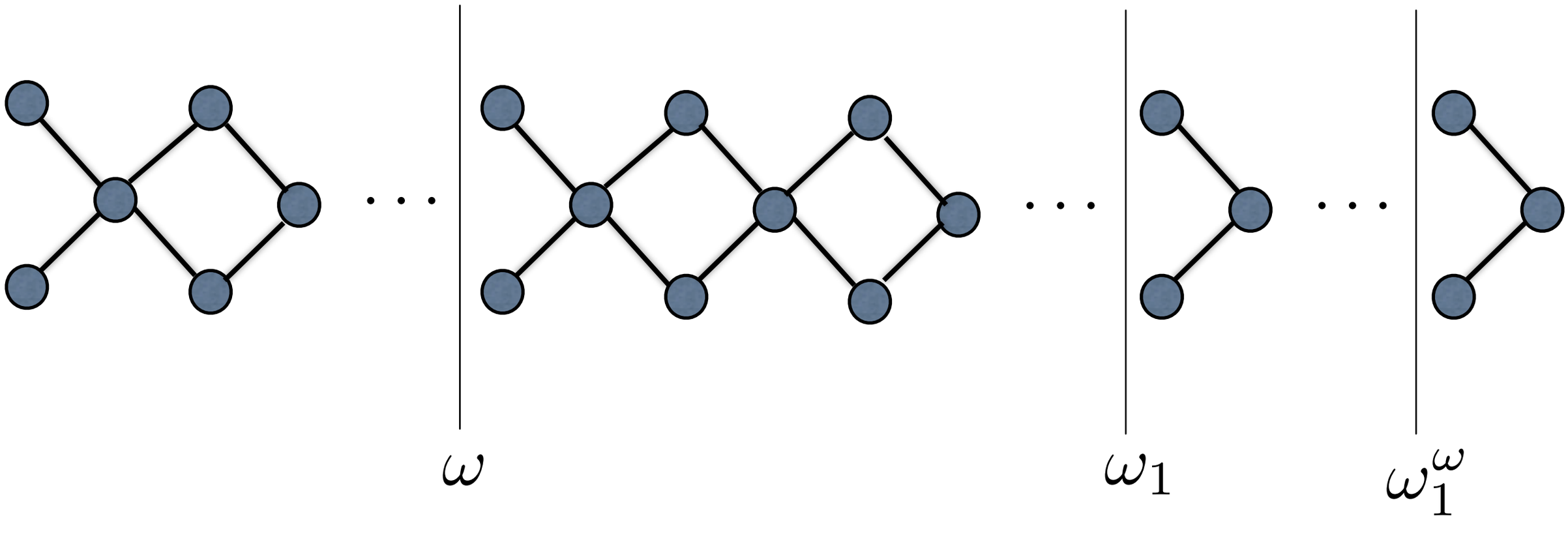
The Wadge Hierarchy



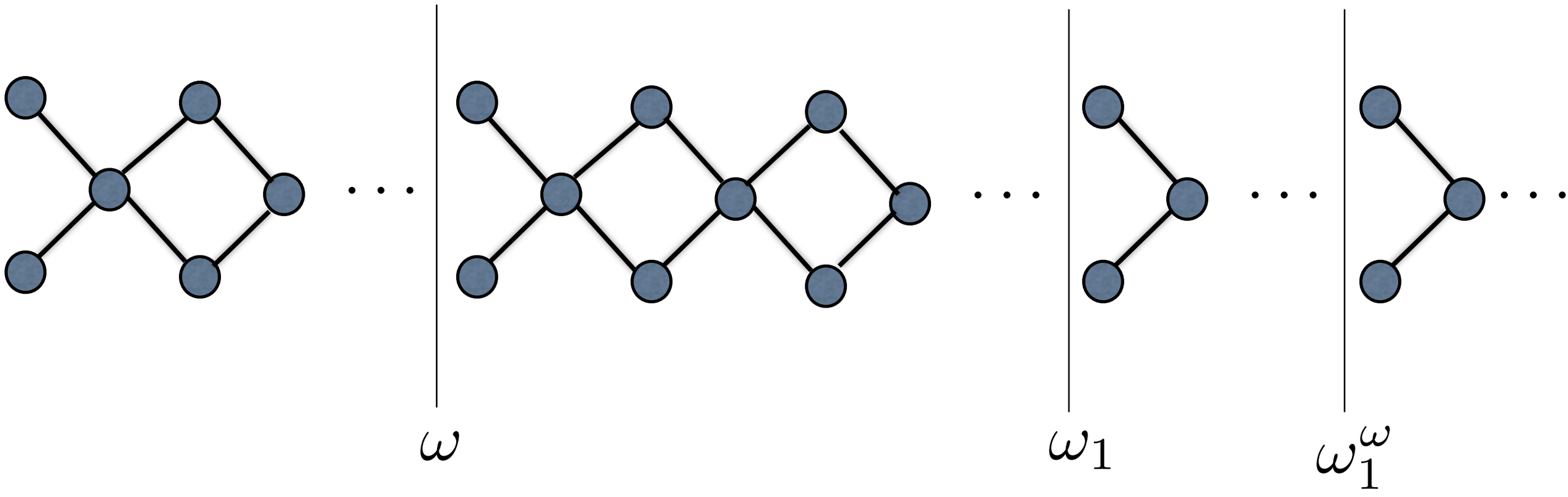
The Wadge Hierarchy



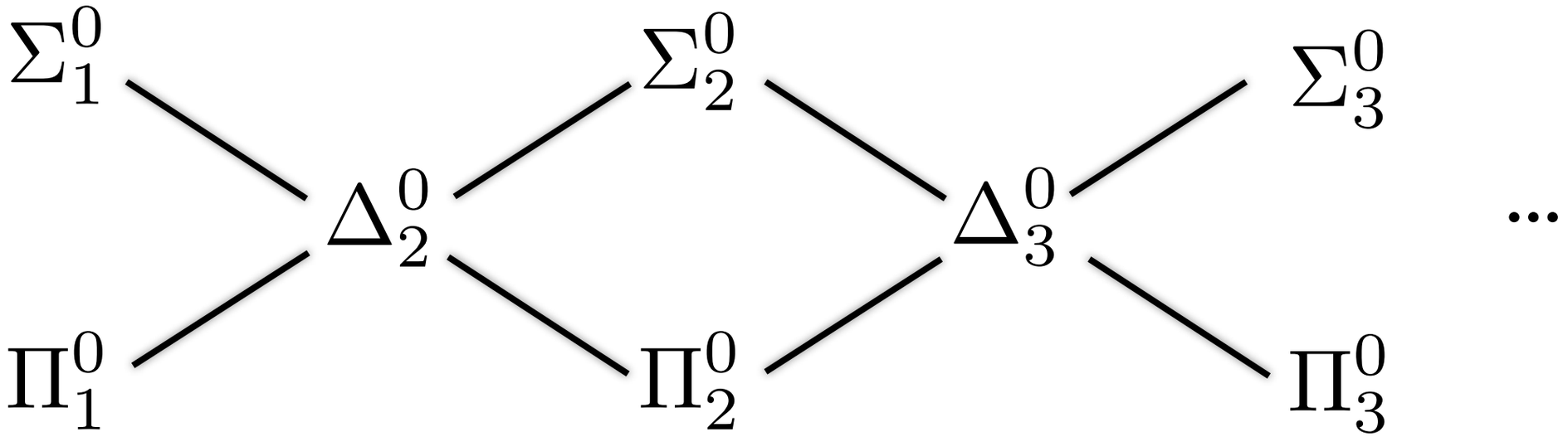
The Wadge Hierarchy



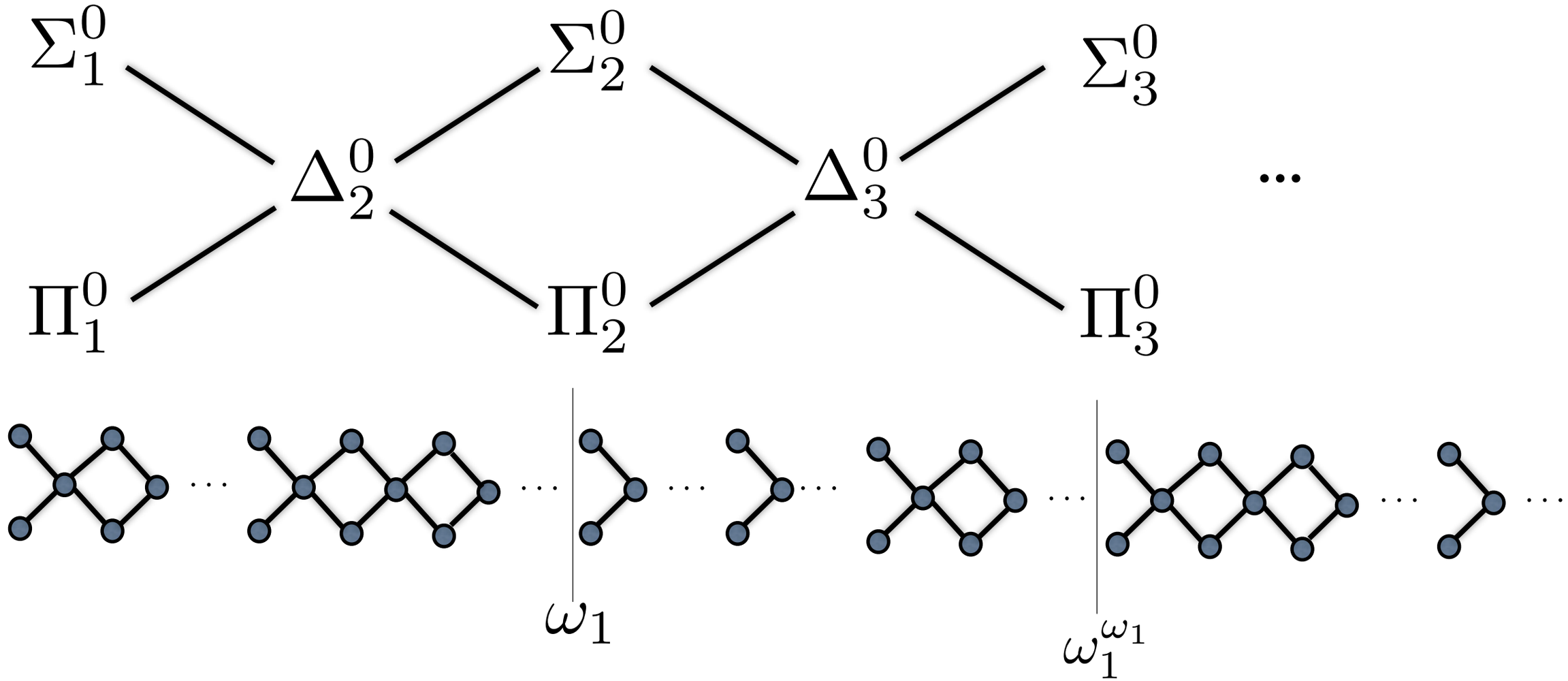
The Wadge Hierarchy



The Borel Hierarchy



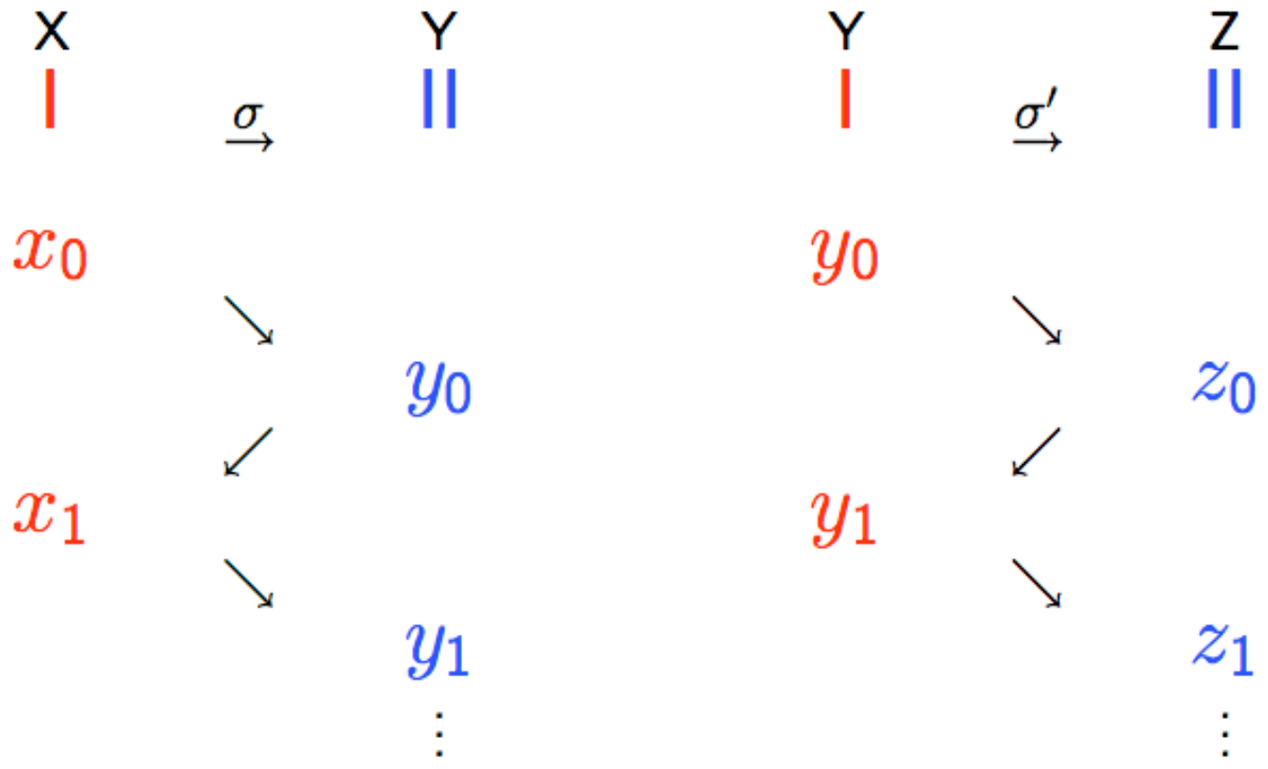
The Borel Hierarchy

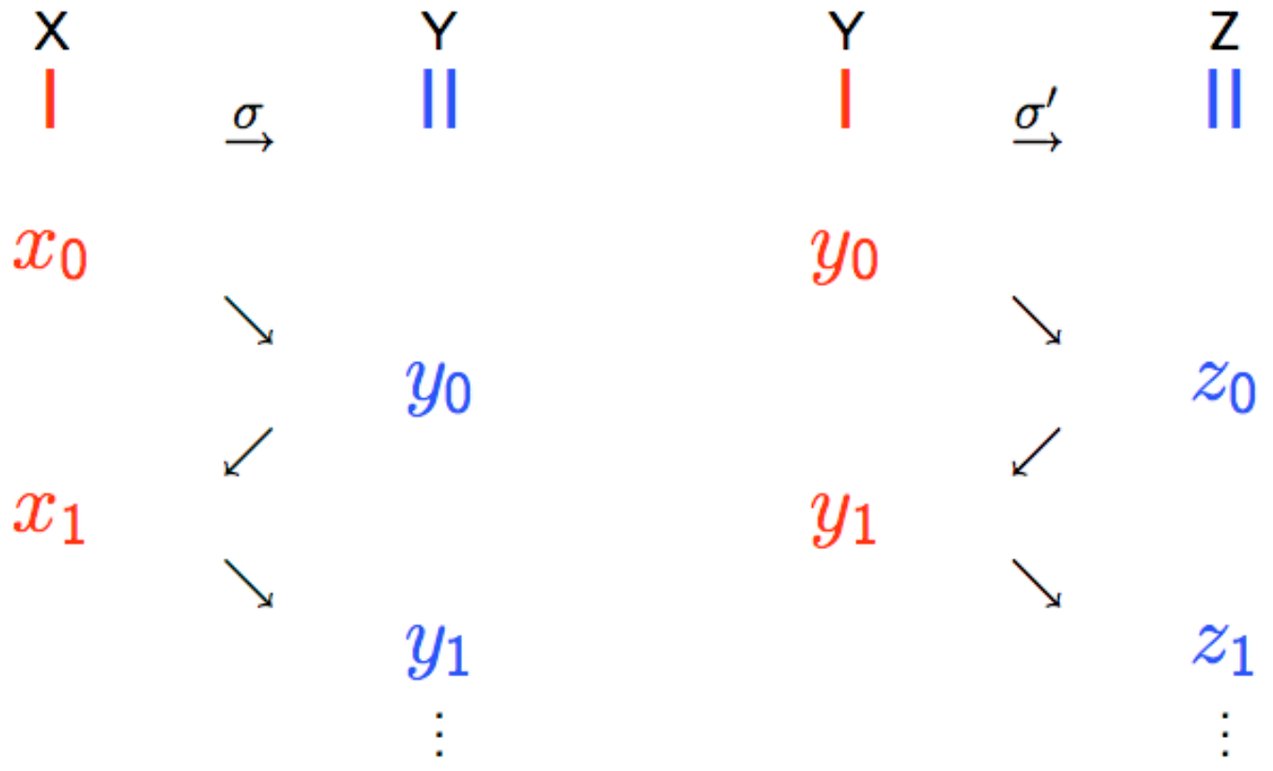


Basic Properties of the Wadge Ordering

Lemma

- $\leq_w$ *is reflexive* $\quad X \leq_w X$
- $\leq_w$ *is transitive* $\quad X \leq_w Y \leq_w Z \implies X \leq_w Z$


X
Y
Y
Z
$\xrightarrow{\sigma}$
$\xrightarrow{\sigma'}$
x_0
x_1
y_0
y_1
y_0
y_1
z_0
z_1



$$x \in X \quad \leftrightarrow \quad y \in Y$$

$$y \in Y \quad \leftrightarrow \quad z \in Z$$

Basic Properties of the Wadge Ordering

Lemma

- $\leq_w$ *is reflexive* $\quad X \leq_w X$
- $\leq_w$ *is transitive* $\quad X \leq_w Y \leq_w Z \implies X \leq_w Z$

- $\leq_w$ is a partial ordering
- with Determinacy . . . . . .

Definition

- $X \equiv_w Y \iff X \leq_w Y \leq_w X$
- $X <_w Y \iff X \leq_w Y \not\leq_w X$

Wadge Determinacy

Theorem (Louveau, Saint-Raymond (86))

$$\textit{Borel Determinacy} \overset{\Rightarrow}{\not\Leftarrow} \textit{Borel Wadge Determinacy}$$

Theorem (Harrington (78))

$$\Sigma^1_1\textit{-Determinacy} \iff \Sigma^1_1\textit{-Wadge Determinacy}$$

Theorem (Martin, Monk, Wadge)

Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 {}_W > X_1 {}_W > X_2 {}_W > \ldots {}_W > X_k {}_W > X_{k+1} {}_W > \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$

- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$

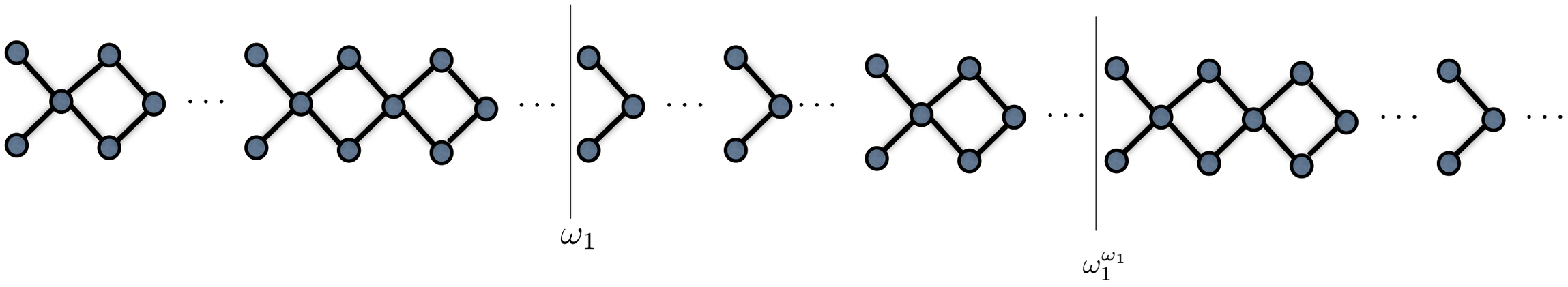
Theorem (Martin, Monk, Wadge)

Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 {}_W > X_1 {}_W > X_2 {}_W > \ldots {}_W > X_k {}_W > X_{k+1} {}_W > \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$
- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$



Theorem (Martin, Monk, Wadge)

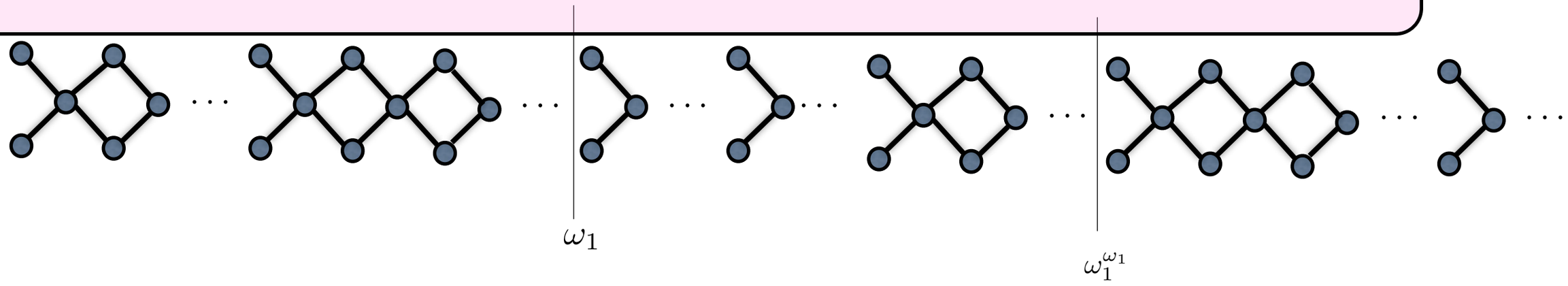
Assuming Determinacy,

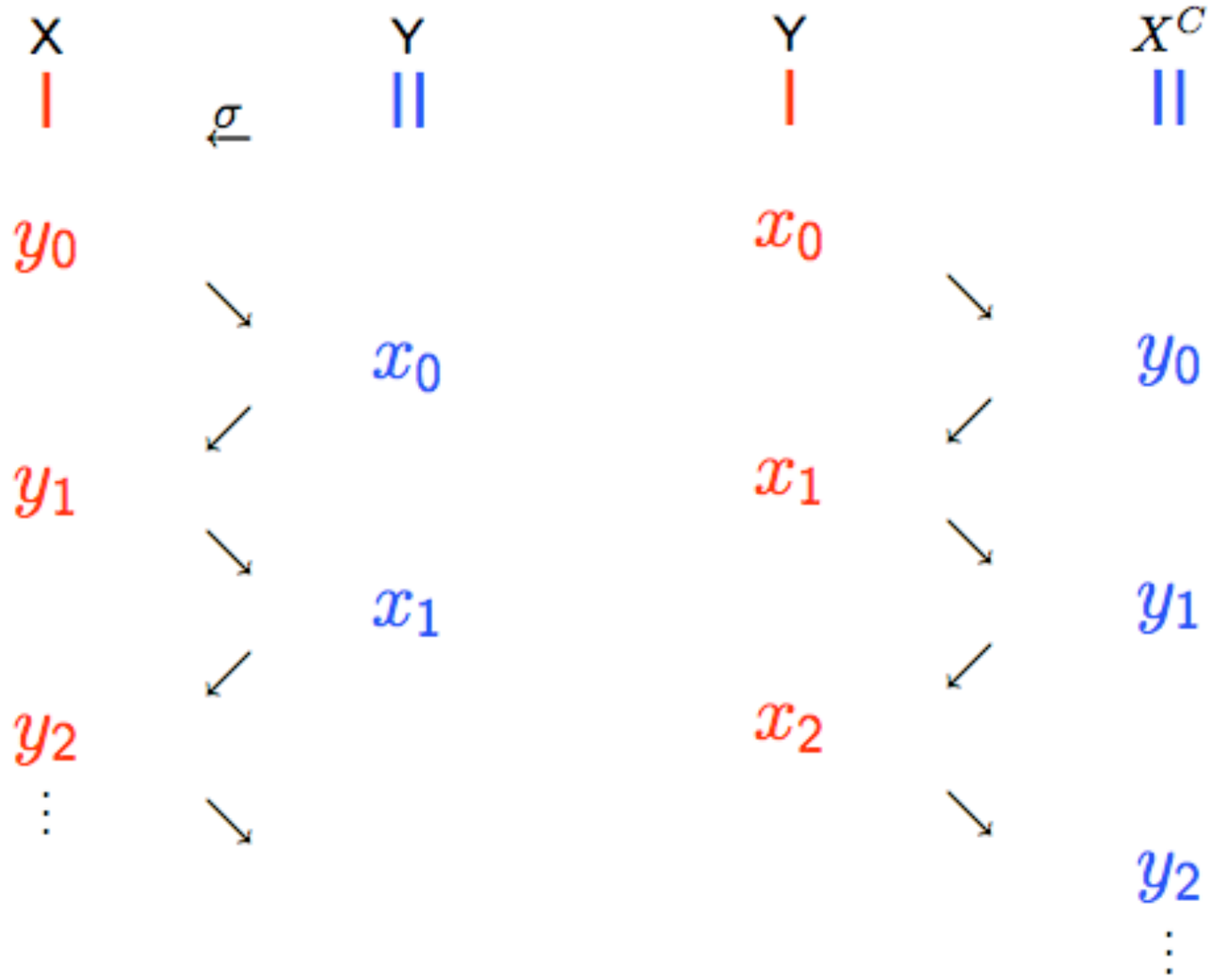
- $<_W$ *is wellfounded, i.e. there is no*

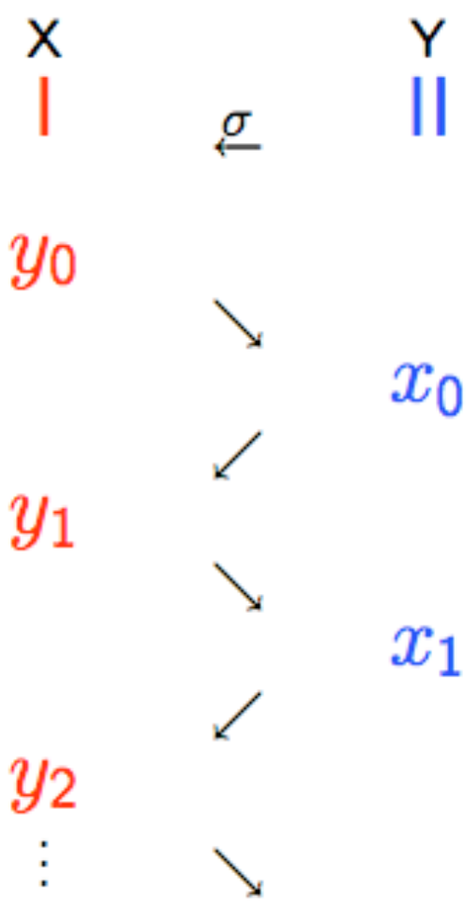
$$X_0 {}_W > X_1 {}_W > X_2 {}_W > \ldots {}_W > X_k {}_W > X_{k+1} {}_W > \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$

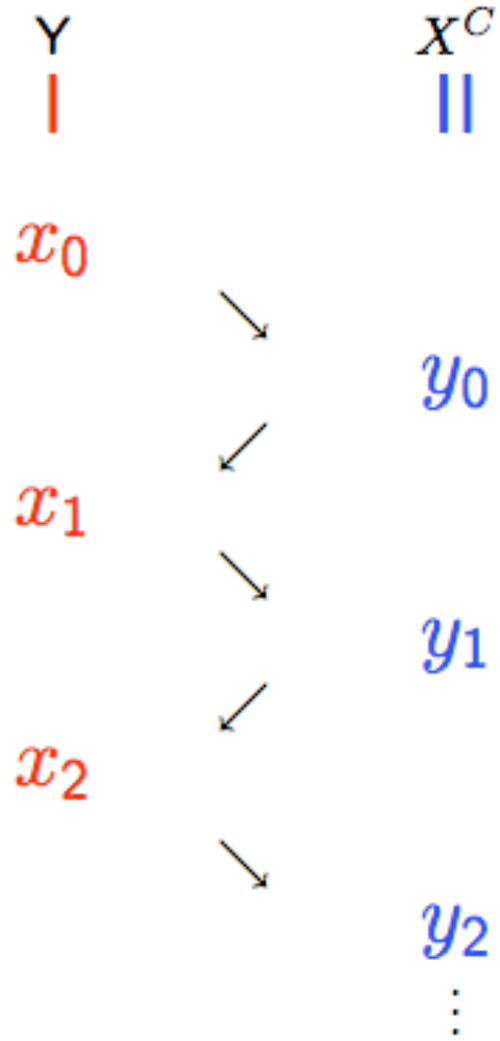
- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$




X
Y
σ
y_0
x_0
y_1
x_1
y_2
Y
X^C
x_0
y_0
x_1
y_1
x_2
y_2



$$y \notin X \quad \leftrightarrow \quad x \in Y$$



$$x \in Y \quad \leftrightarrow \quad y \in X^C$$

Theorem (Martin, Monk, Wadge)

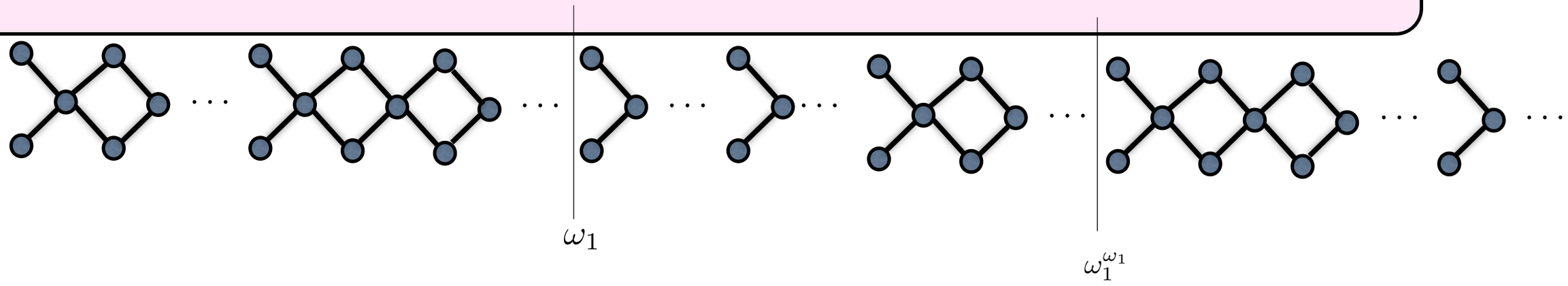
Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 {}_W > X_1 {}_W > X_2 {}_W > \ldots {}_W > X_k {}_W > X_{k+1} {}_W > \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$

- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$



Theorem (Martin, Monk, Wadge)

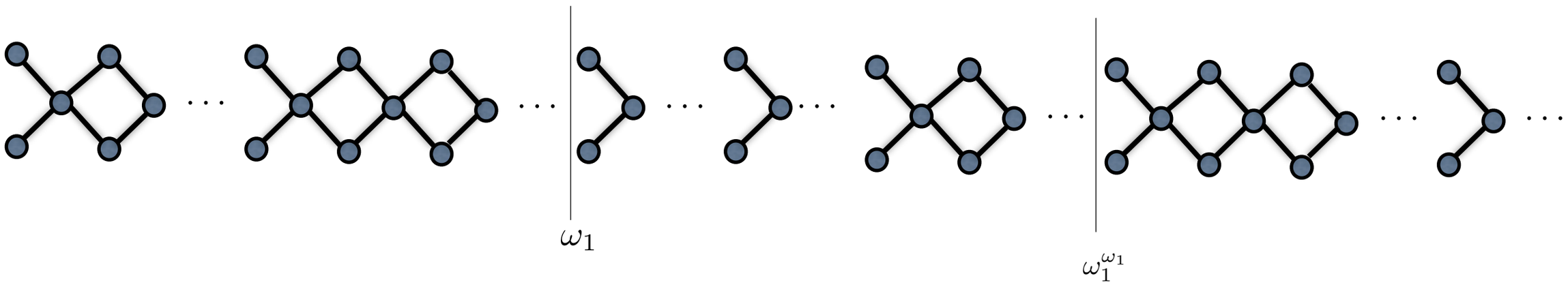
Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 {}_W > X_1 {}_W > X_2 {}_W > \ldots {}_W > X_k {}_W > X_{k+1} {}_W > \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$

- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$



- $\leq_w$-antichains have length at most 2.
- A and B are incomparable $\iff (A \not\leq_w B \wedge B \not\leq_w A)$
- There are no three A, B, C incomparable sets.
-
  - $A \not\leq_w B \implies B \leq_w A^{\complement}$,
  - $C \not\leq_w A \implies A \leq_w C^{\complement} \implies A^{\complement} \leq_w C$,
  - $\implies B \leq_w C$.

Theorem (Martin, Monk, Wadge)

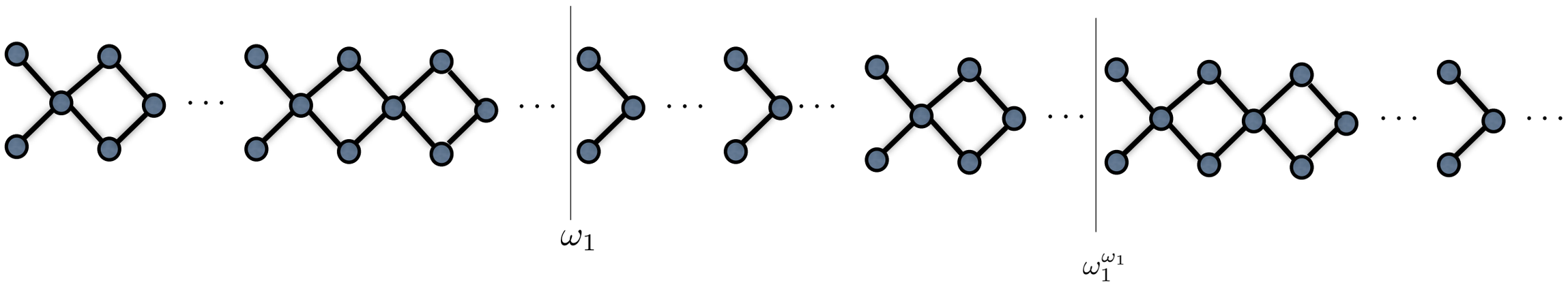
Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 {}_W > X_1 {}_W > X_2 {}_W > \ldots {}_W > X_k {}_W > X_{k+1} {}_W > \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$

- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$



Theorem (Martin, Monk, Wadge)

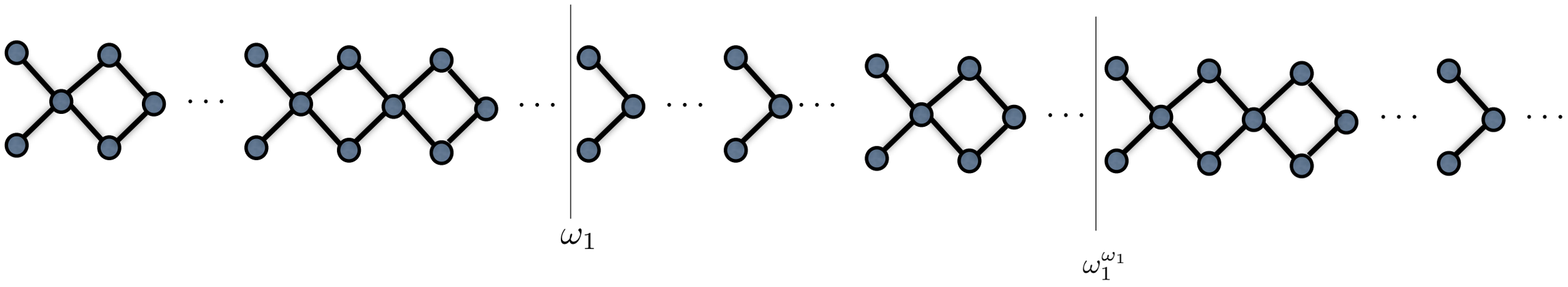
Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$X_0 >_W X_1 >_W X_2 >_W \ldots >_W X_k >_W X_{k+1} >_W \ldots$$

- *there are* **no three** *incomparable* $X,\ Y,\ Z$

- $X \not\leq_W Y \implies Y \leq_W X^C$



- $X <_w Y \implies X <_w Y^{\mathsf{C}}$
  - $\implies$ I has a w.s. in $\mathbb{W}(Y, X)$
    $\implies$ $I\!I$ has a w.s. in $\mathbb{W}\left(X, Y^{\mathsf{C}}\right)$
    $\implies X \leq_w Y^{\mathsf{C}}$,
- $\implies Y^{\mathsf{C}} \not\leq_w X$, otherwise
  - $Y^{\mathsf{C}} \leq_w X \leq_w Y \implies Y^{\mathsf{C}} \leq_w Y \implies Y \leq_w Y^{\mathsf{C}}$,
  - $\implies Y \leq_w X$, a contradiction

- $X <_w Y \implies$
  - $Y \not\leq_w X \implies$ I has a w.s. in $\mathbb{W}(Y, X)$,
  - $\implies$ I has a w.s. in $\mathbb{W}\left(Y^{\mathsf{C}}, X^{\mathsf{C}}\right)$,
  - $Y^{\mathsf{C}} \not\leq_w X \implies$ I has a w.s. in $\mathbb{W}\left(Y, X^{\mathsf{C}}\right)$,
  - $\implies$ I has a w.s. in $\mathbb{W}\left(Y^{\mathsf{C}}, X\right)$,

- $X <_w Y \Longrightarrow$
  - $Y \not\leq_w X \Longrightarrow$ I has a w.s. in $\mathbb{W}(Y, X)$,
  - $\Longrightarrow$ I has a w.s. in $\mathbb{W}\left(Y^{\complement}, X^{\complement}\right)$,
  - $Y^{\complement} \not\leq_w X \Longrightarrow$ I has a w.s. in $\mathbb{W}\left(Y, X^{\complement}\right)$,
  - $\Longrightarrow$ I has a w.s. in $\mathbb{W}\left(Y^{\complement}, X\right)$,

$$A > A_0 > A_1 > \ldots > A_n > A_{n+1} > \ldots$$

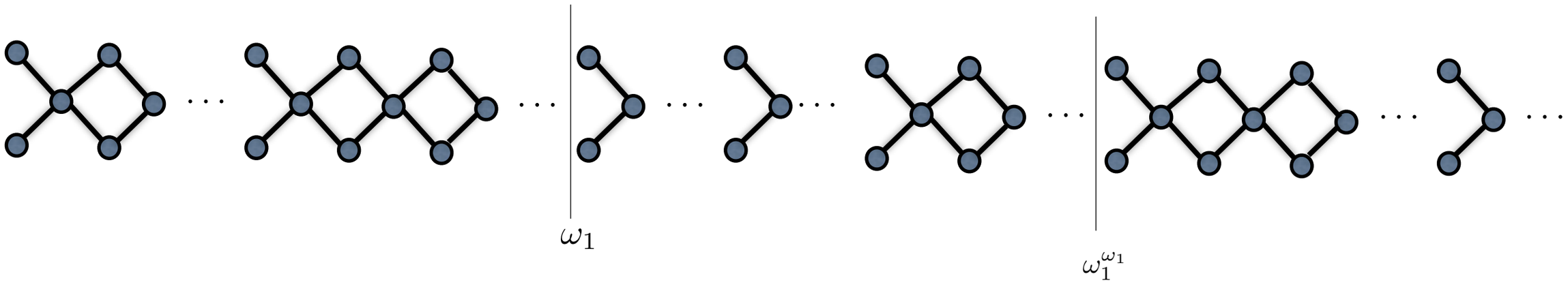
Theorem (Martin, Monk, Wadge)

Assuming Determinacy,

- $<_W$ *is wellfounded, i.e. there is no*

$$A > A_0 > A_1 > \ldots > A_n > A_{n+1} > \ldots$$

- *there are* **no three** *incomparable* X, Y, Z
- $X \not\leq_W Y \implies Y \leq_W X^{\complement}$



I

$I_{\parallel}$ $II_{\parallel}$

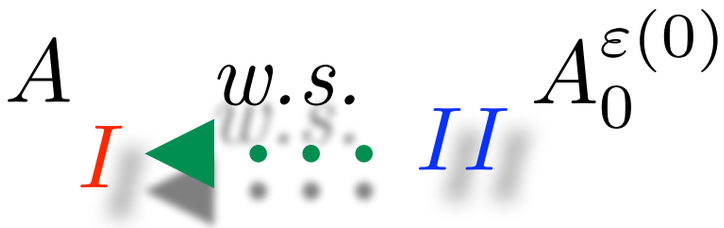
A

$I_{\parallel}$ $II_{\parallel}$

A_{I} A_{II} $A_0^{\varepsilon(0)}$

$$A \quad A_0^{\varepsilon(0)}$$

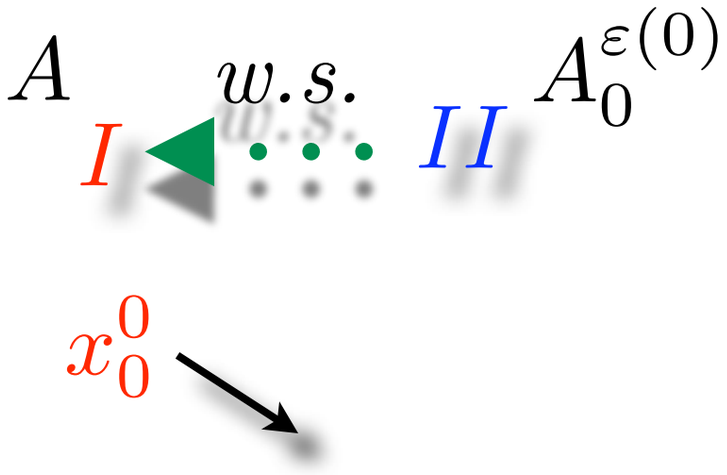


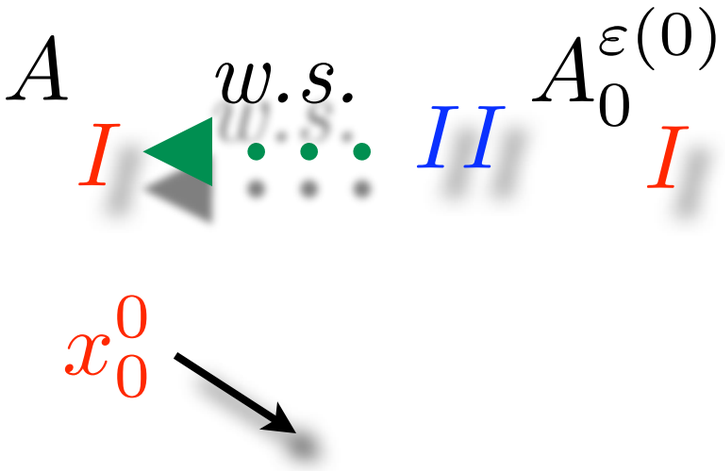

A
$w.s.$
$A_0^{\varepsilon(0)}$
I
II

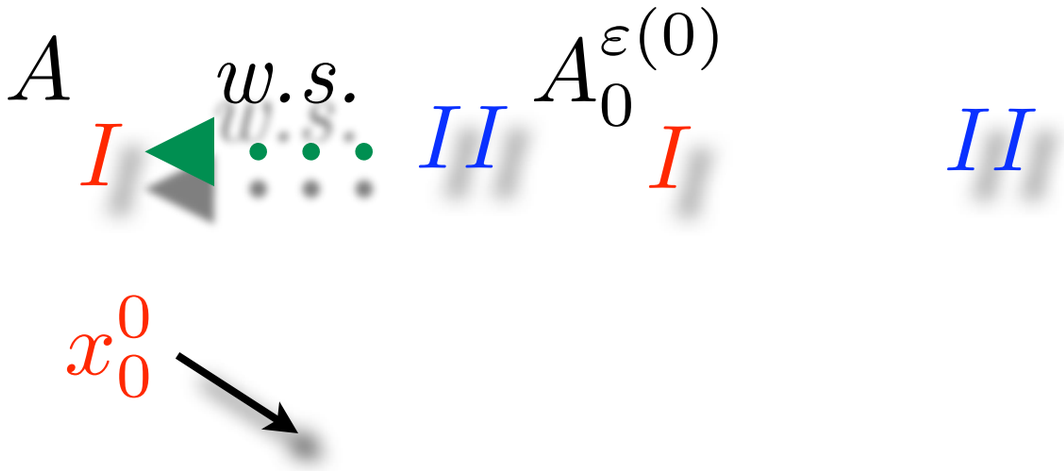
A $w.s.$ $A_0^{\varepsilon(0)}$

I $w.s.$ II

x_0^0


A
$w.s.$
$A_0^{\varepsilon(0)}$
I
II
x_0^0

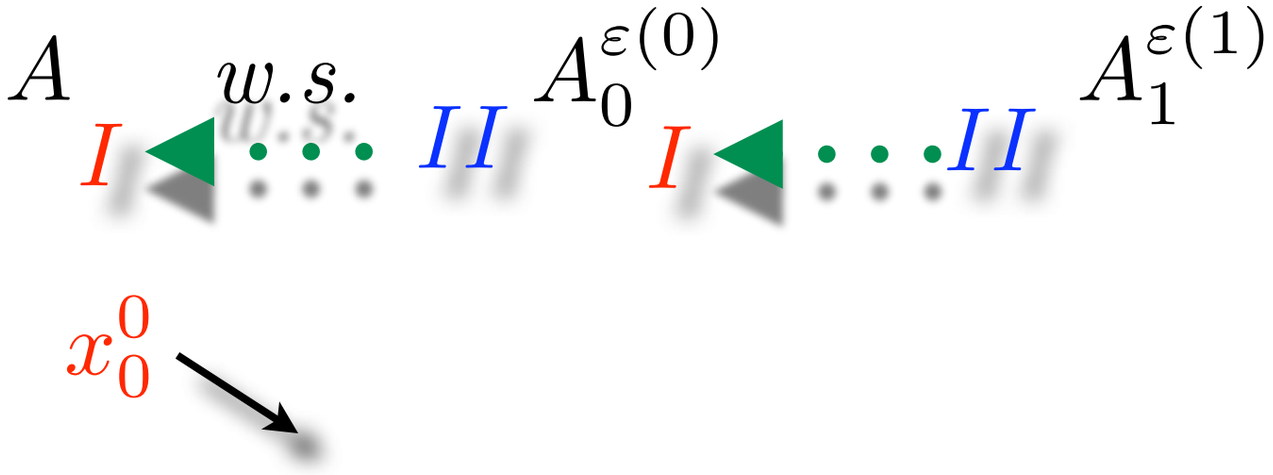

A
$w.s.$
$A_0^{\varepsilon(0)}$
I
II
I
x_0^0

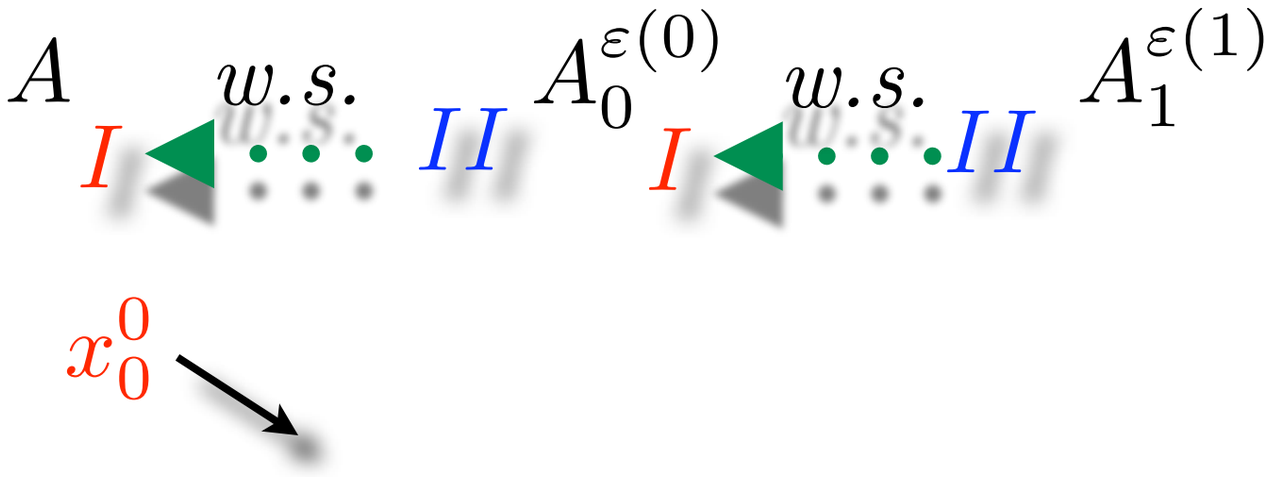

A
$w.s.$
$A_0^{\varepsilon(0)}$
I
II
I
II
x_0^0

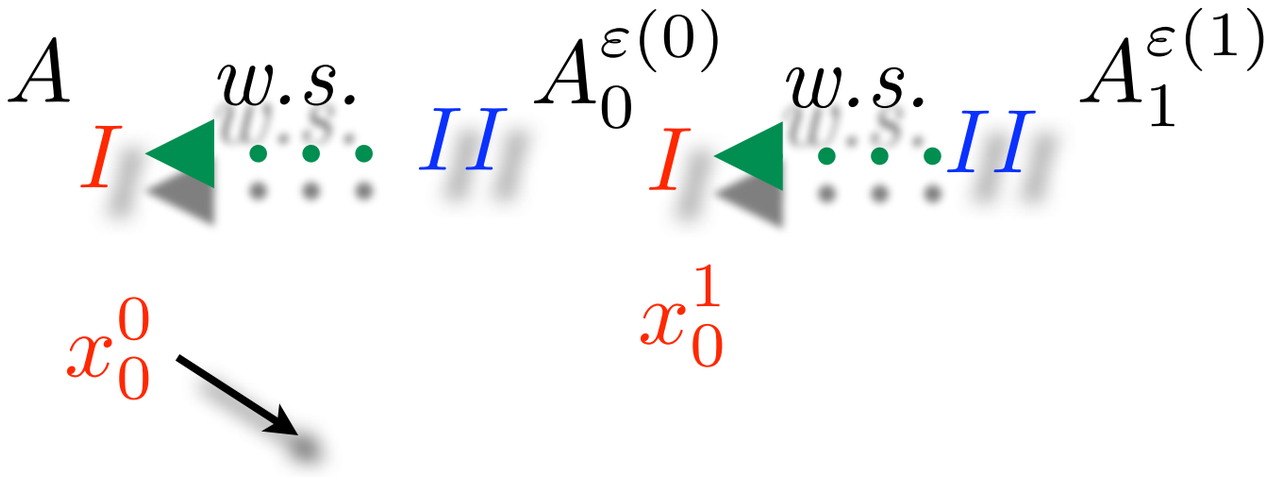
A $w.s.$ $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$

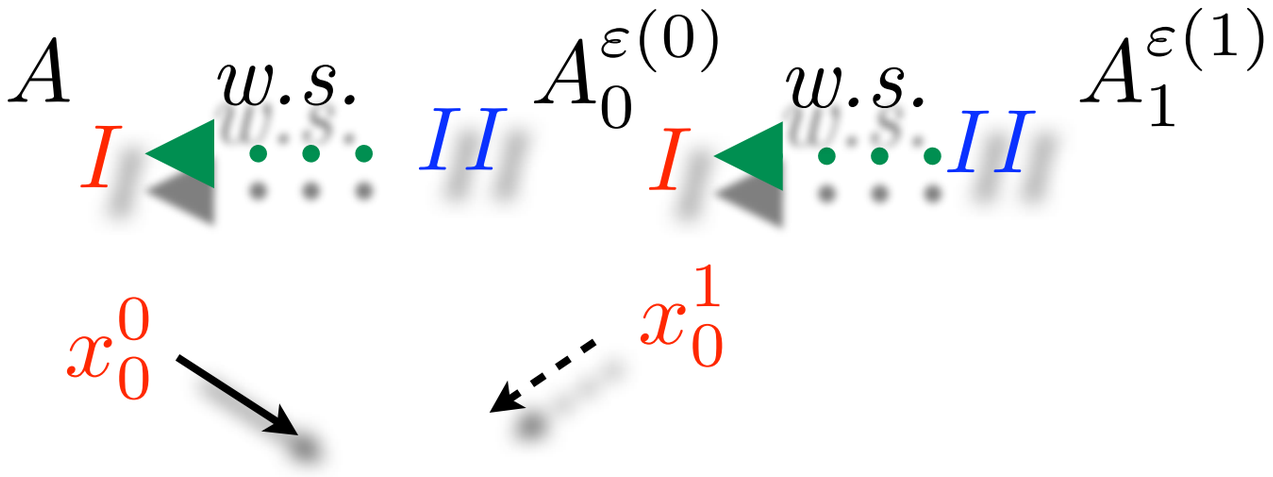


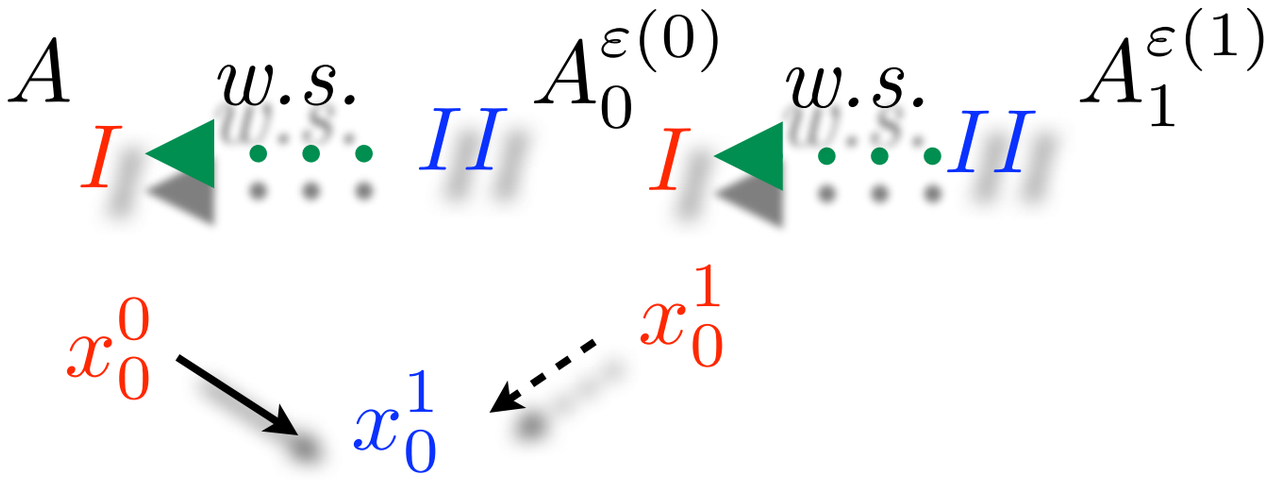


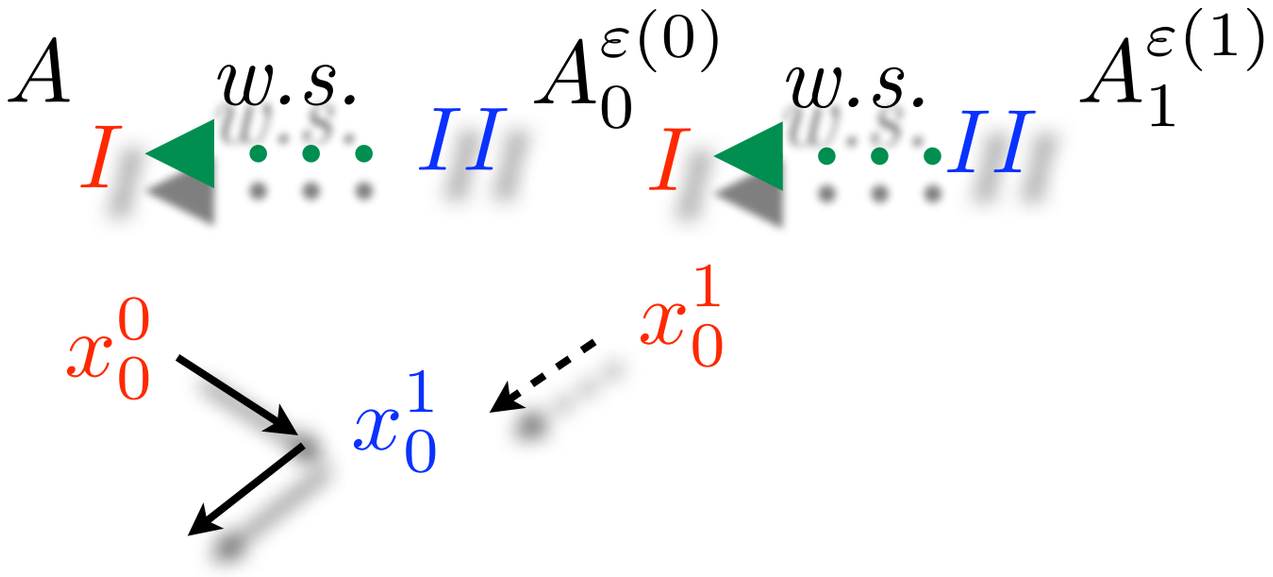

A
$w.s.$
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
I
I
I
I
x_0^0

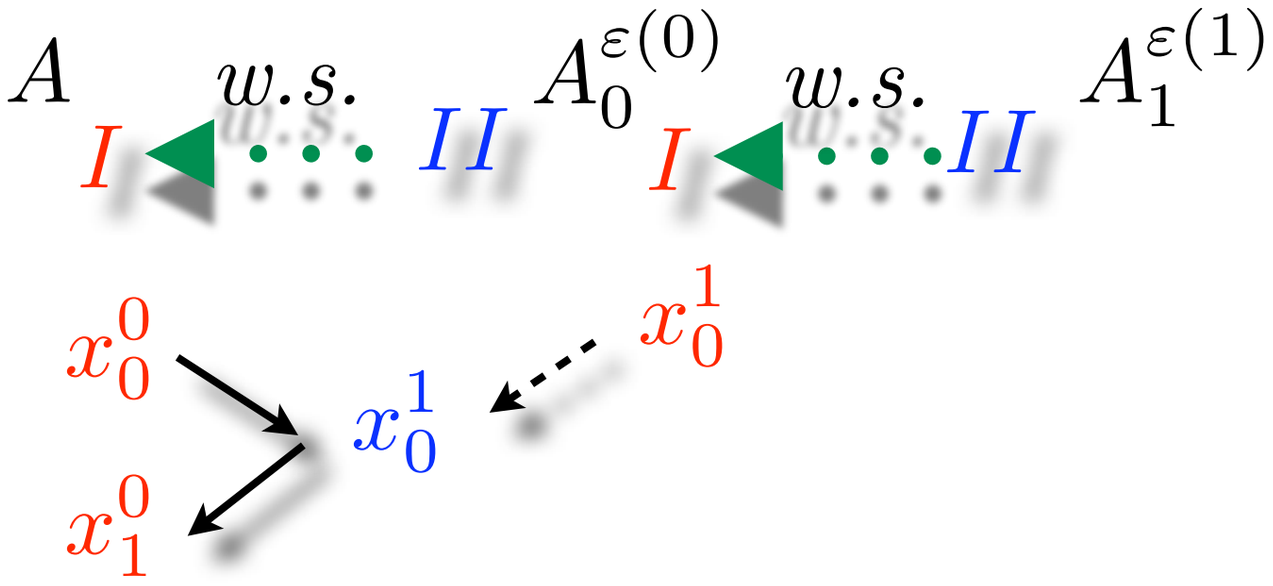

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0

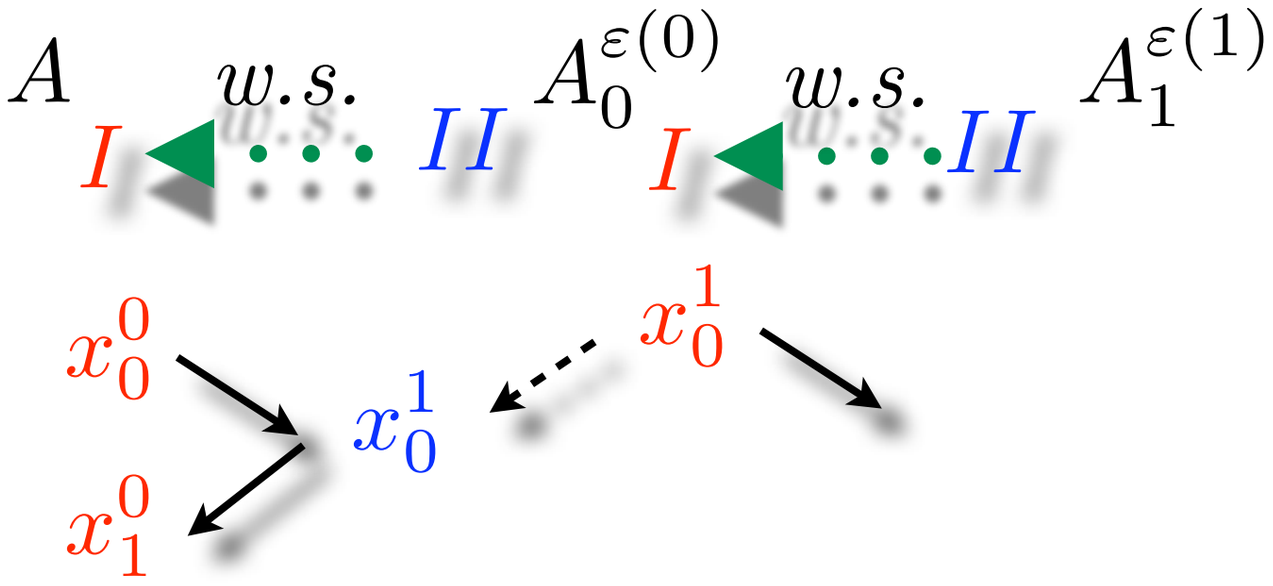

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_0^1

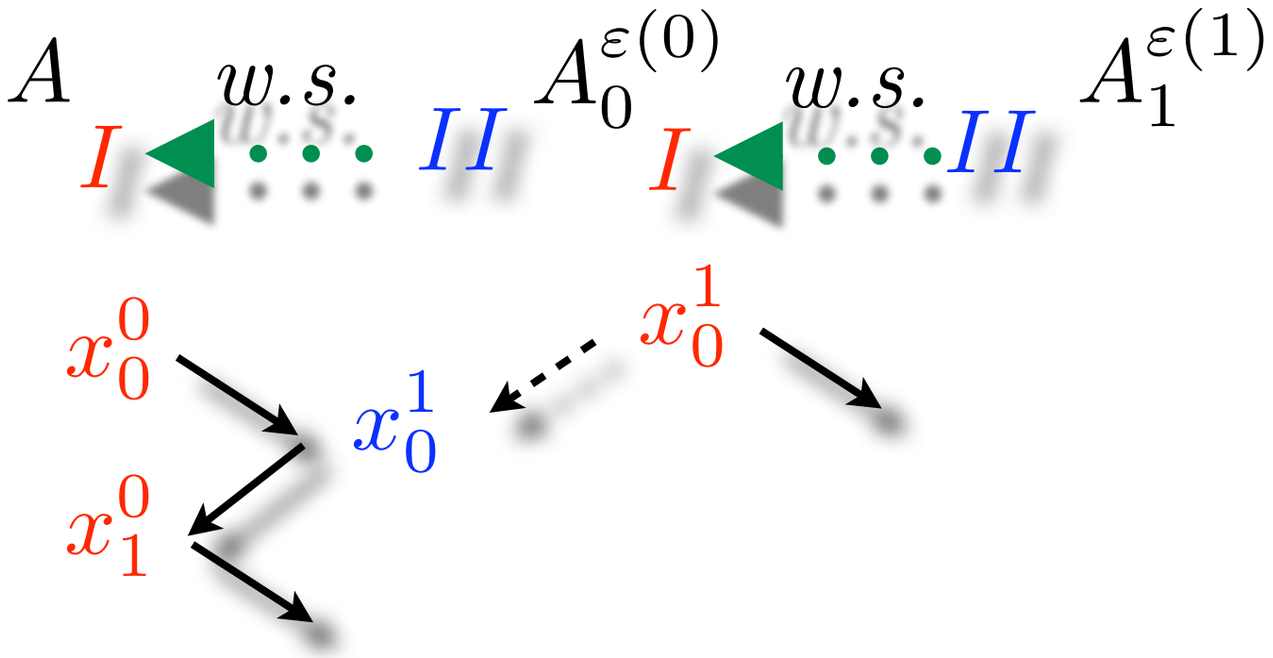

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_0^1

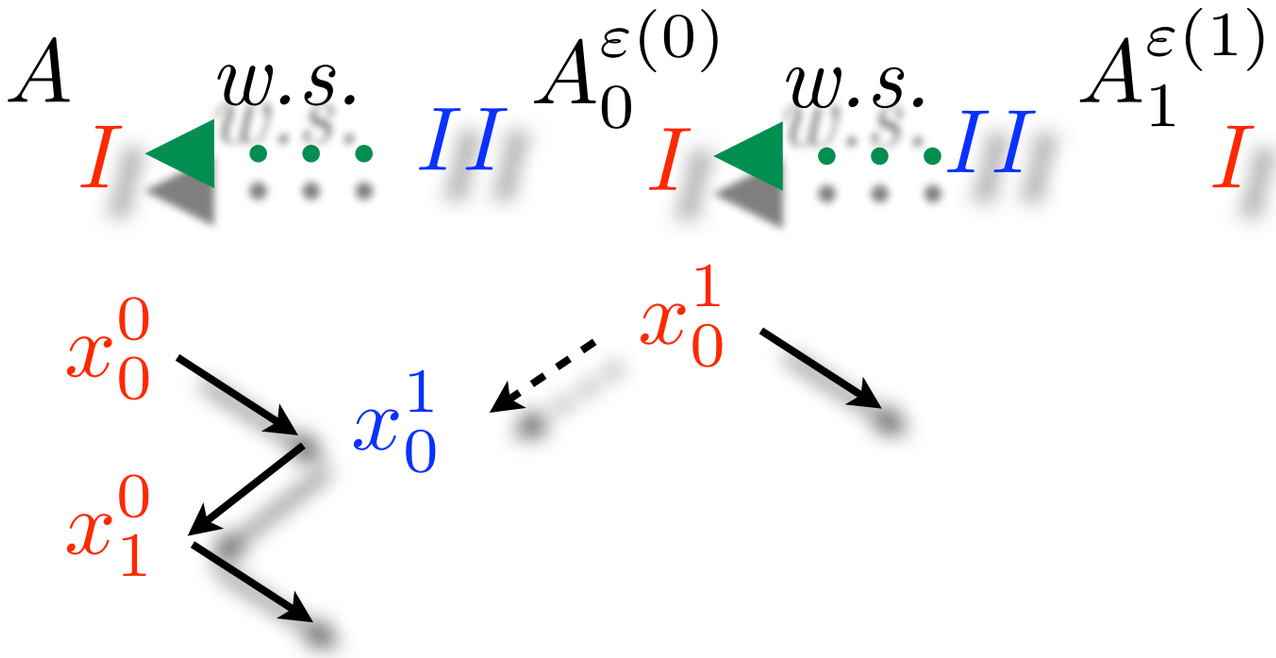

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_0^1
x_0^1

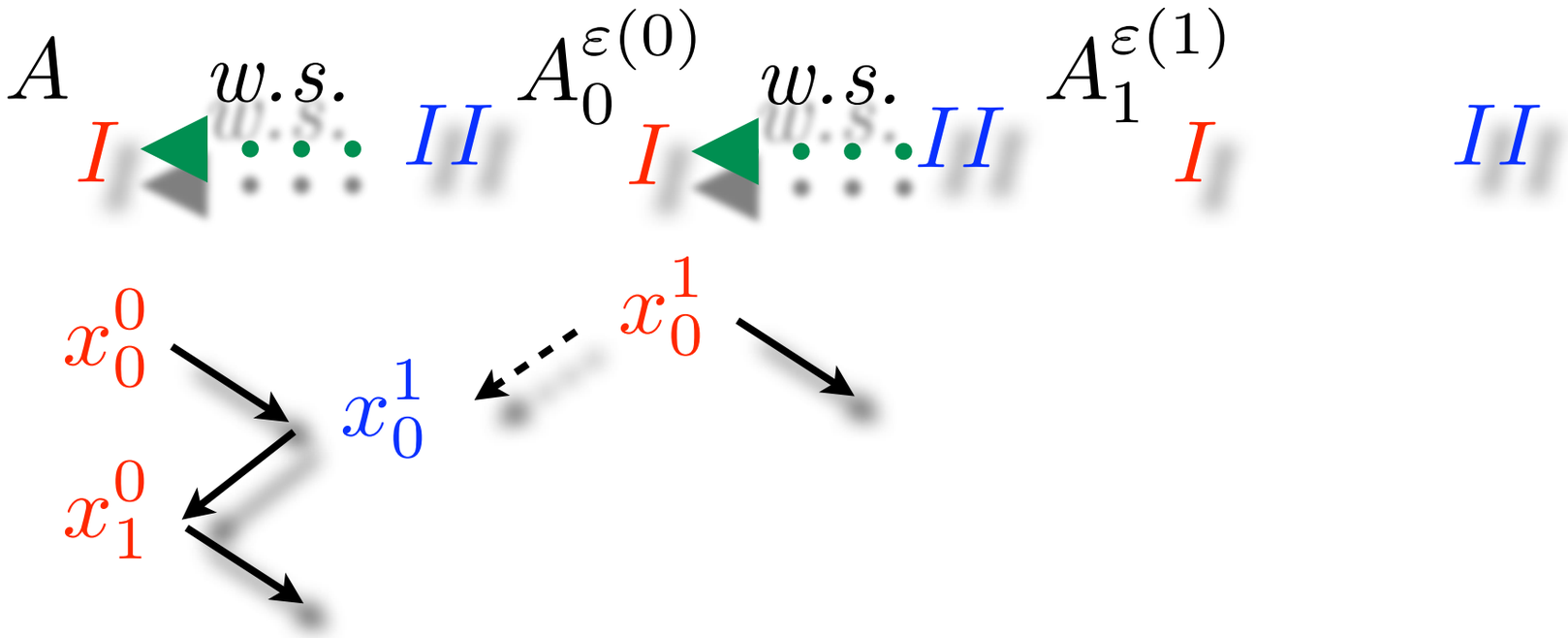

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_0^1
x_0^1

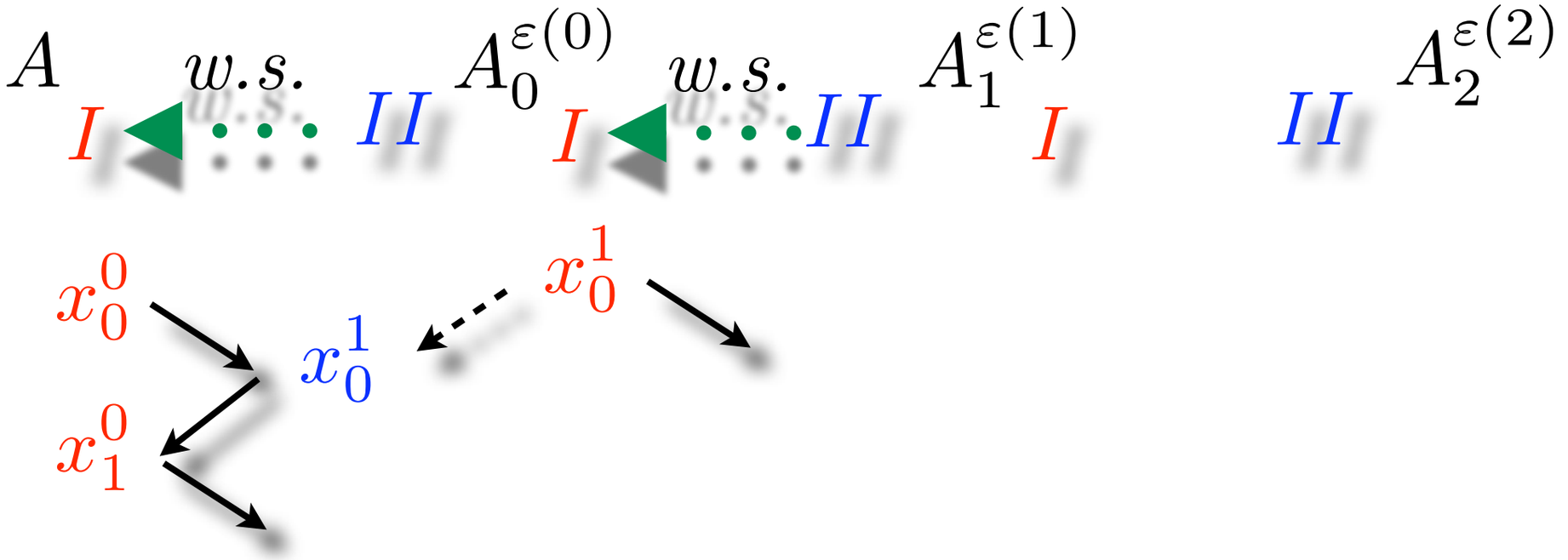

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_1^0
x_0^1
x_0^1

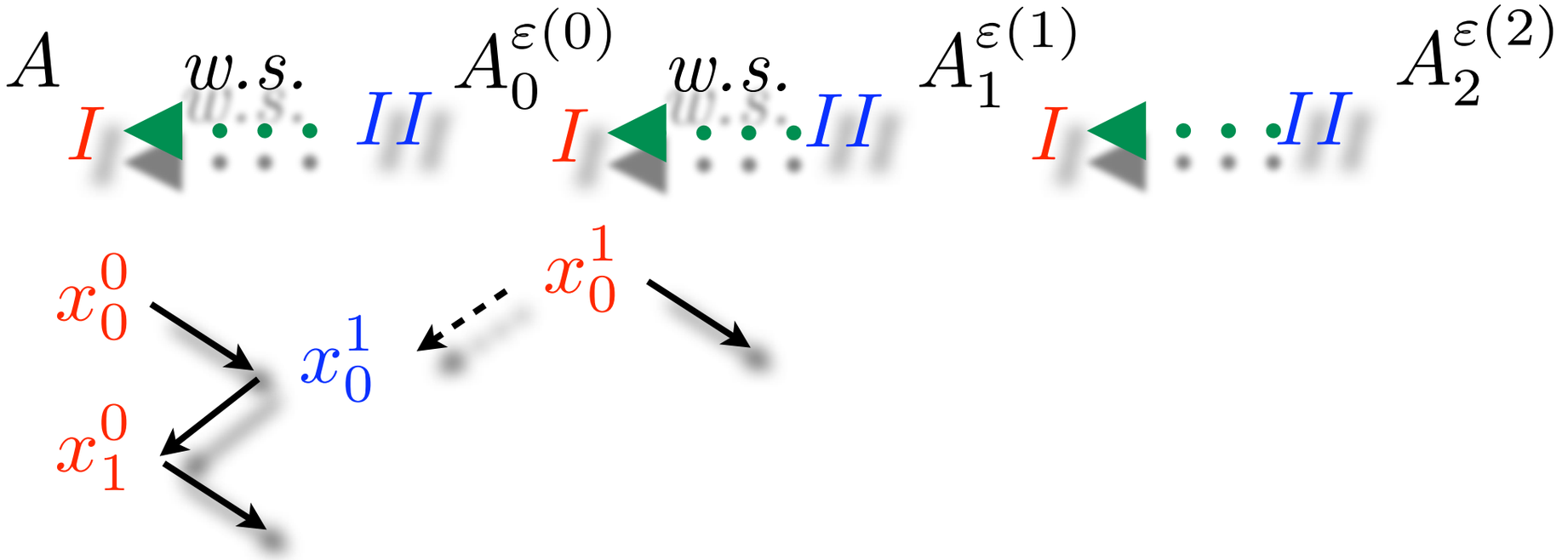

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_1^0
x_0^1
x_0^1

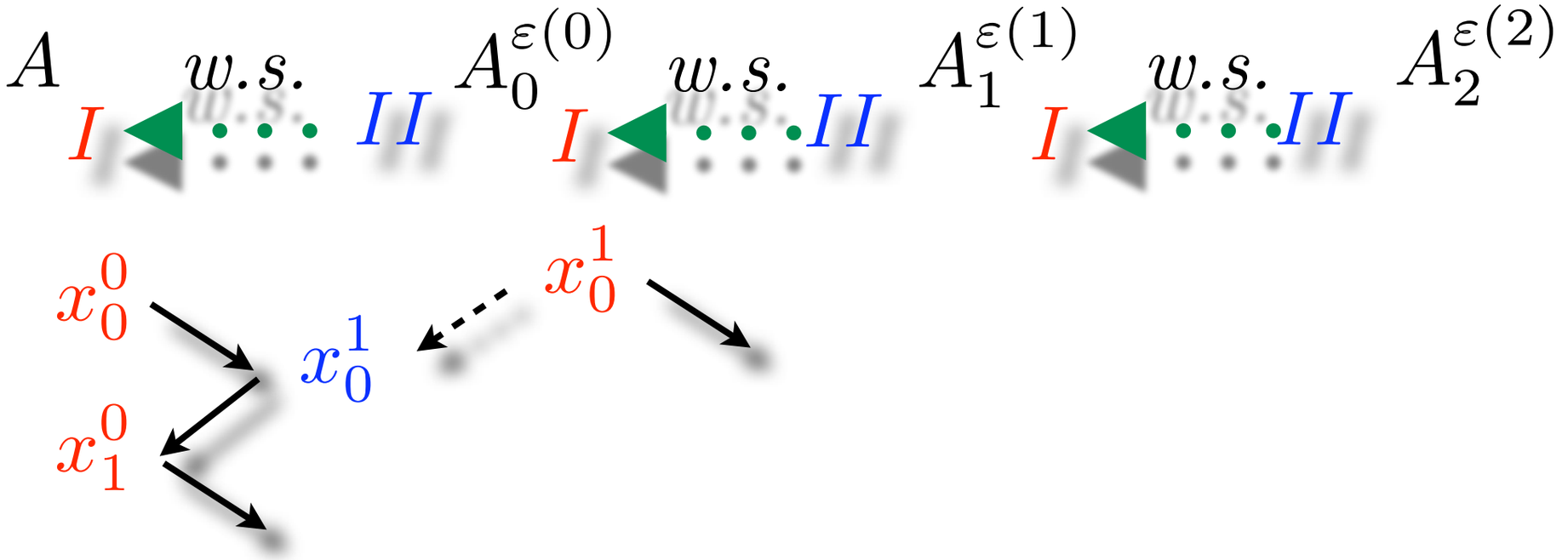

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
x_0^0
x_1^0
x_0^1
x_0^1

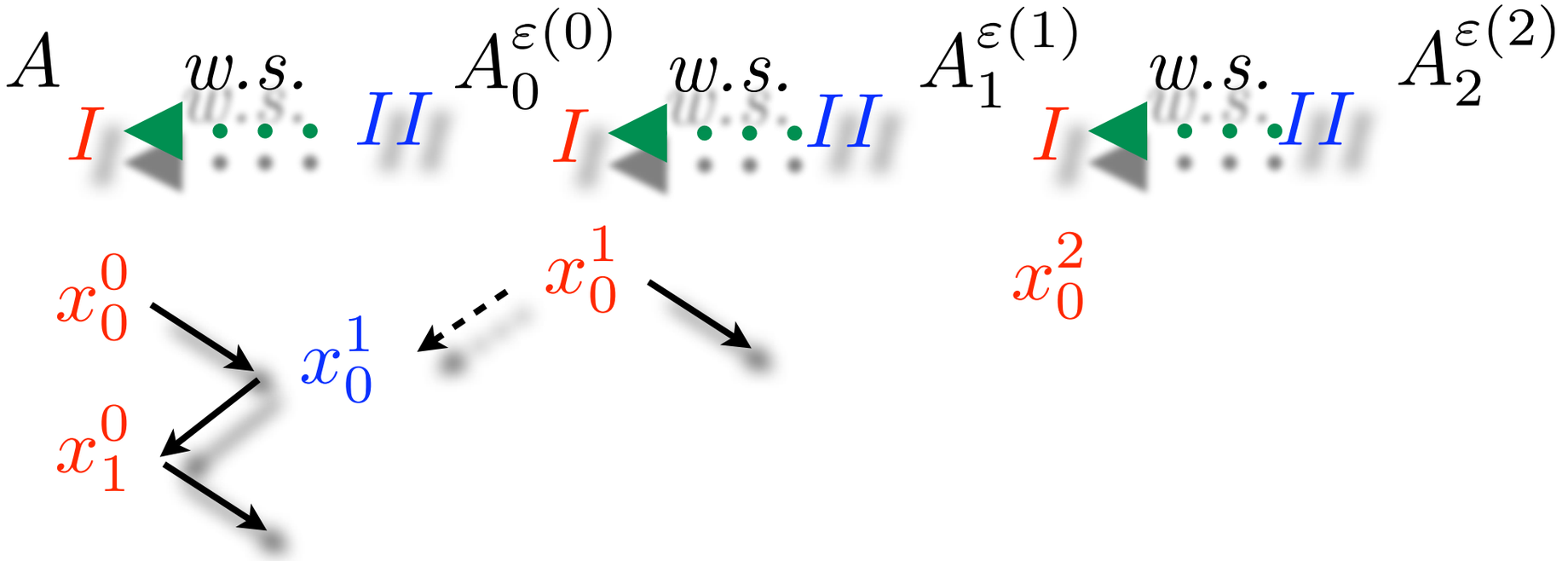

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
I
x_0^0
x_1^0
x_0^1
x_0^1

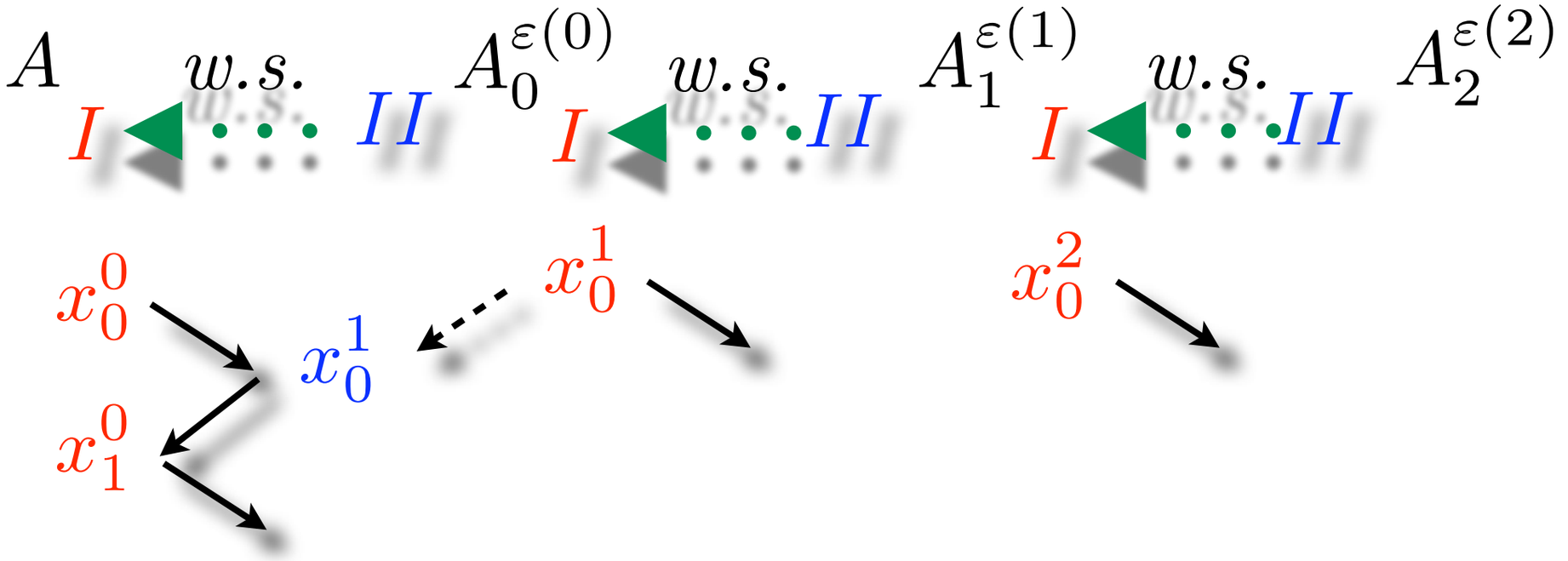

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
I
II
x_0^0
x_1^0
x_0^1
x_0^1

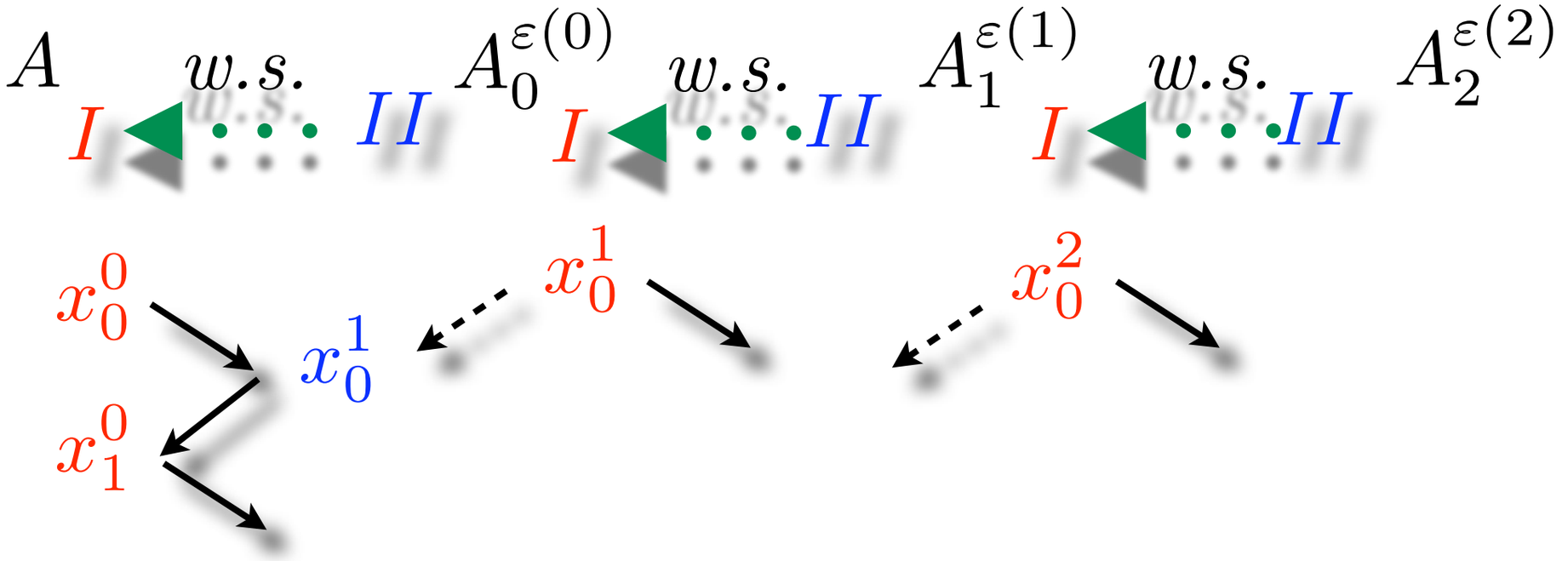

A
$w.s.$
$A_0^{\varepsilon(0)}$
$w.s.$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
I
II
I
II
I
II
x_0^0
x_1^0
x_0^1
x_0^1

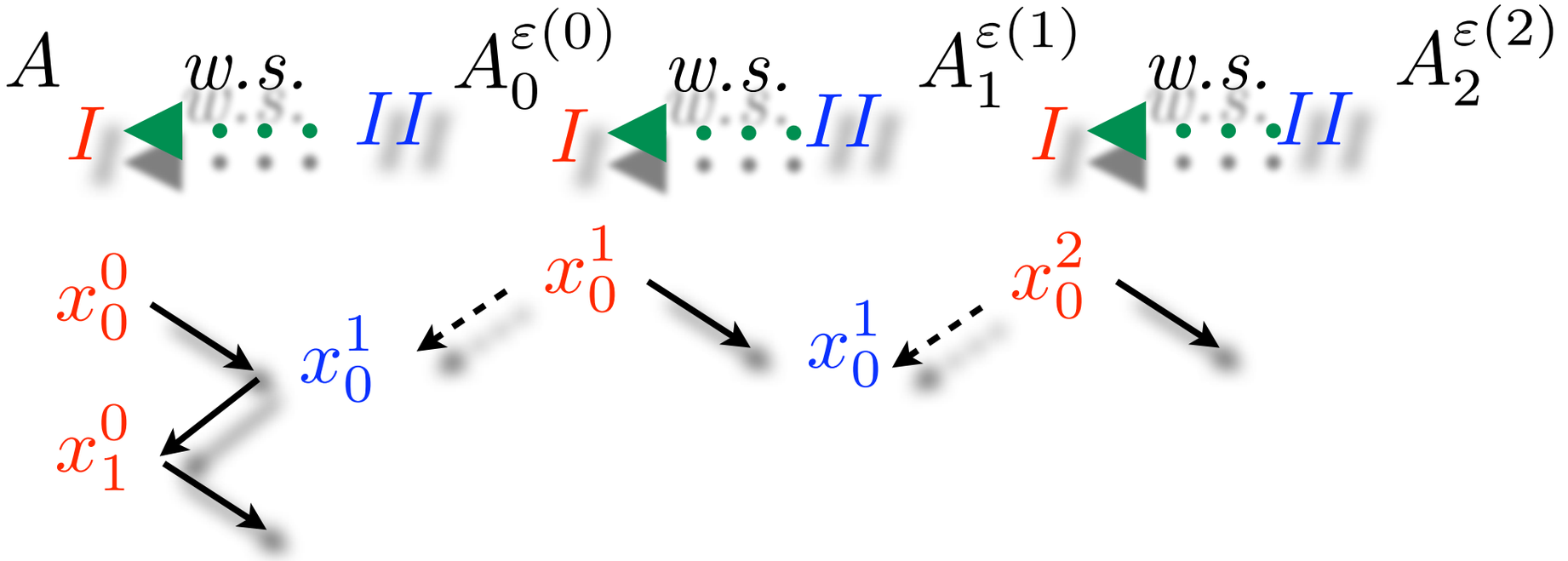

A
$w.s.$
$A_0^{\varepsilon(0)}$
$w.s.$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
I
I
I
I
I
I
x_0^0
x_1^0
x_0^1
x_0^1

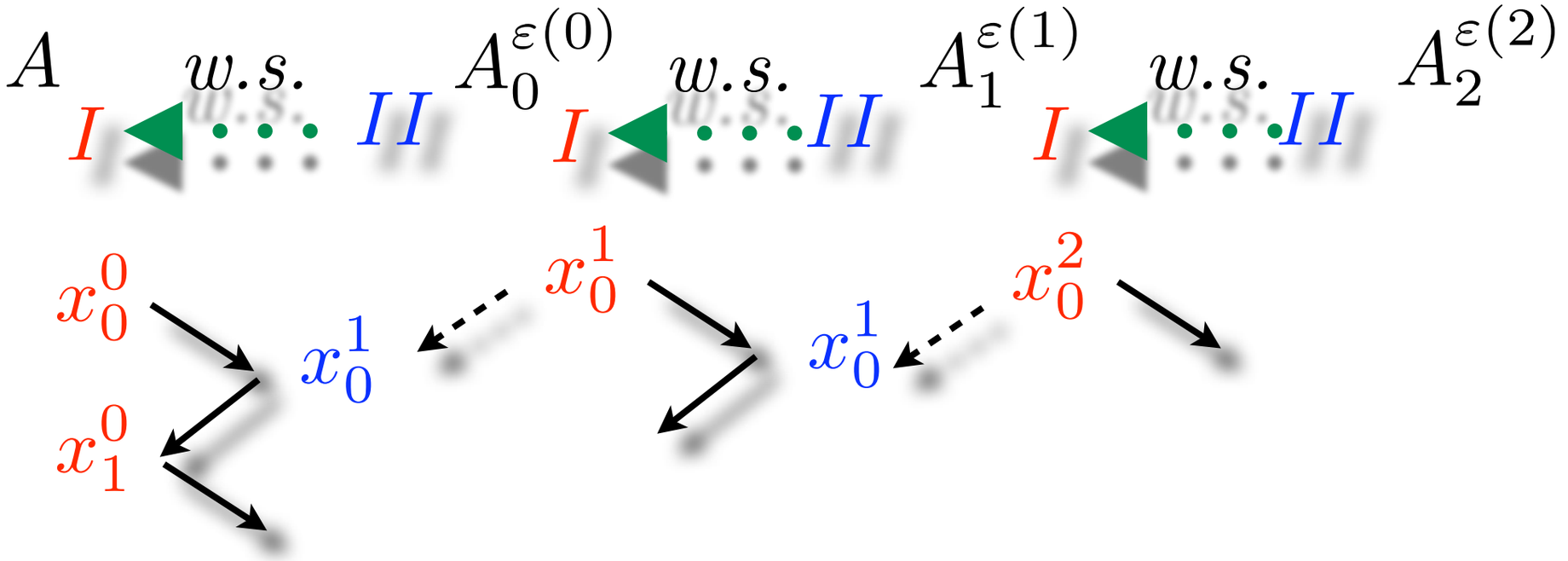

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1

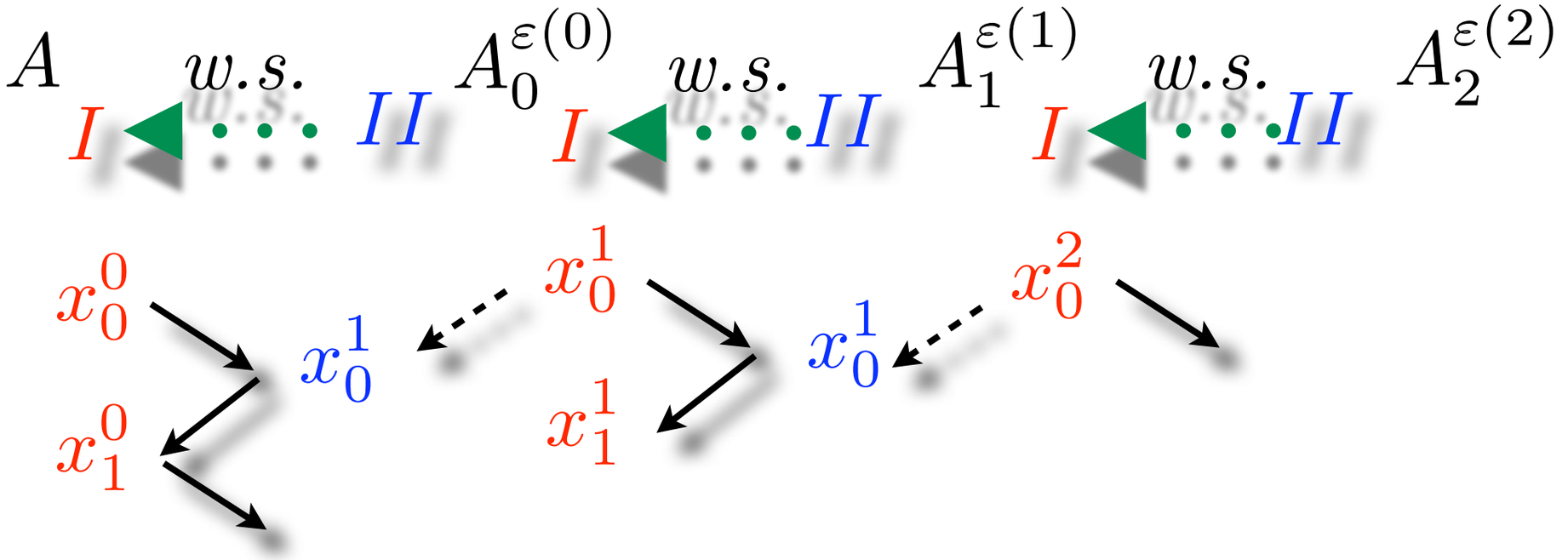

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_0^2

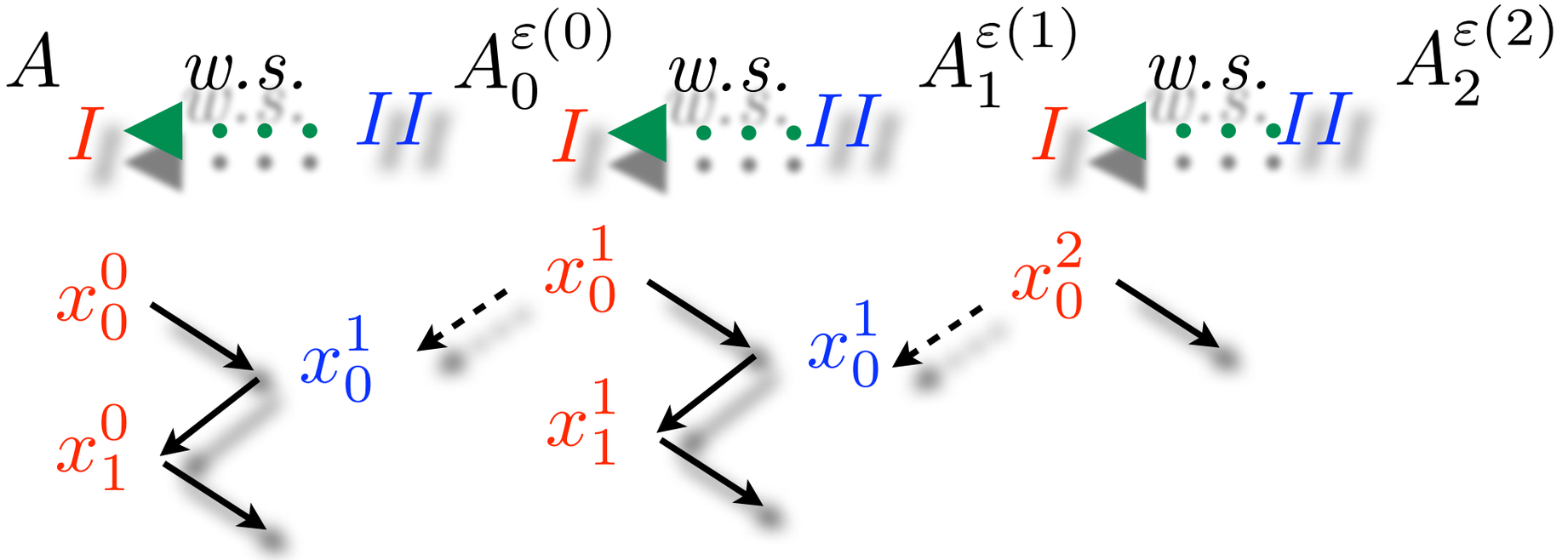

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_0^2

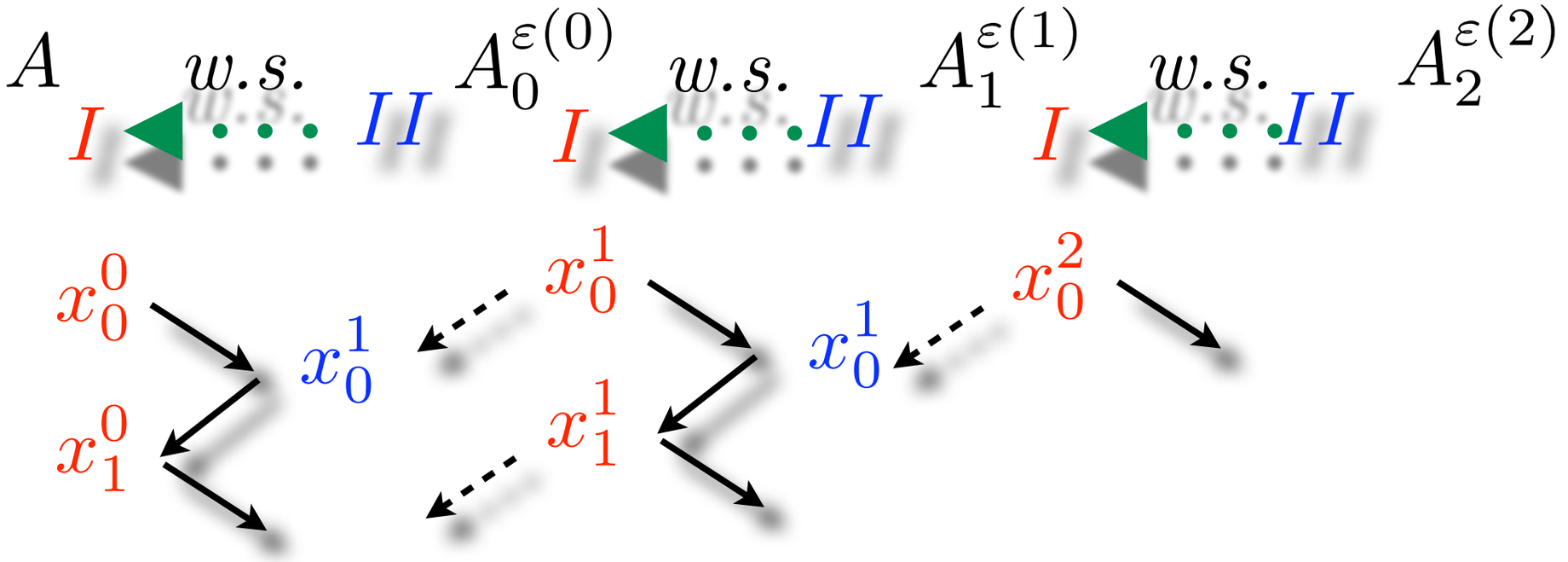

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_0^2

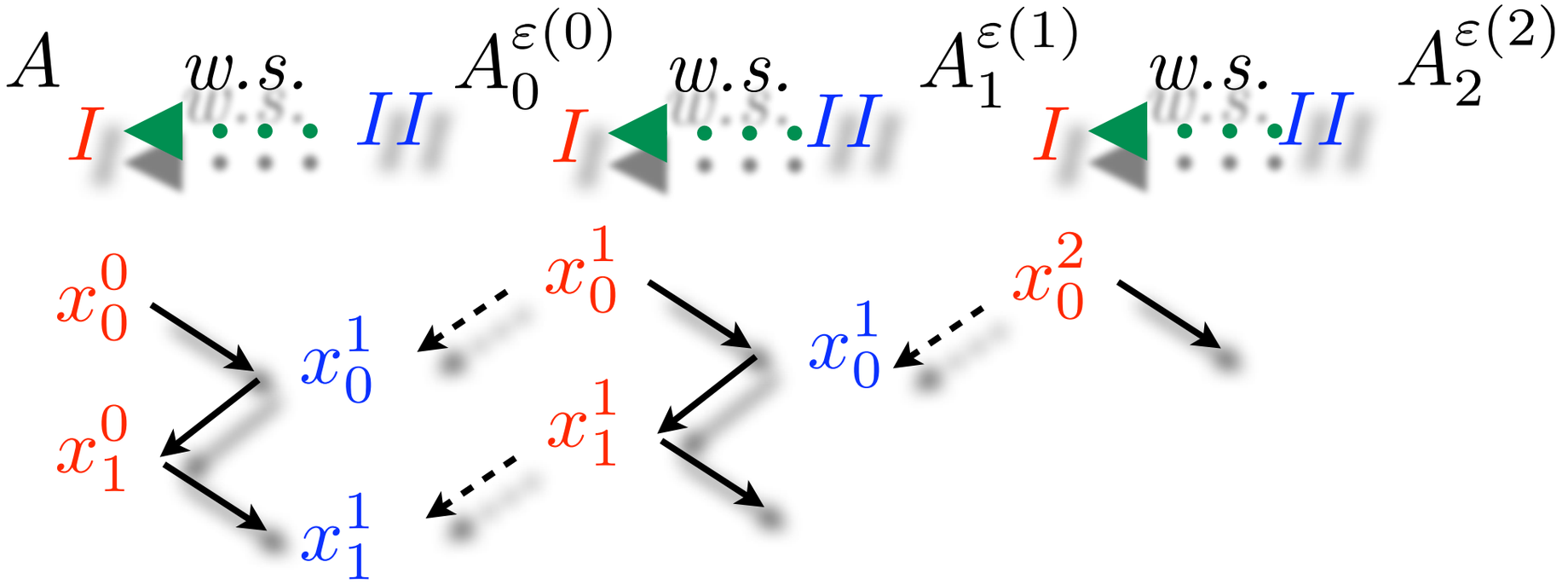

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_0^2

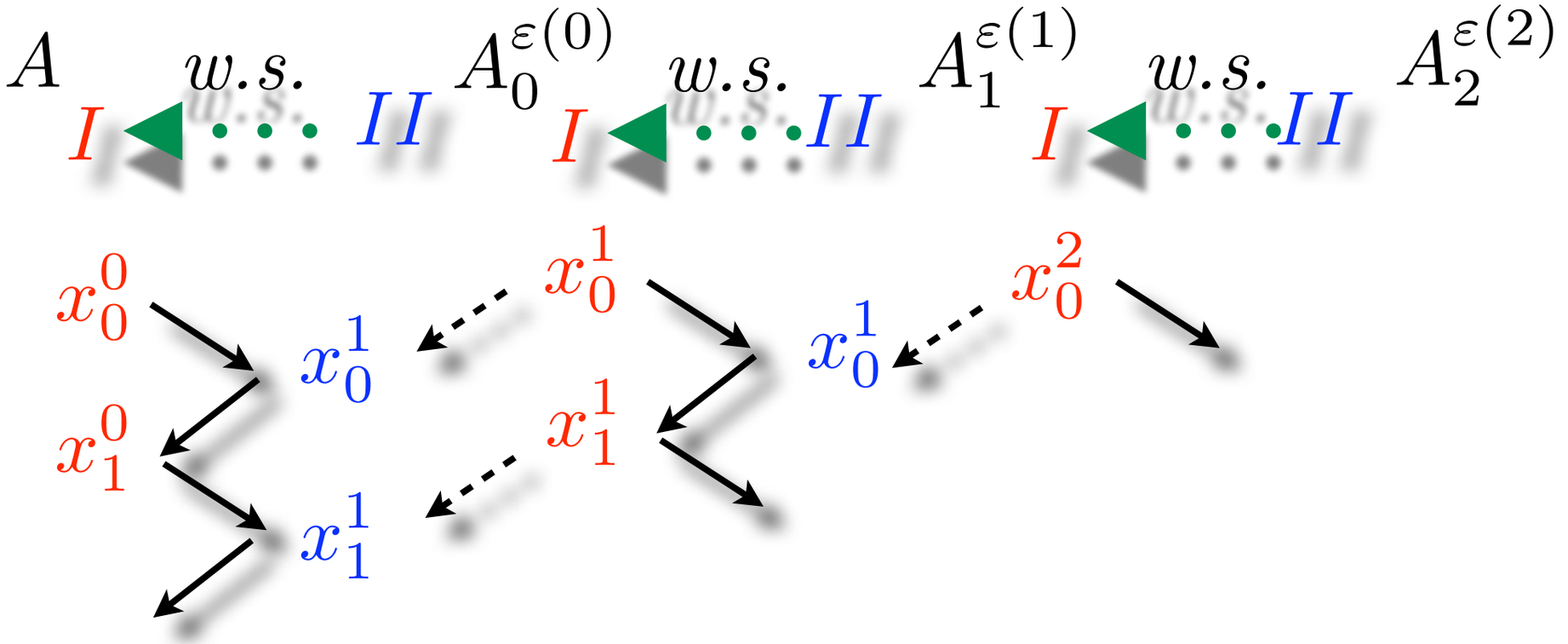

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_0^2
x_0^1

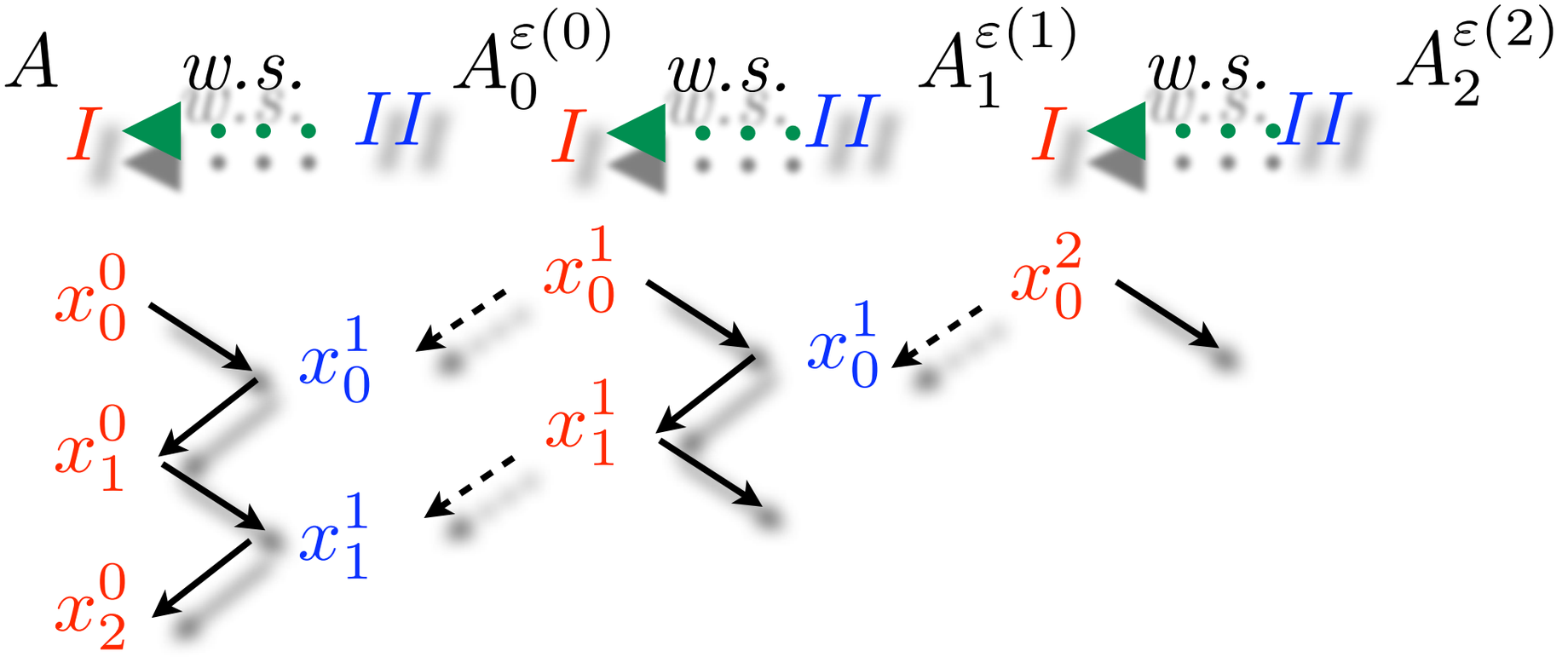

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_1^1
x_0^2
x_0^1

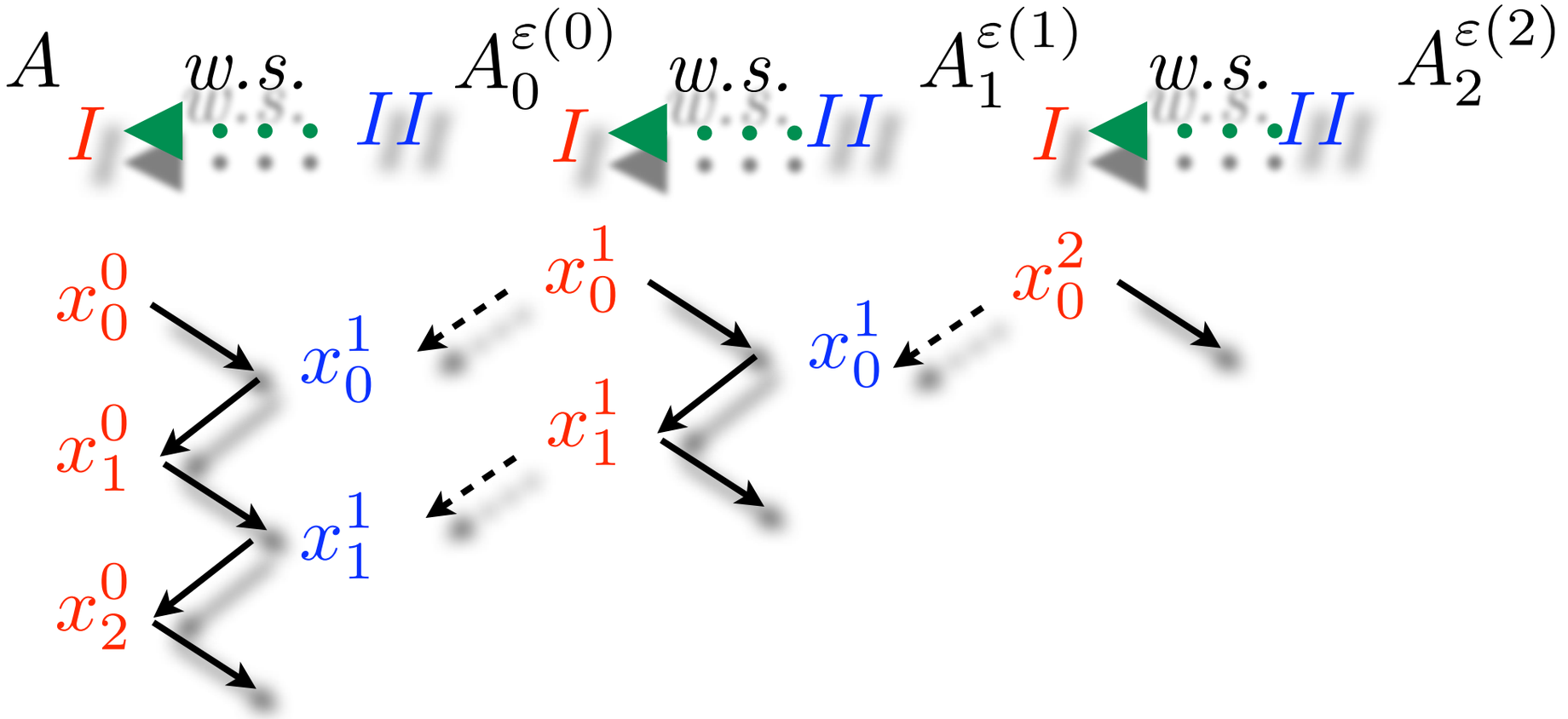

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_1^1
x_0^2
x_0^1

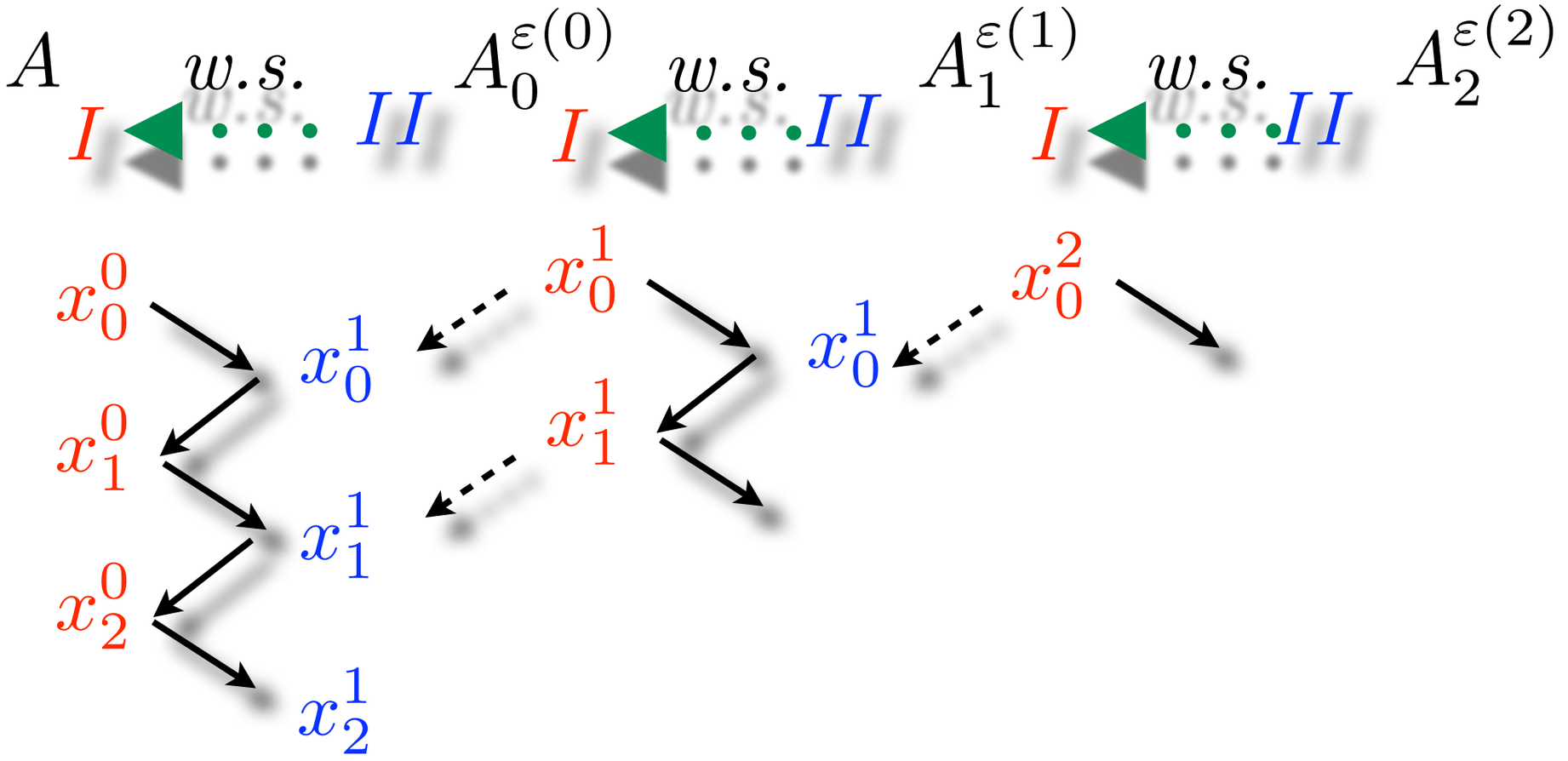

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_0^1
x_1^1
x_0^2
x_0^1

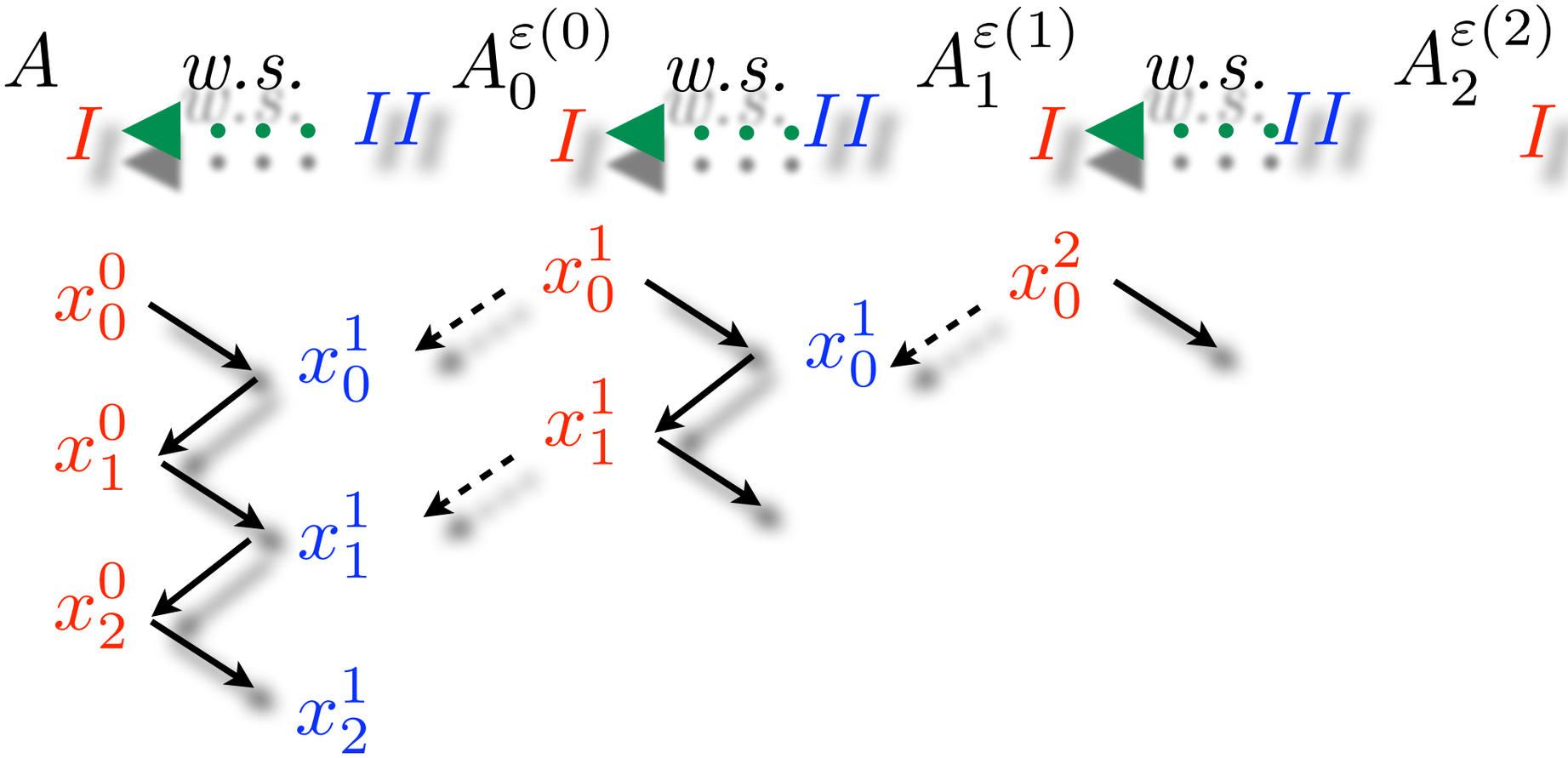

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_1^1
x_0^1
x_1^1
x_0^1
x_0^2

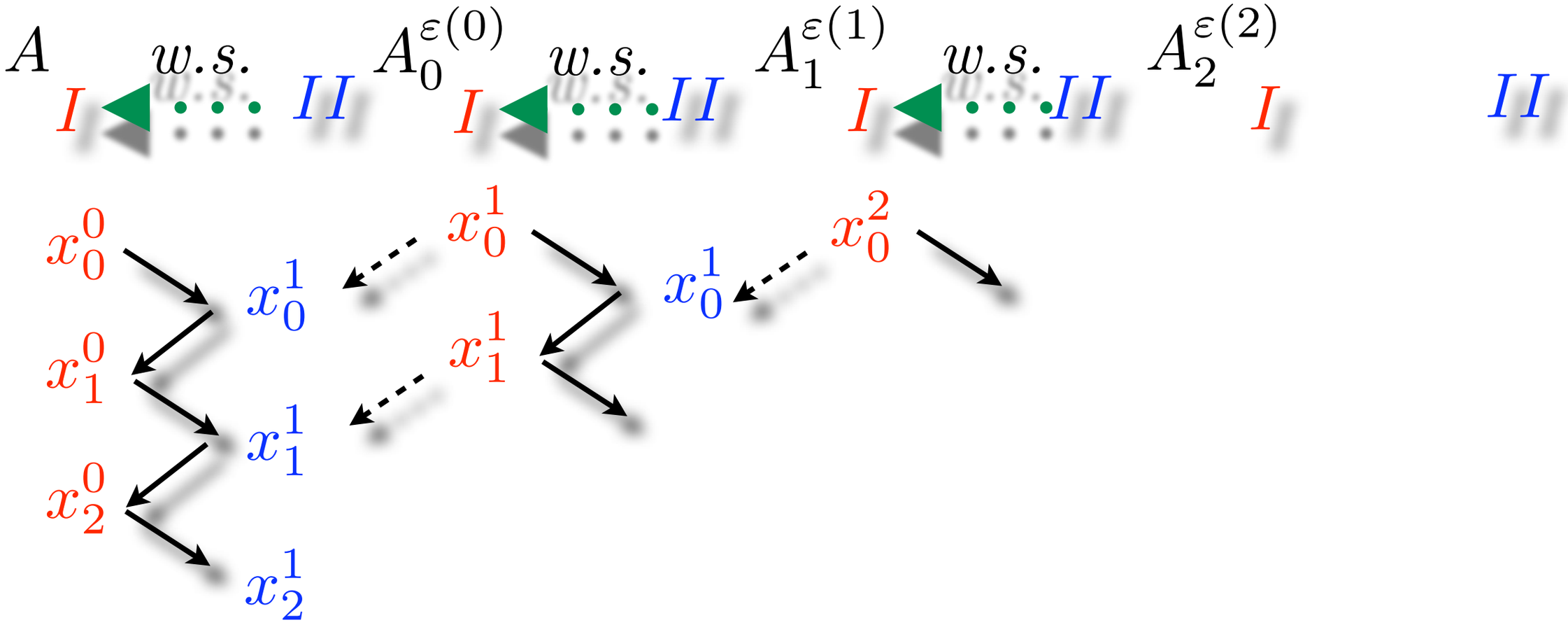

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_0^1
x_1^1
x_0^1
x_1^1
x_0^2
x_0^1

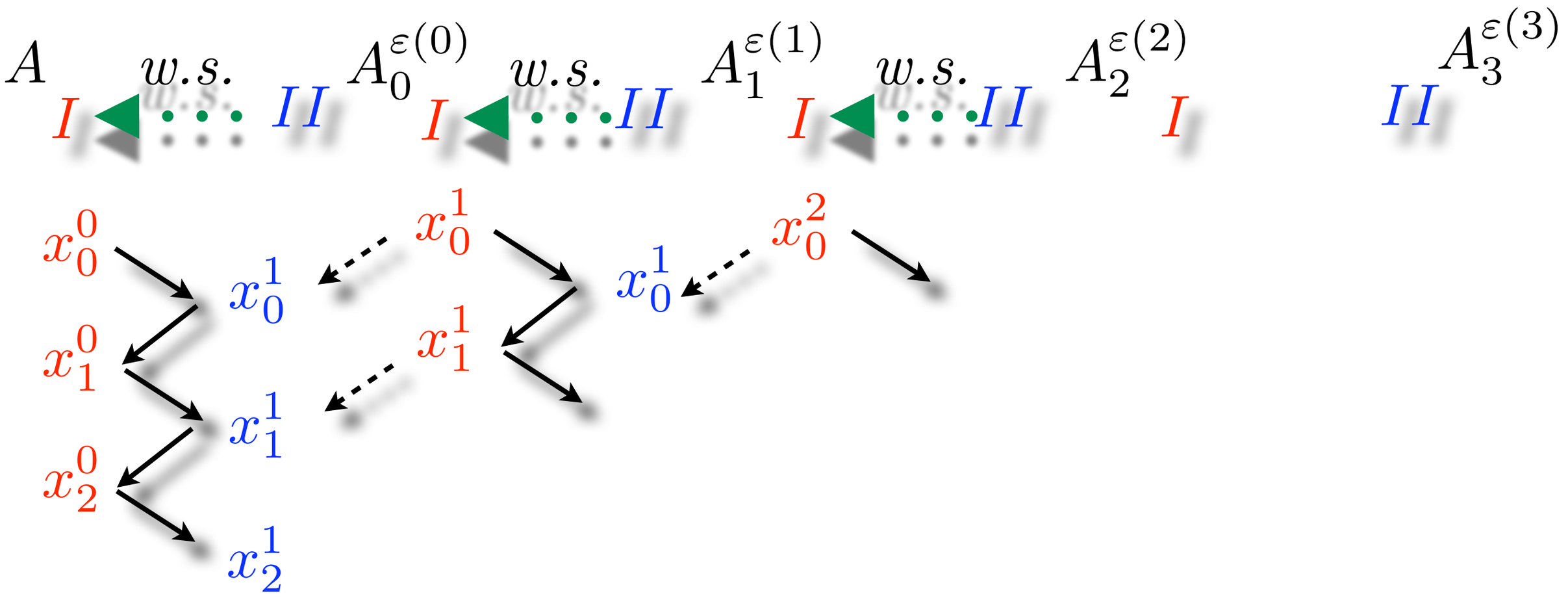

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_0^1
x_1^1
x_0^2
x_0^1

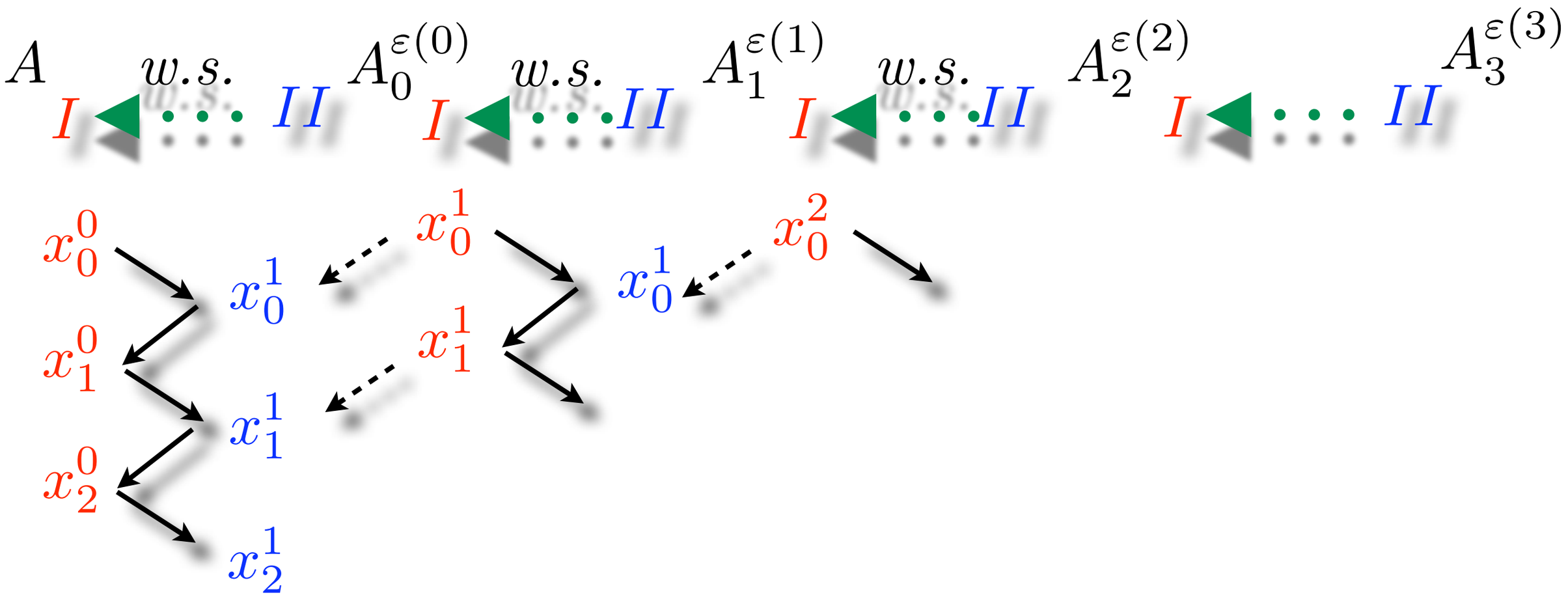

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_0^1
x_1^1
x_0^2
x_0^1

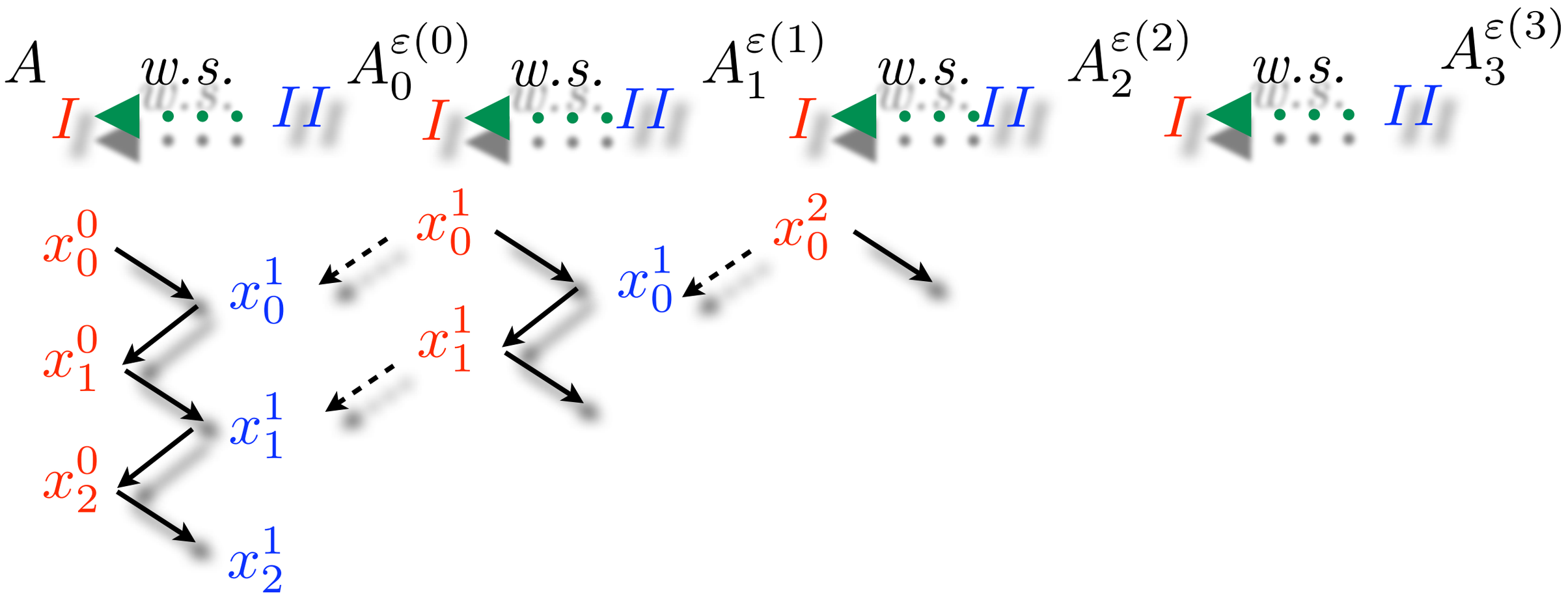

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1

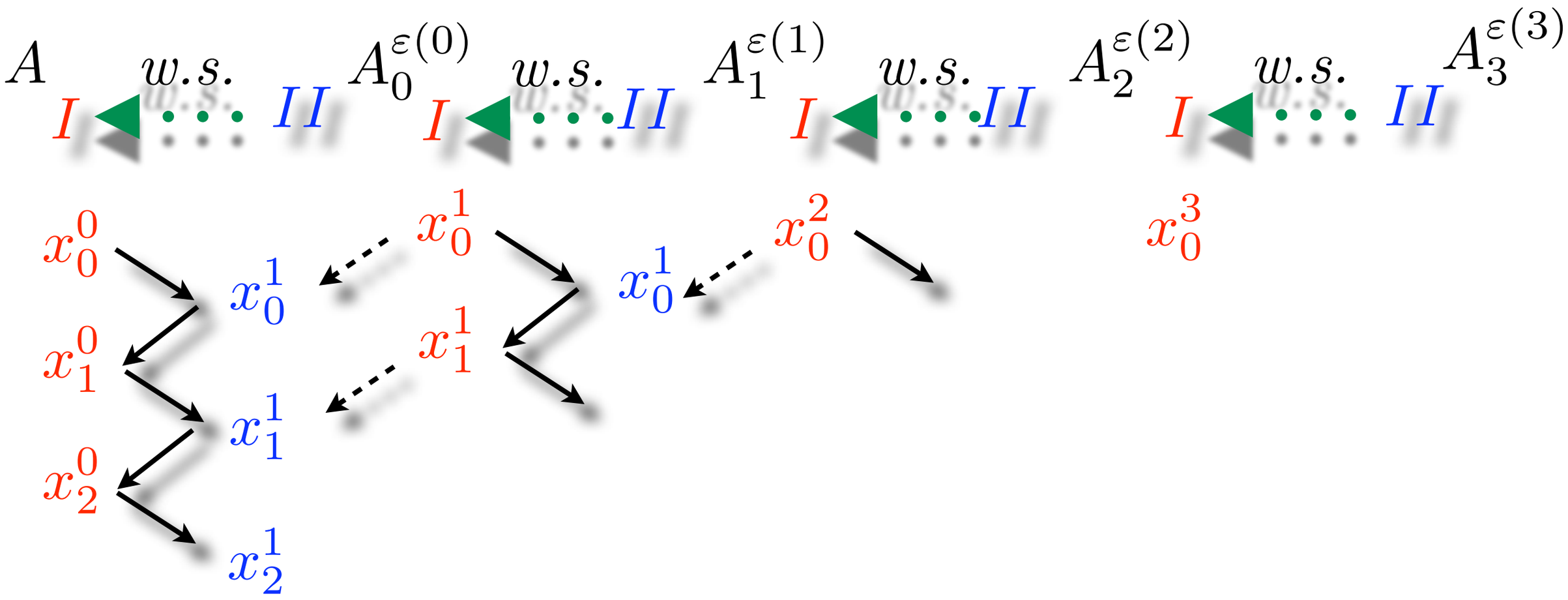

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
I
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1

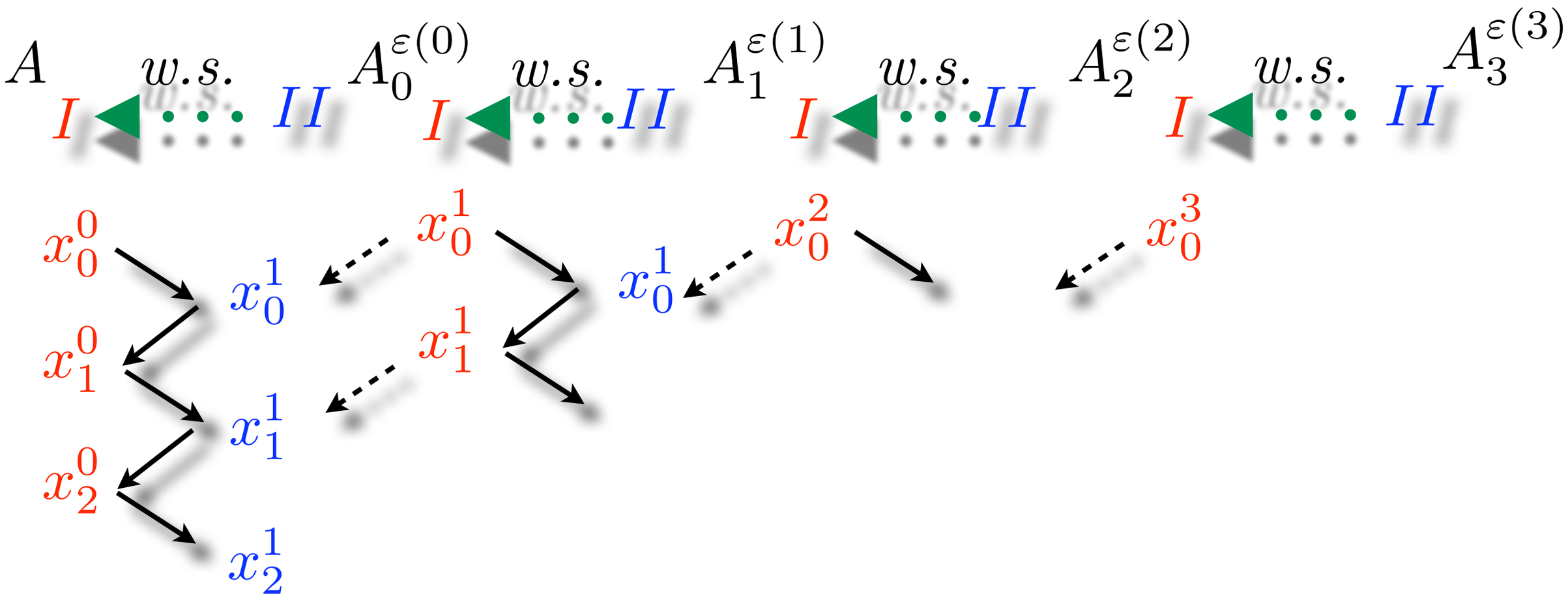

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
I
II
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1

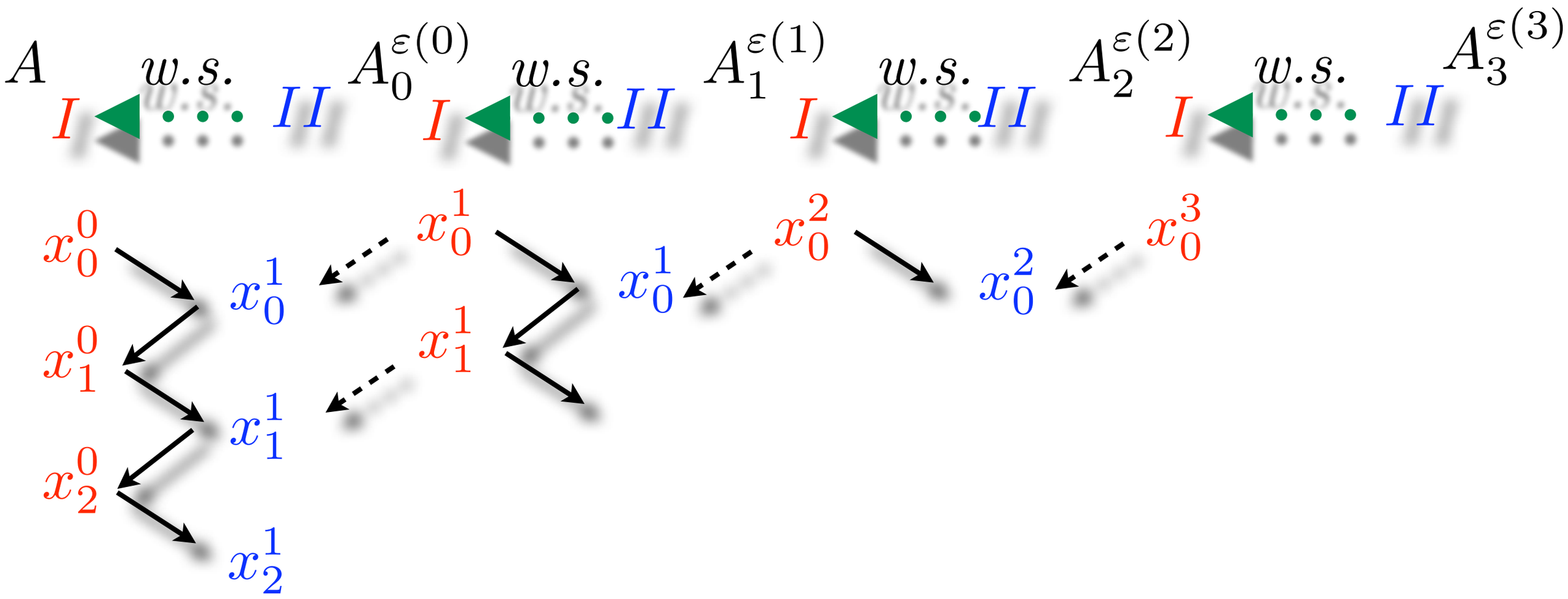

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1

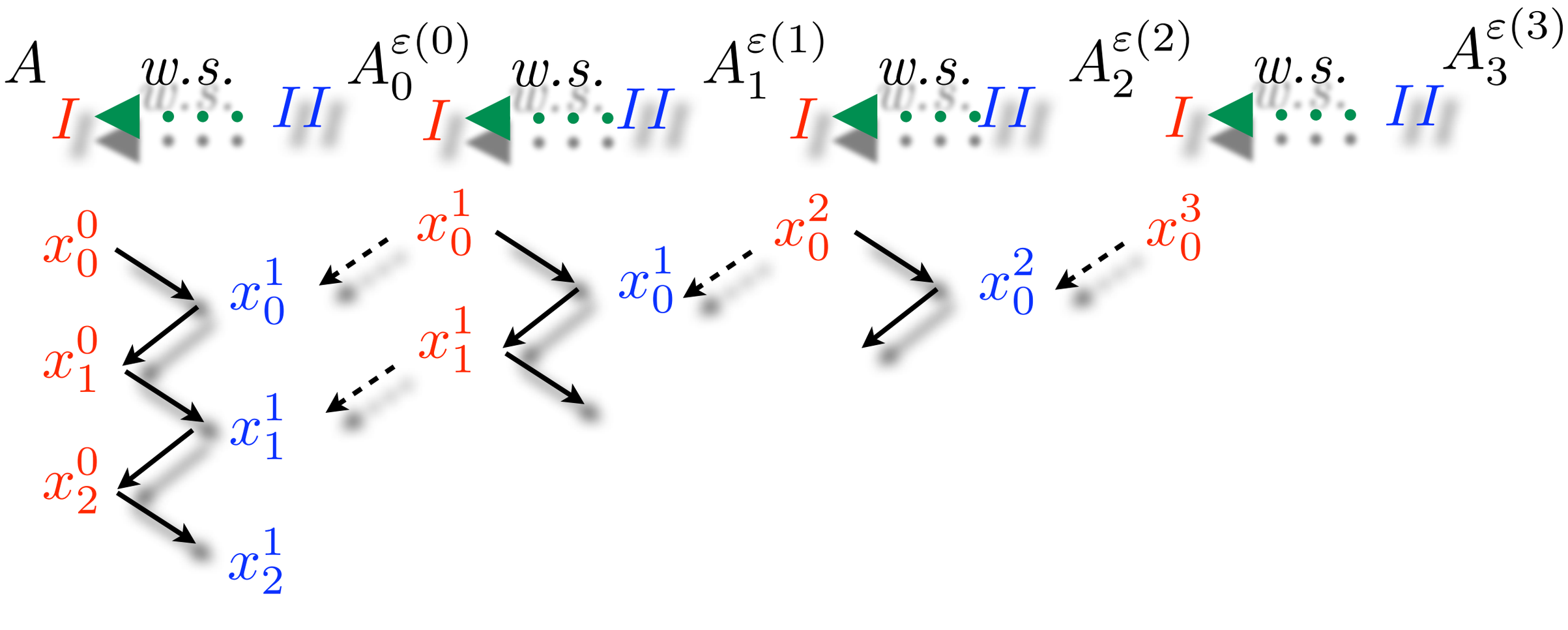

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1

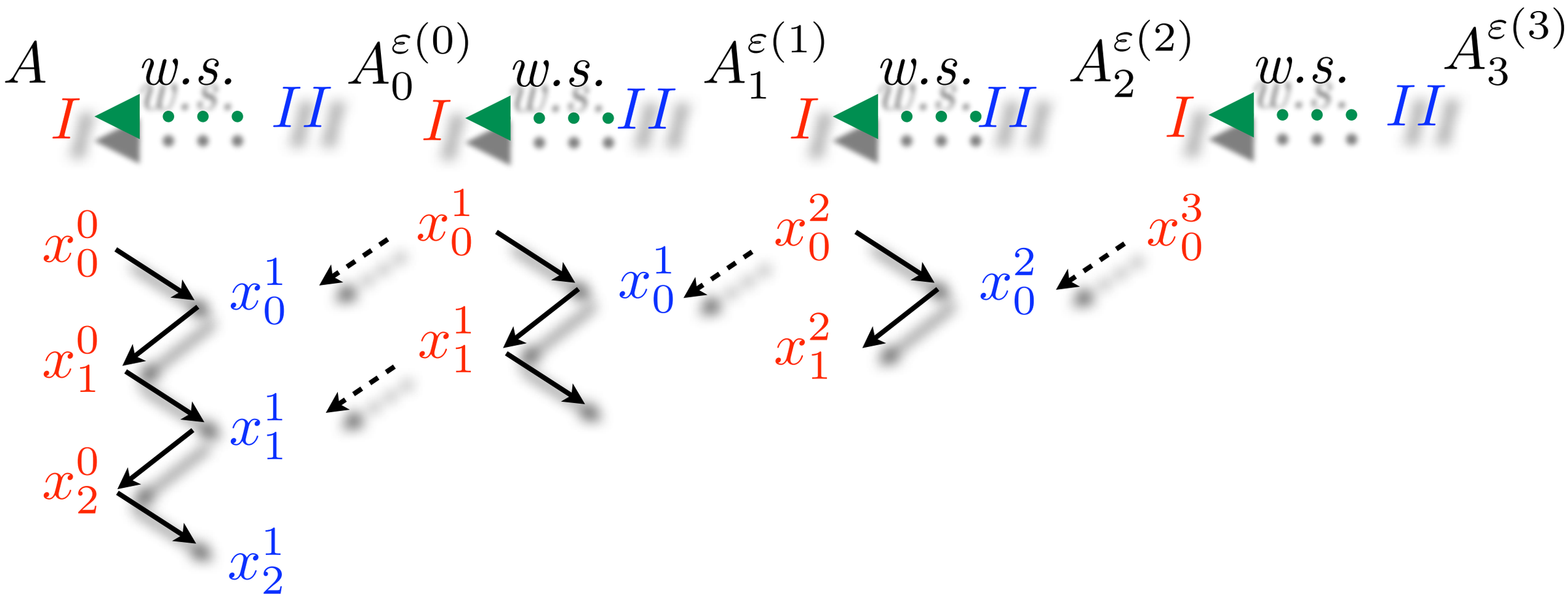

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1

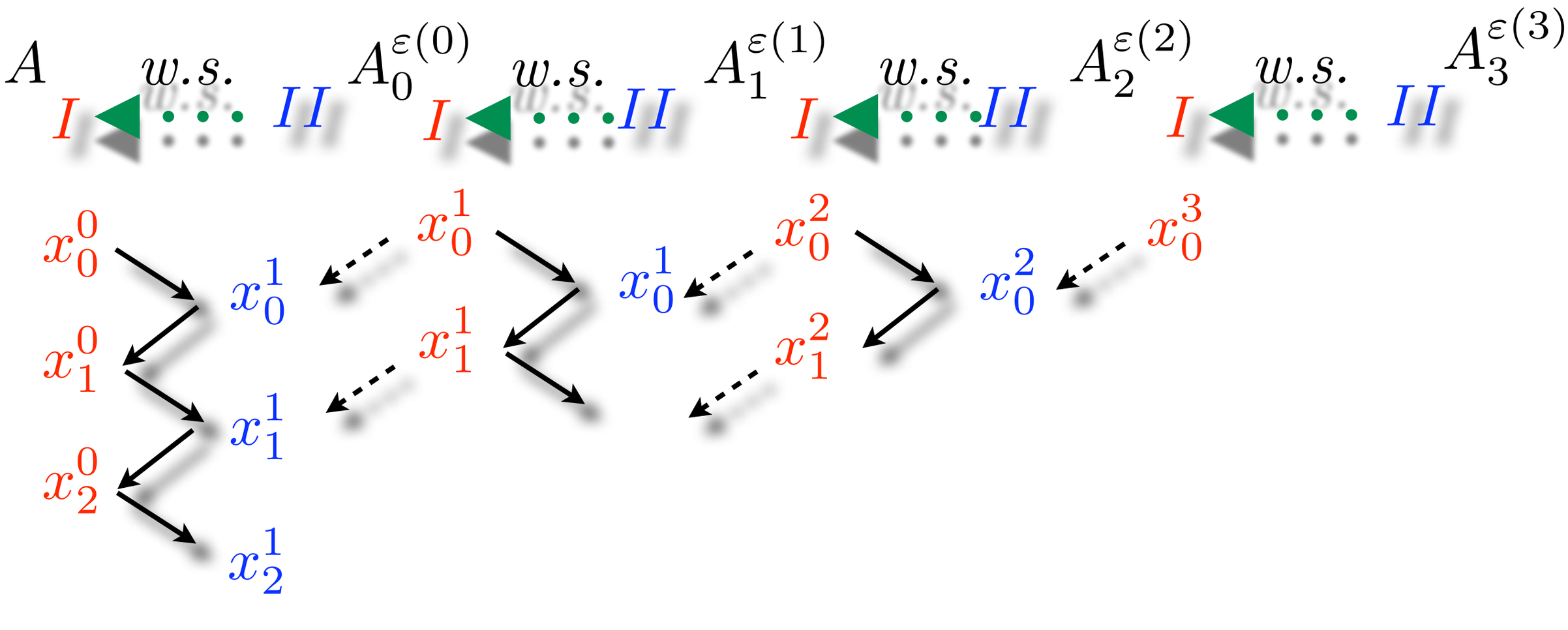

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_0^1
x_0^3

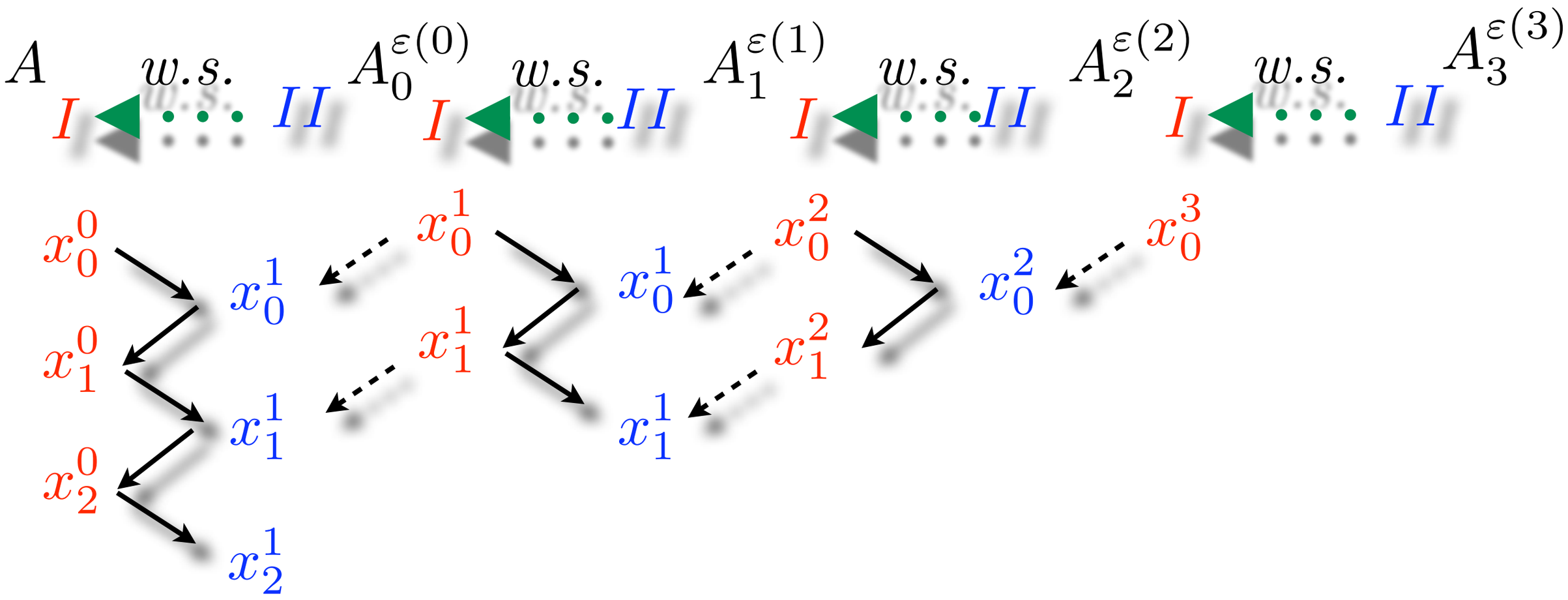

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^1
x_0^2
x_0^3

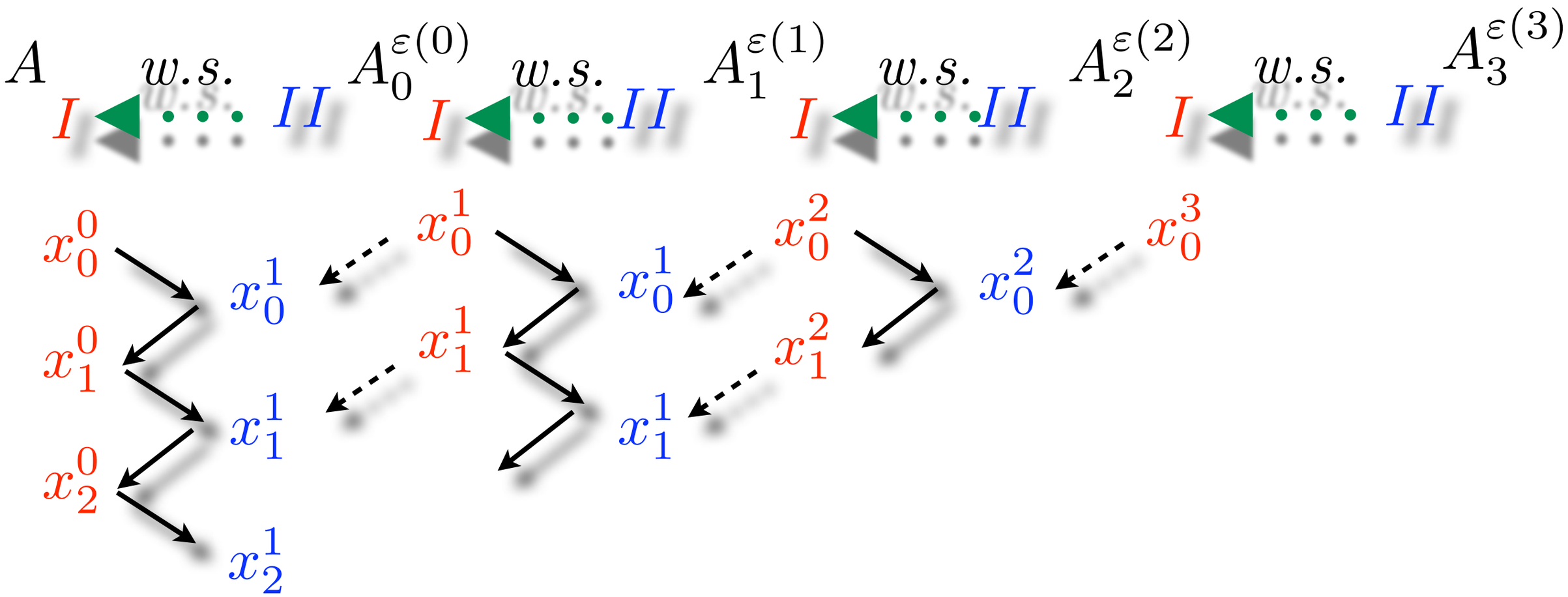

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^1
x_0^2
x_0^2
x_0^3

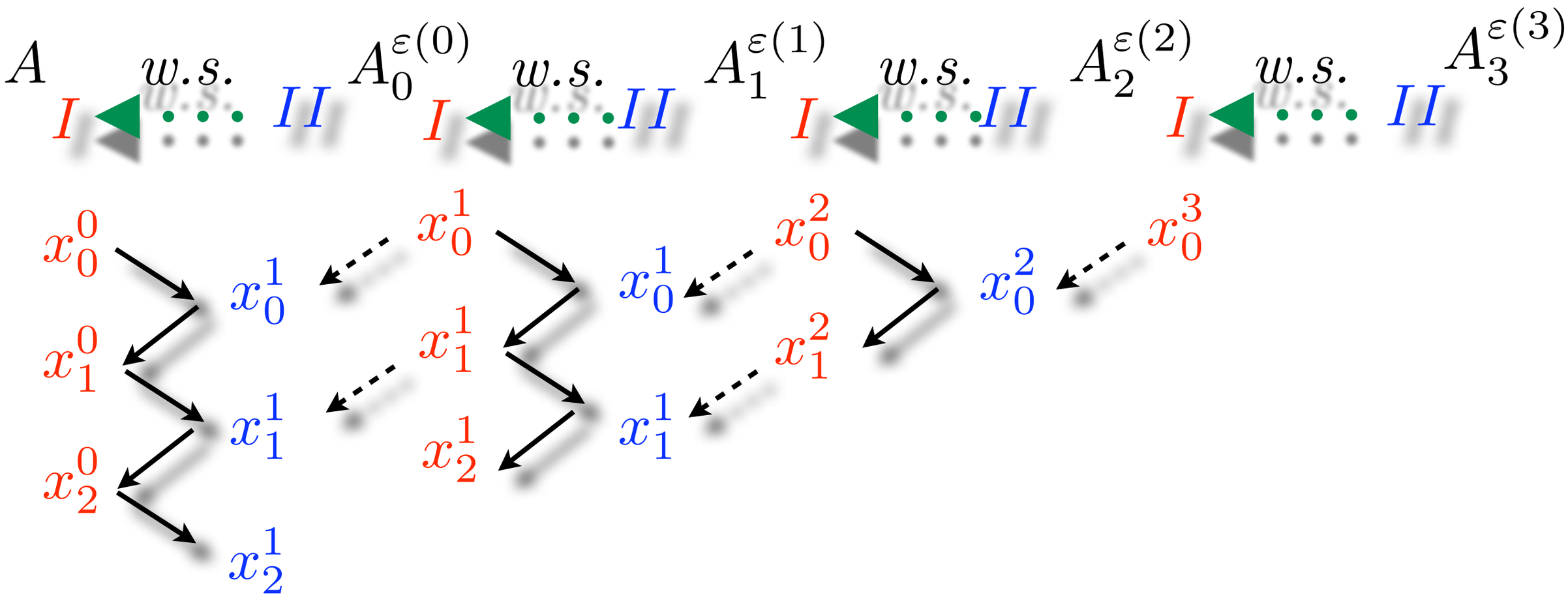

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^1
x_0^2
x_0^2
x_0^3

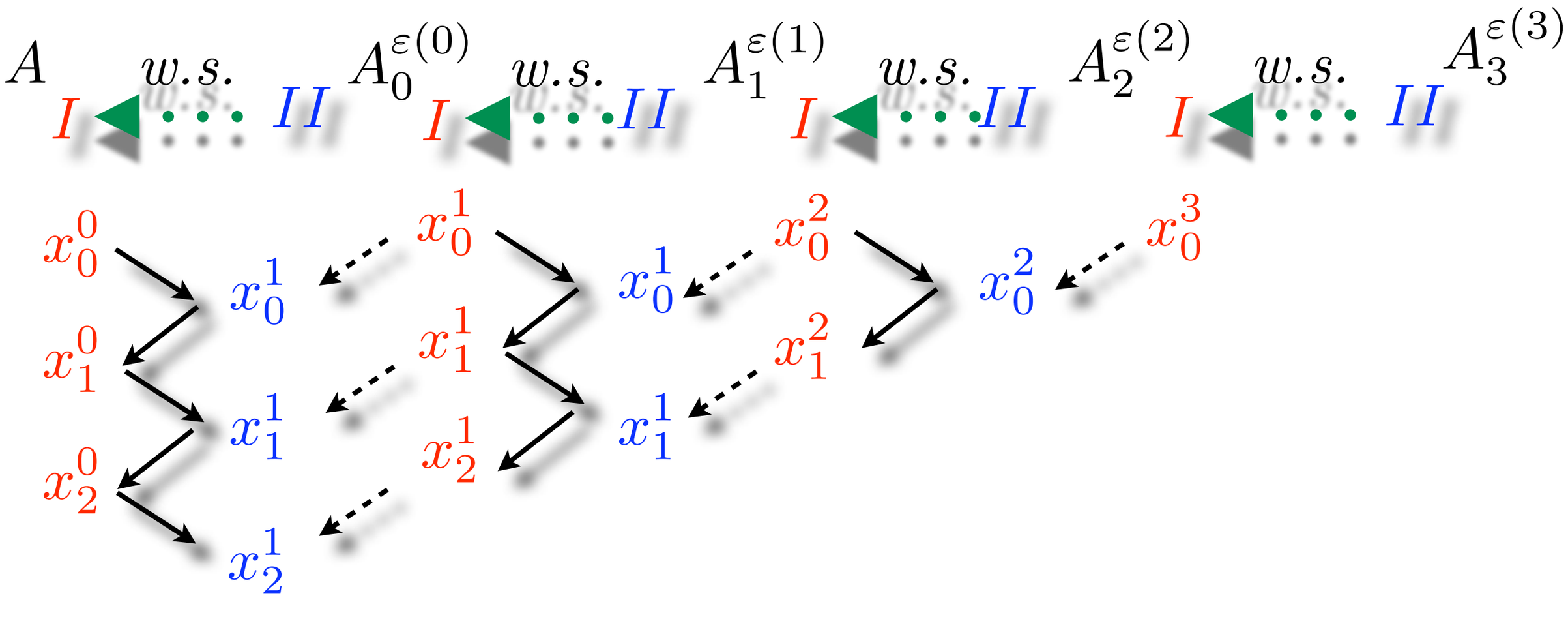

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^1
x_0^2
x_0^3

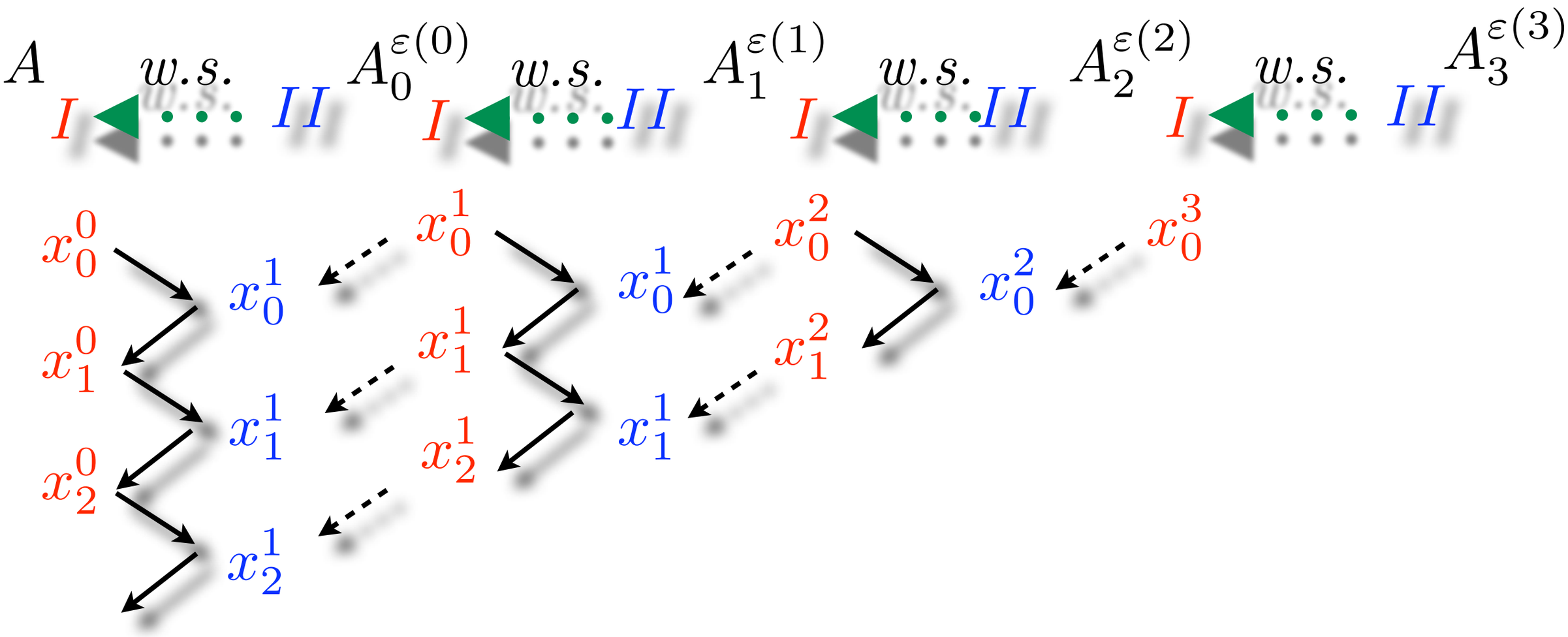

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^1
x_0^2
x_0^2
x_0^3

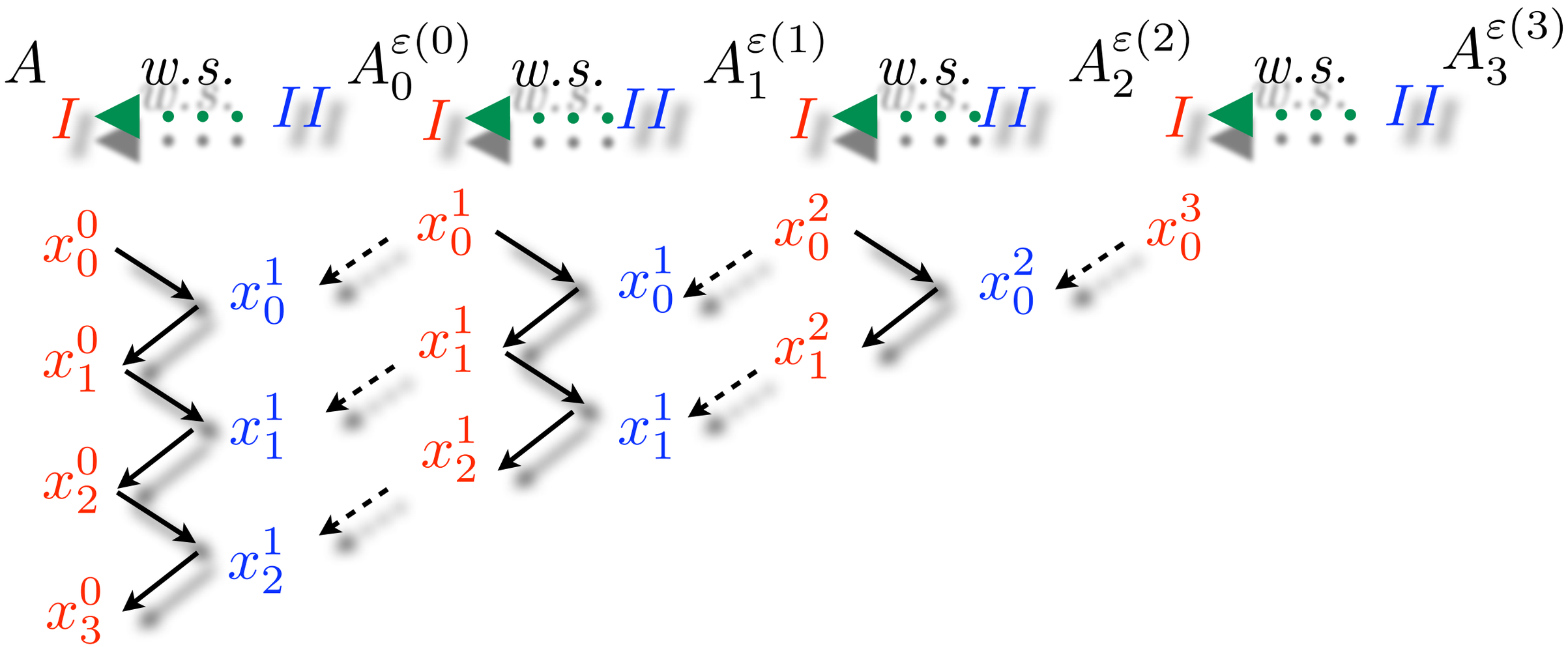

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3

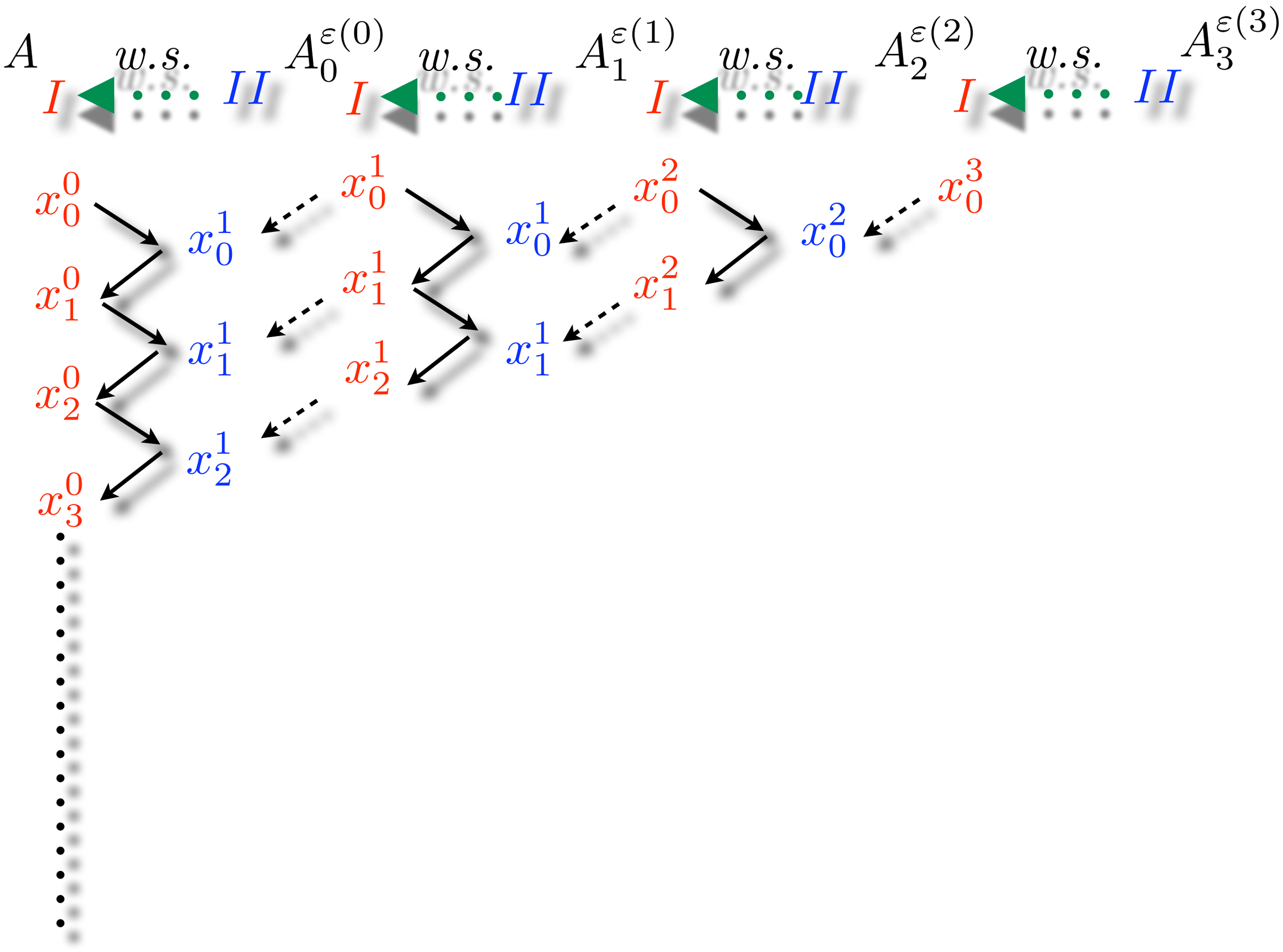

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3

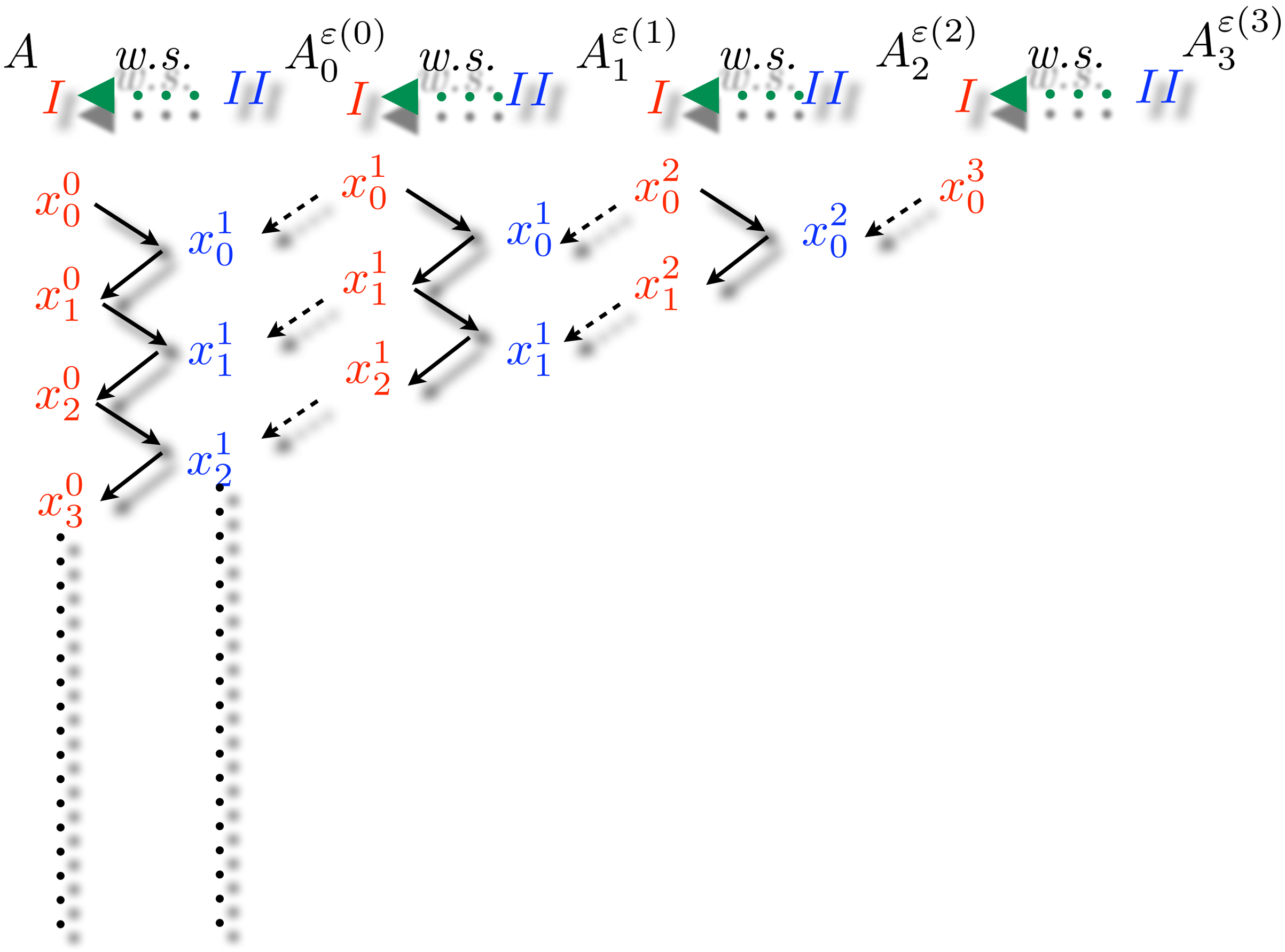

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^1
x_1^1
x_0^2
x_0^3
x_0^2

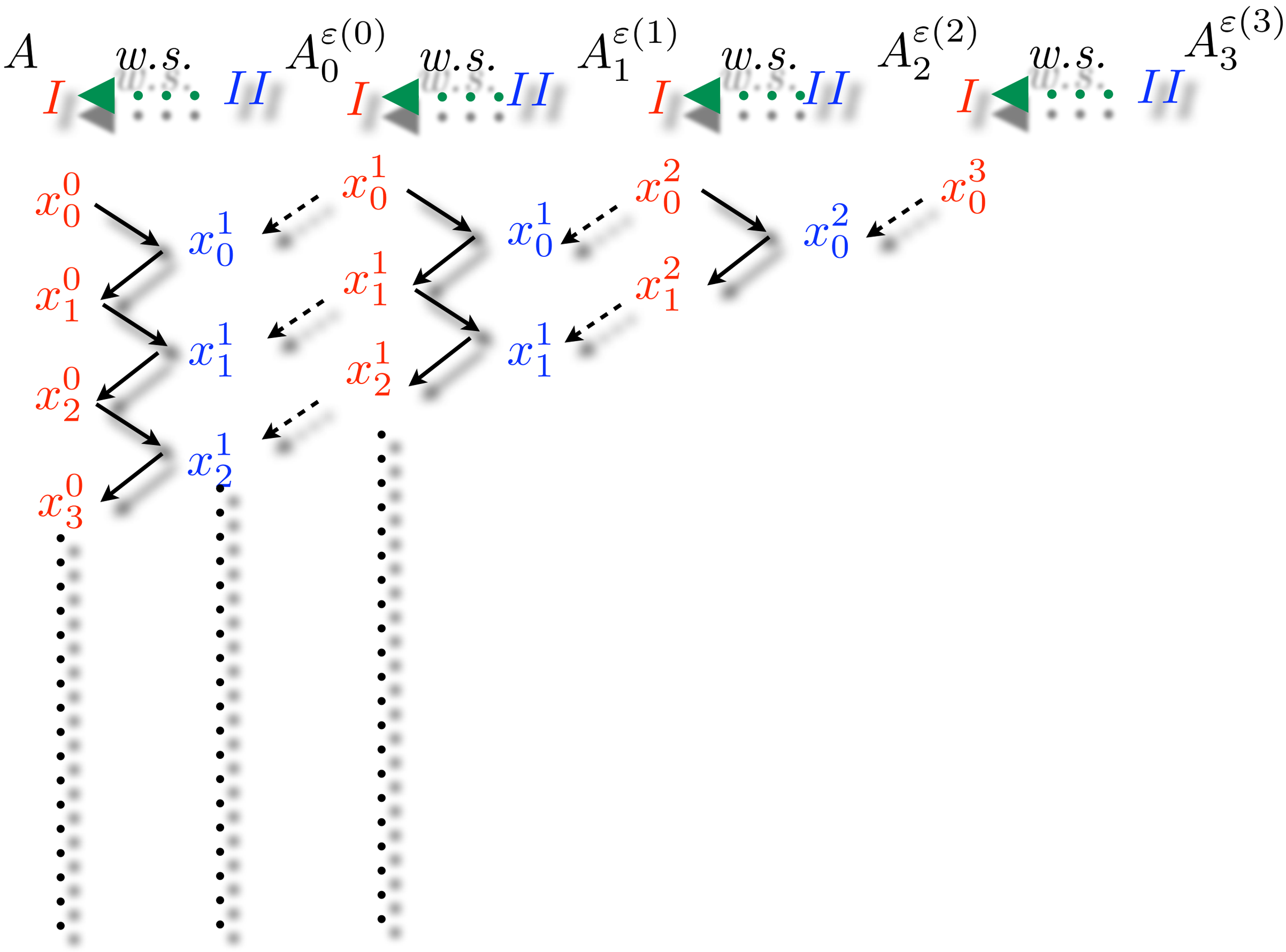

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^1
x_1^1
x_0^2
x_0^3
x_0^2

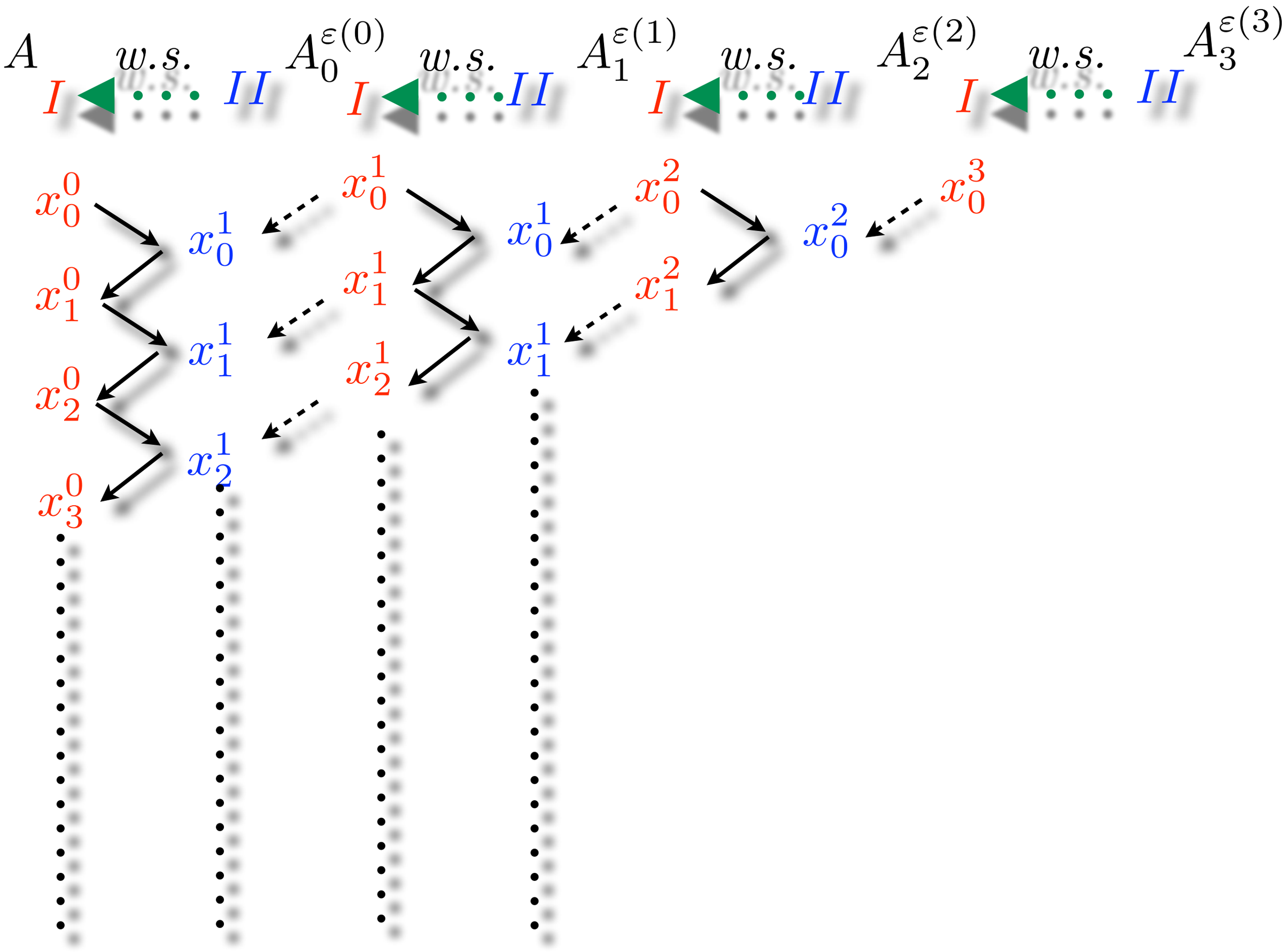

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^1
x_1^1
x_0^2
x_0^3
x_0^2

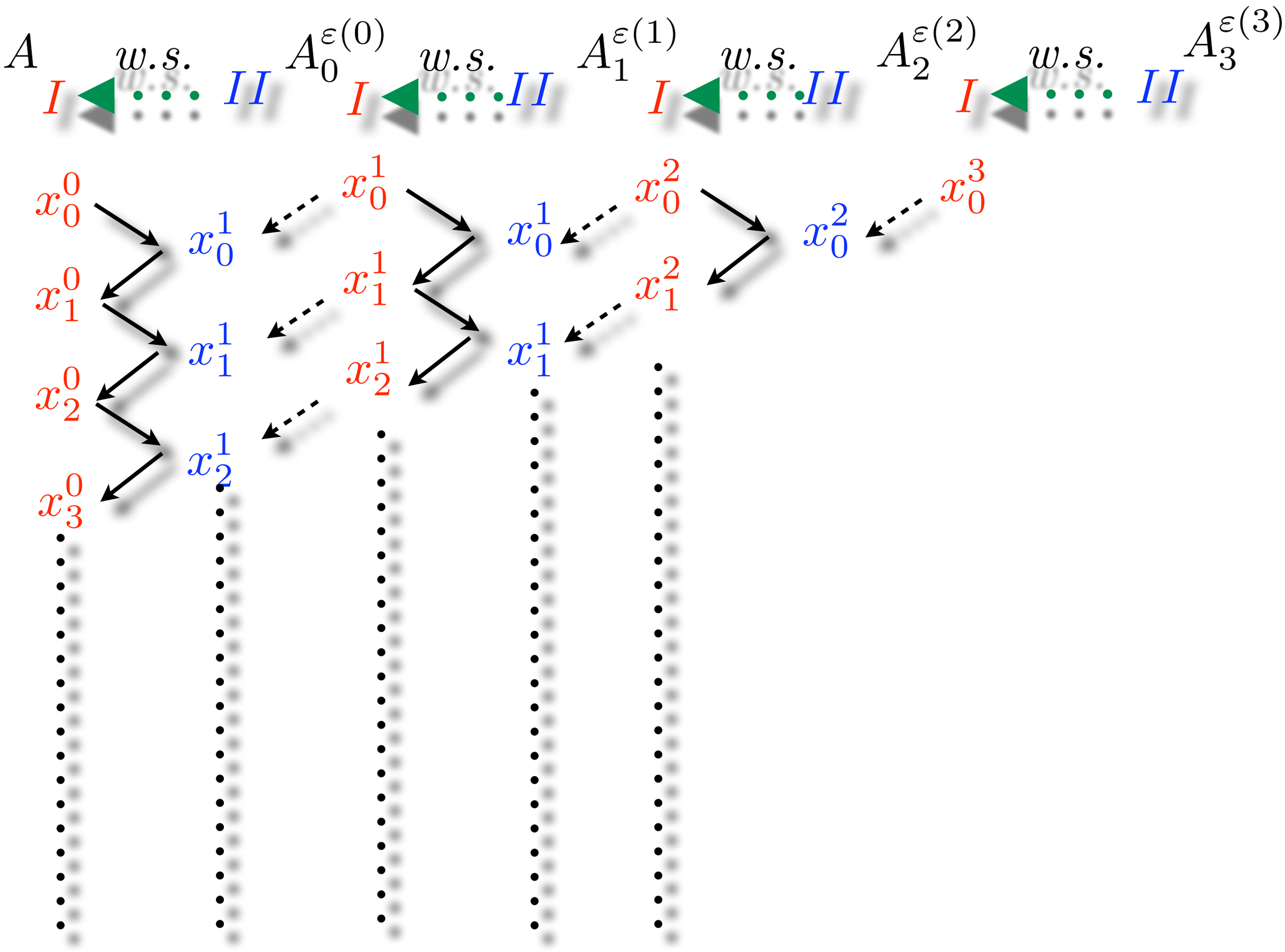

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0 x_1^0 x_2^0 x_3^0
x_0^1 x_1^1 x_2^1
x_0^1 x_1^1
x_0^2 x_1^2
x_0^2
x_0^3

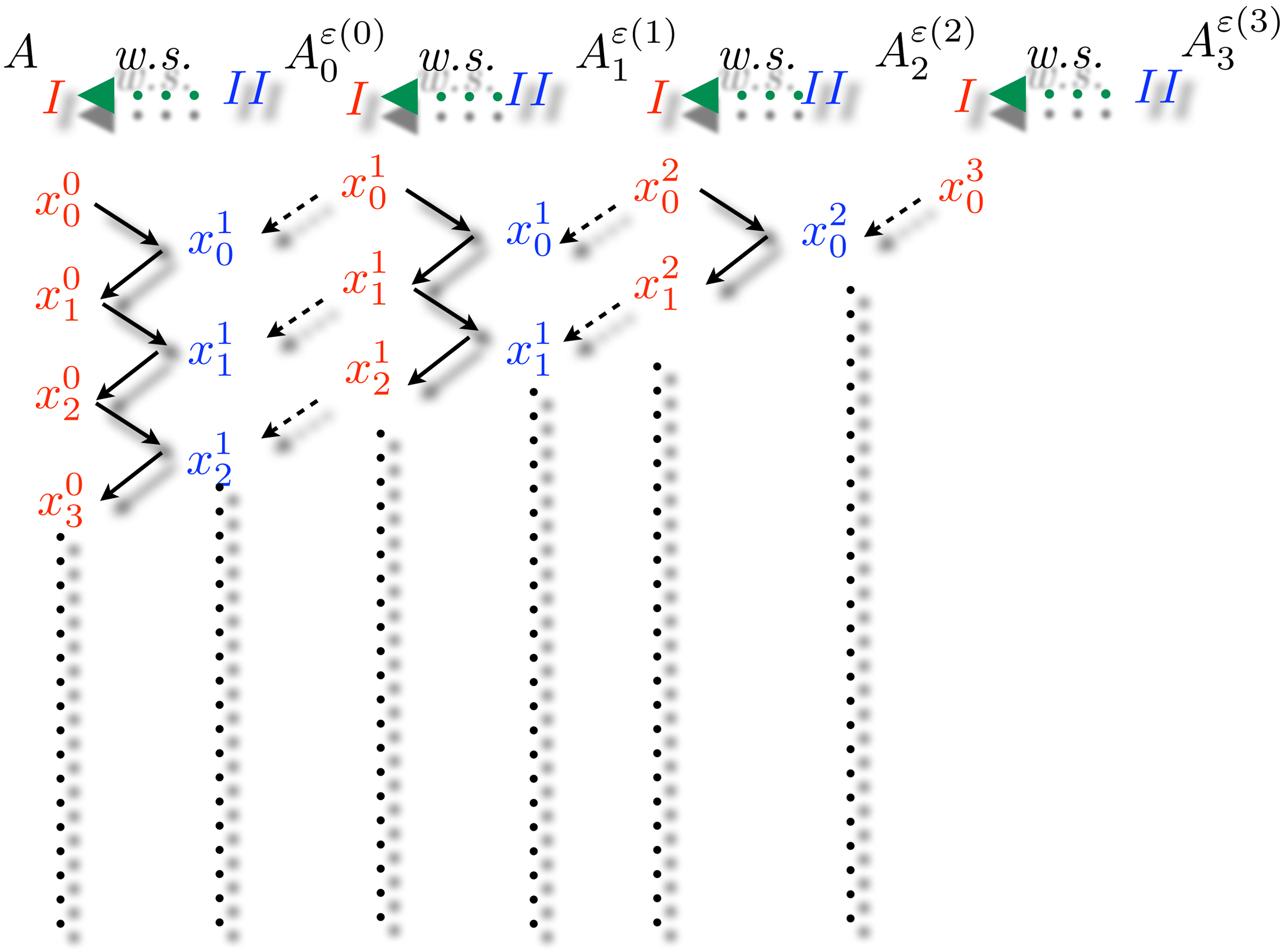

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3

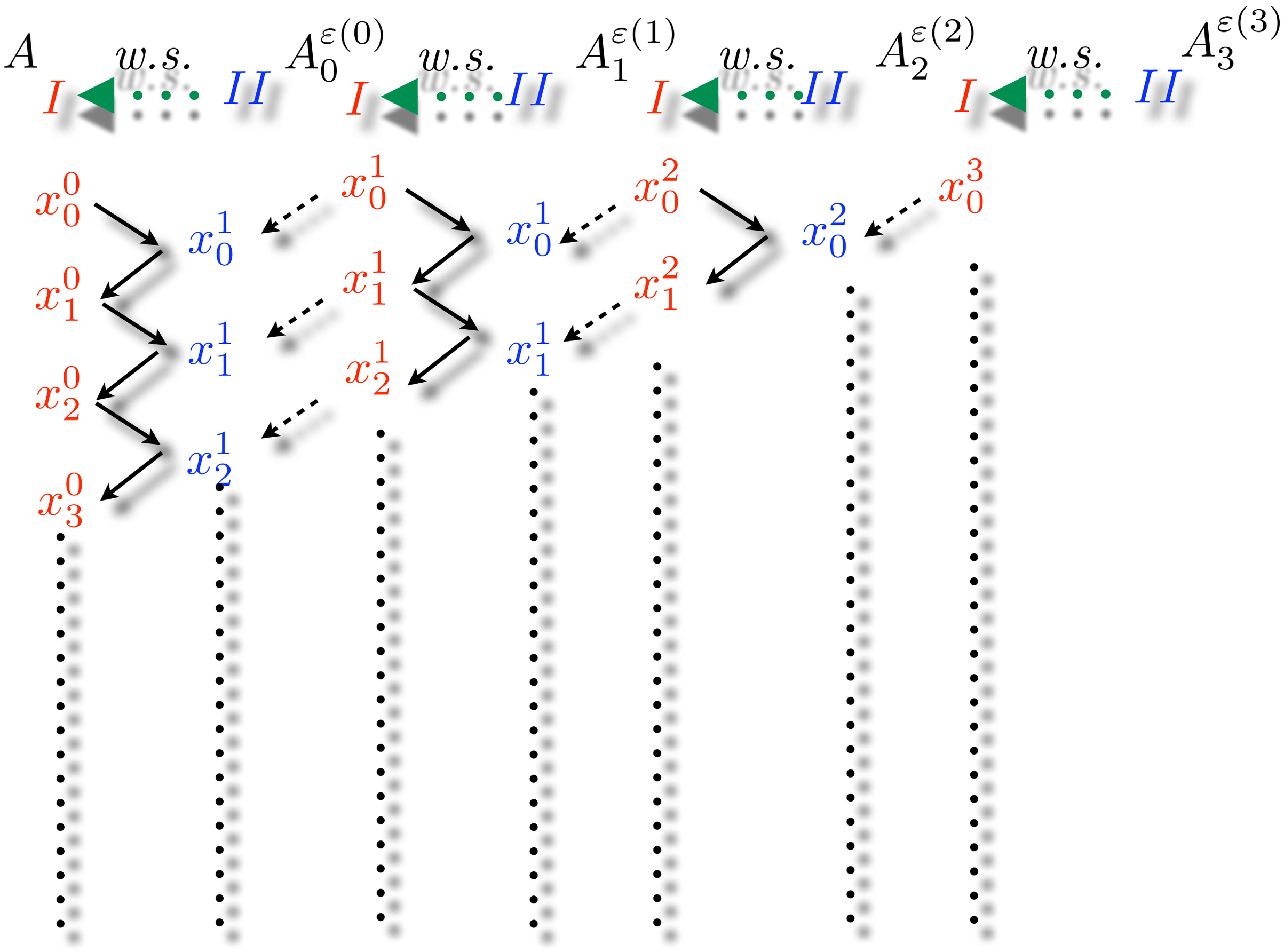

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3

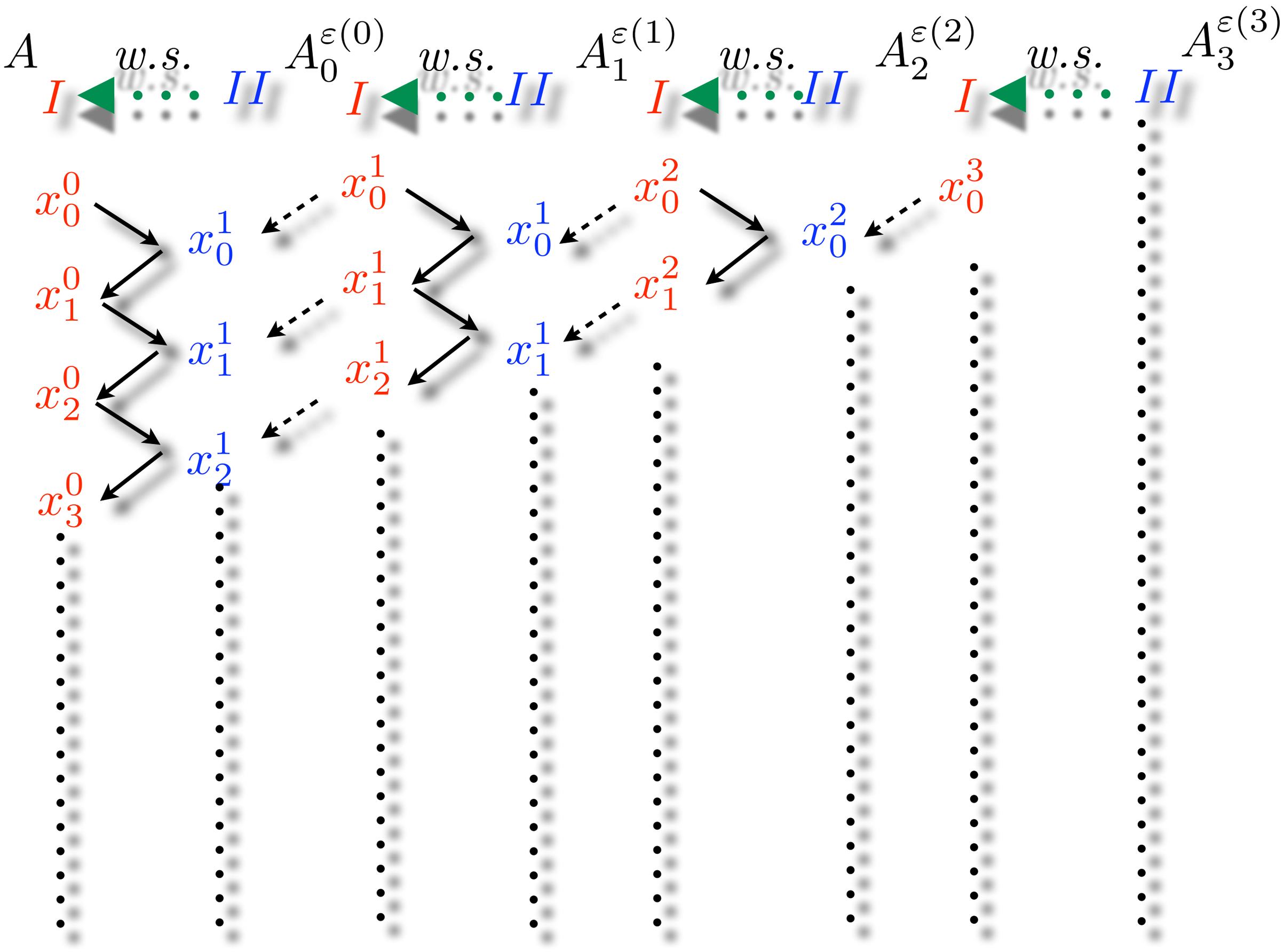

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^3

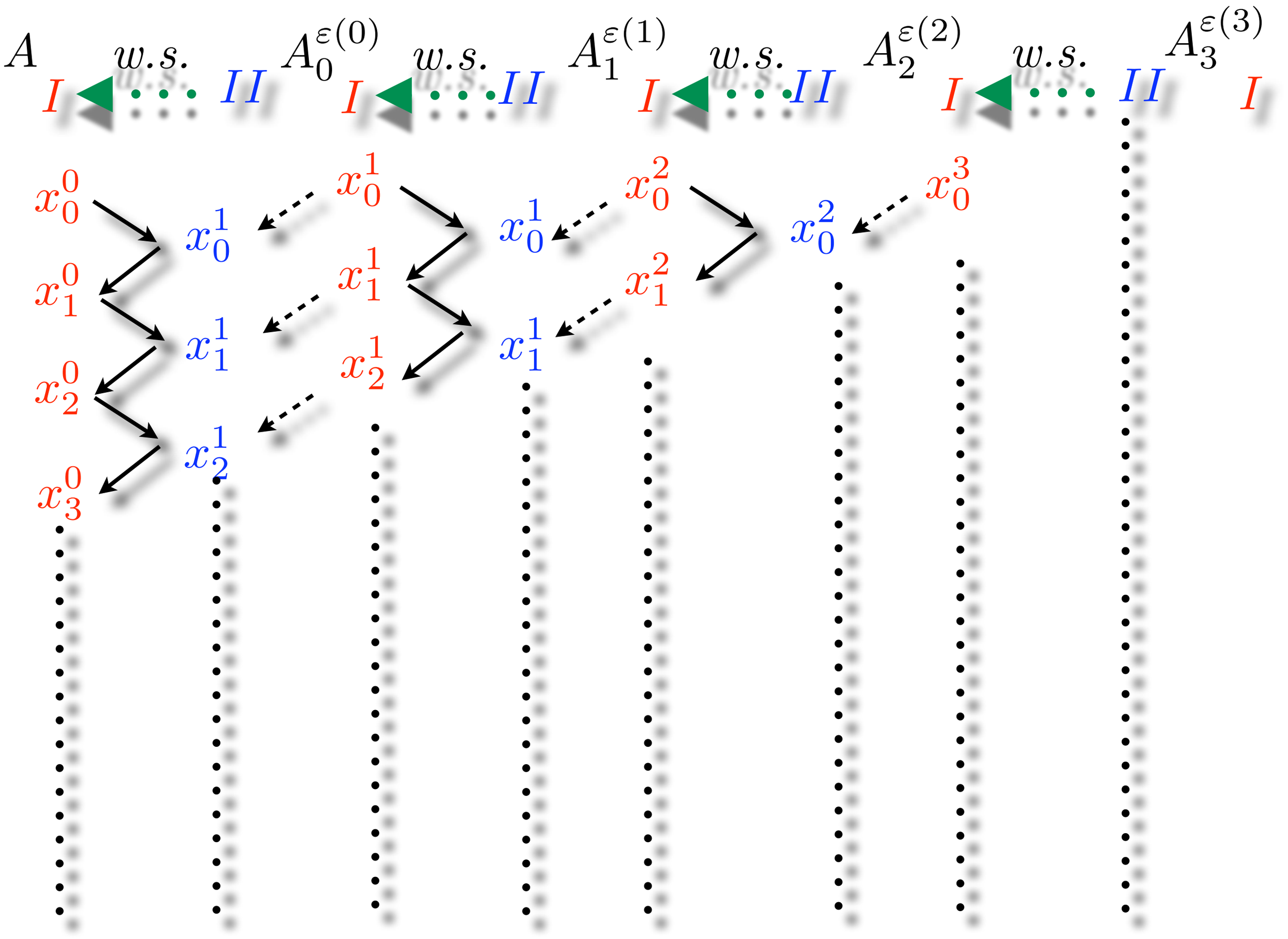

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^3

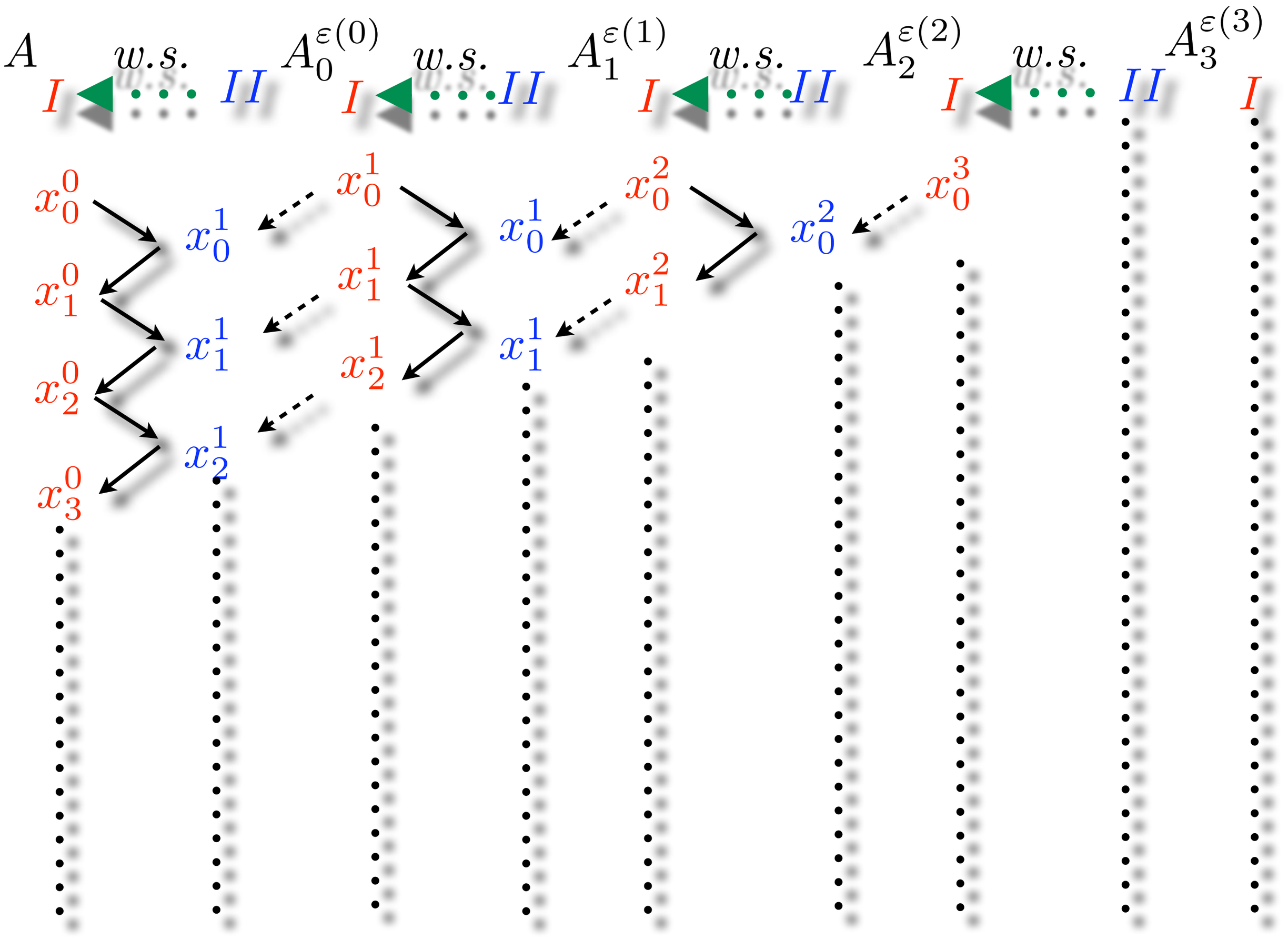

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^3

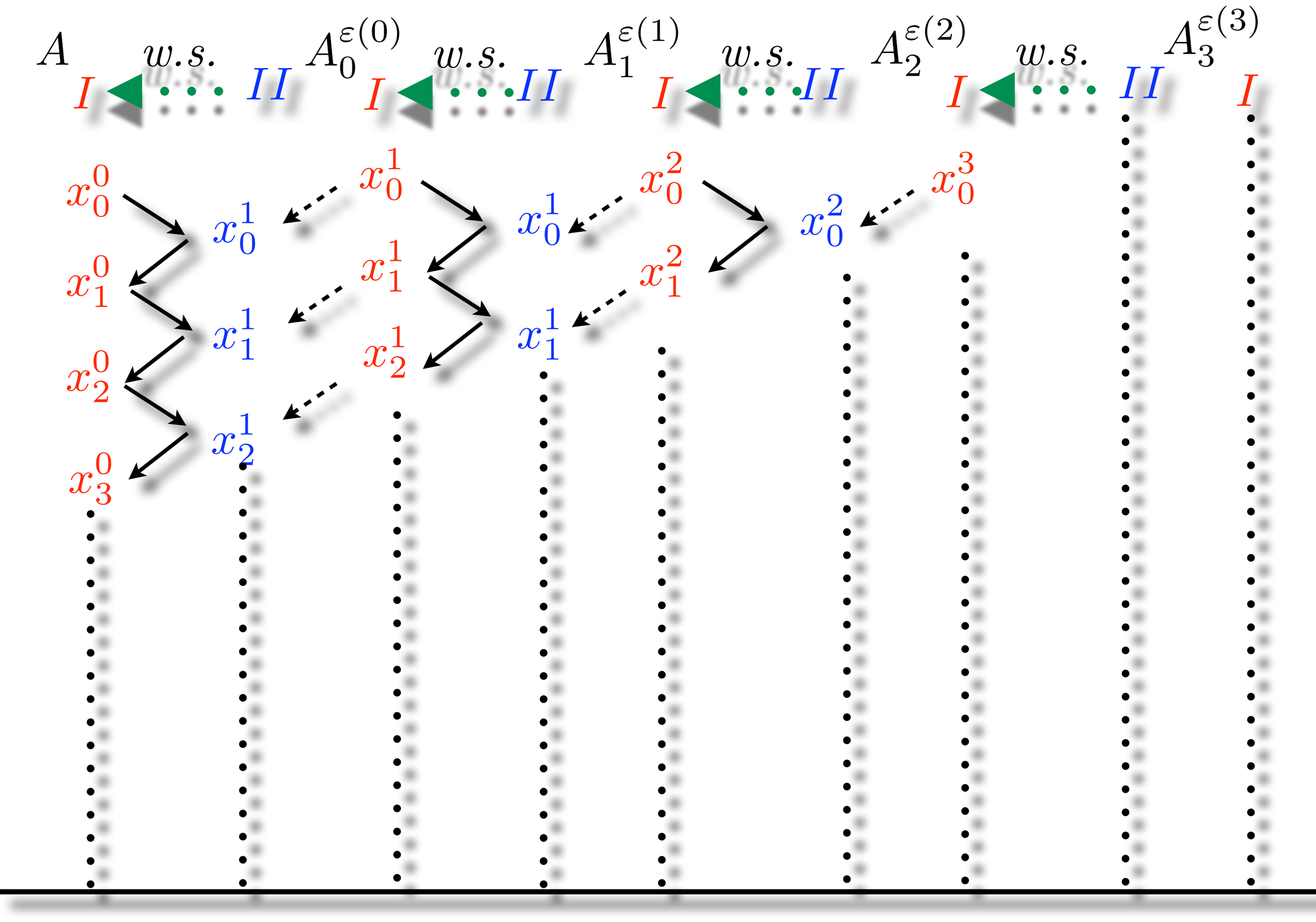

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$A_1^{\varepsilon(1)}$
$A_2^{\varepsilon(2)}$
$A_3^{\varepsilon(3)}$
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^2
x_1^2
x_0^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3

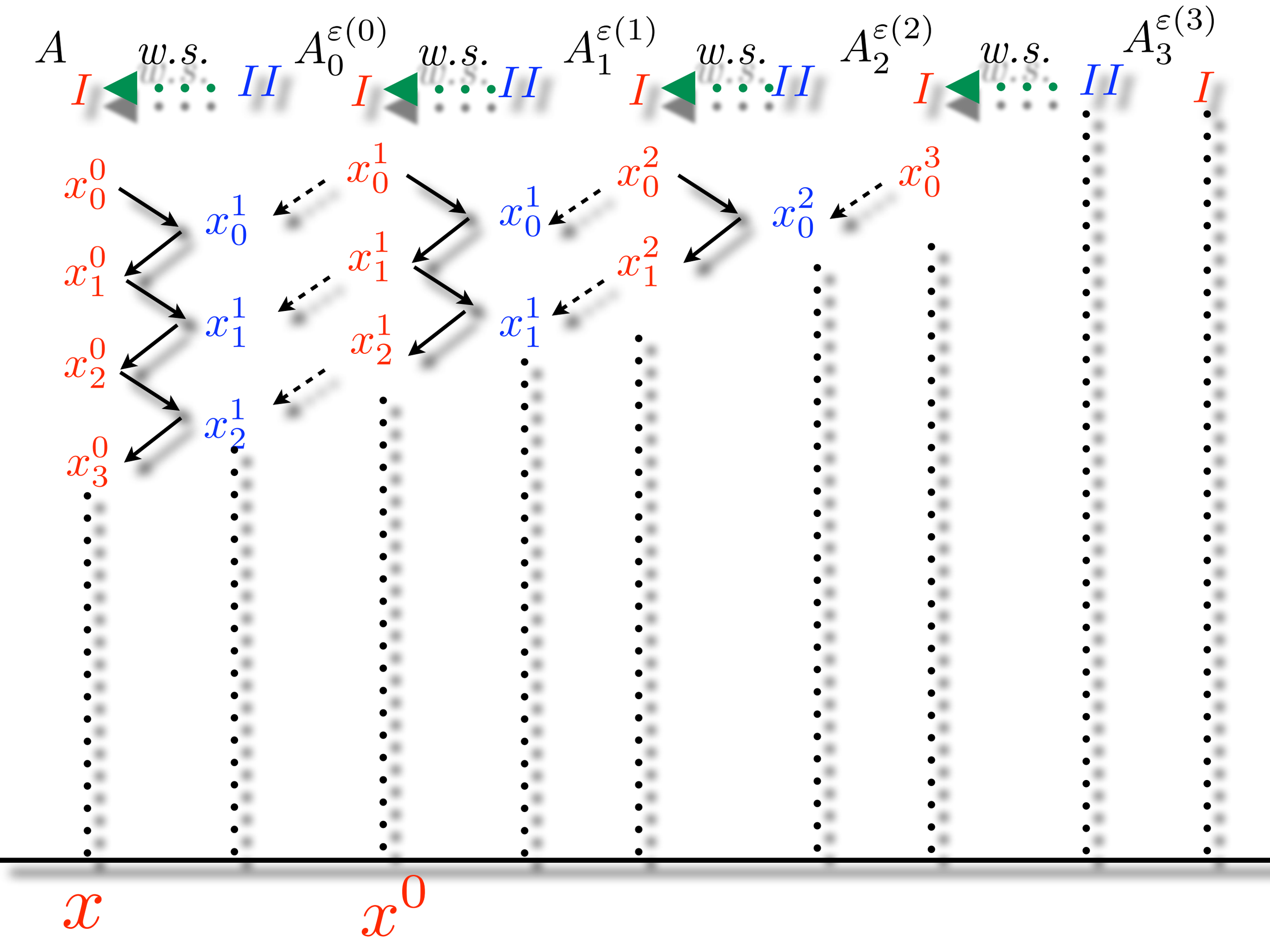

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3

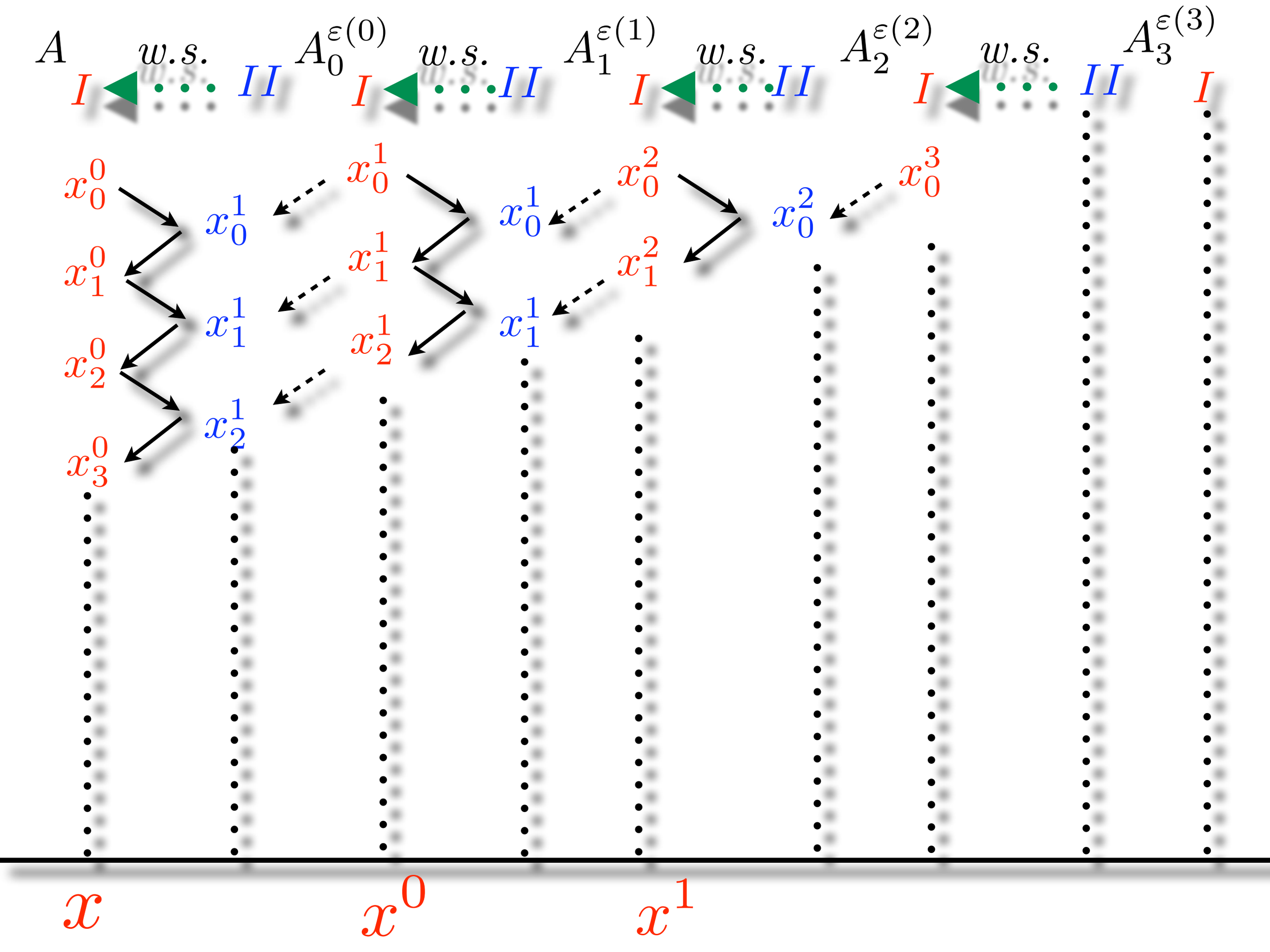
A $w.s.$ $A_0^{\varepsilon(0)}$ $w.s.$ $A_1^{\varepsilon(1)}$ $w.s.$ $A_2^{\varepsilon(2)}$ $w.s.$ $A_3^{\varepsilon(3)}$

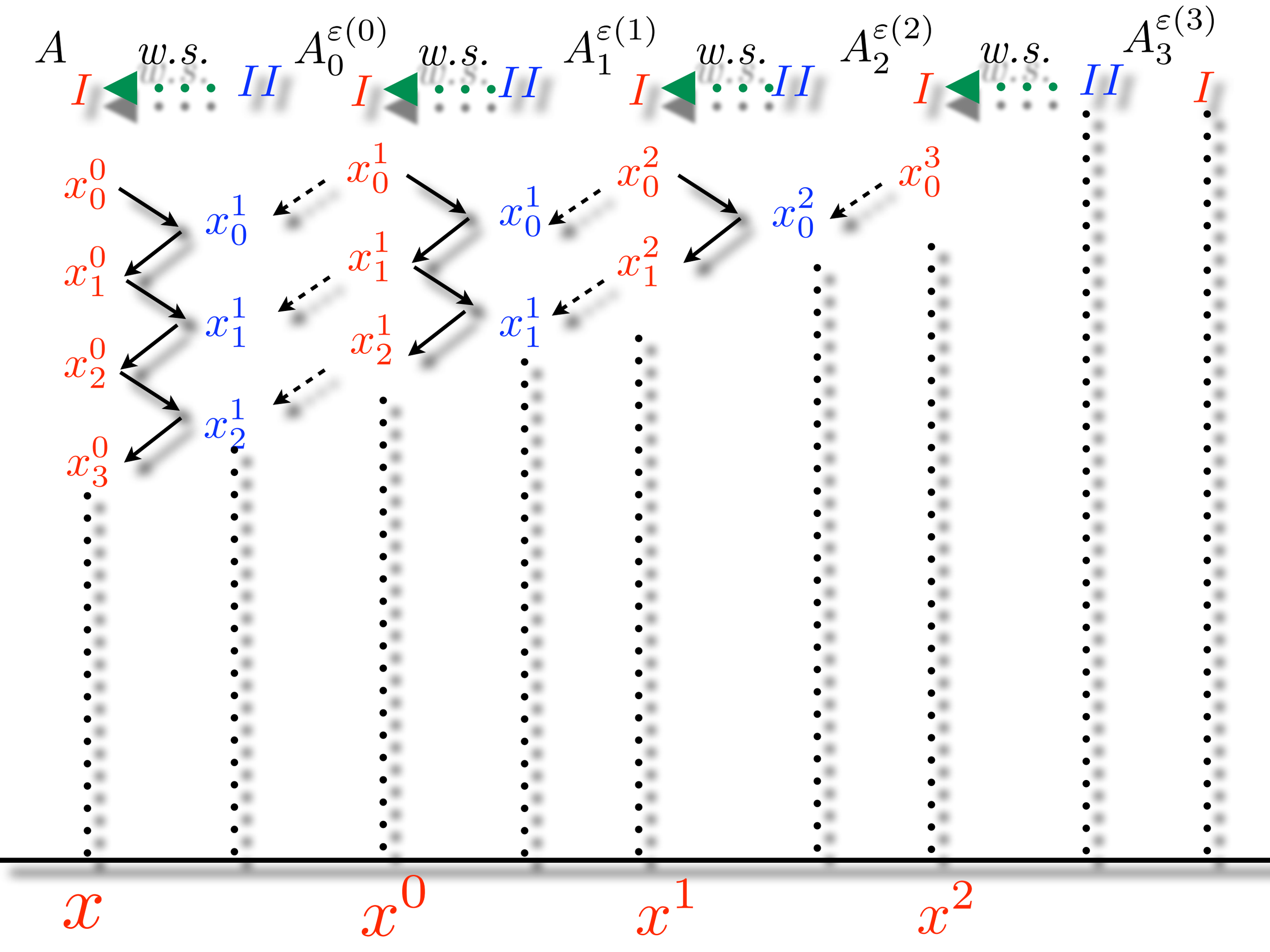
I II I II I II I II I

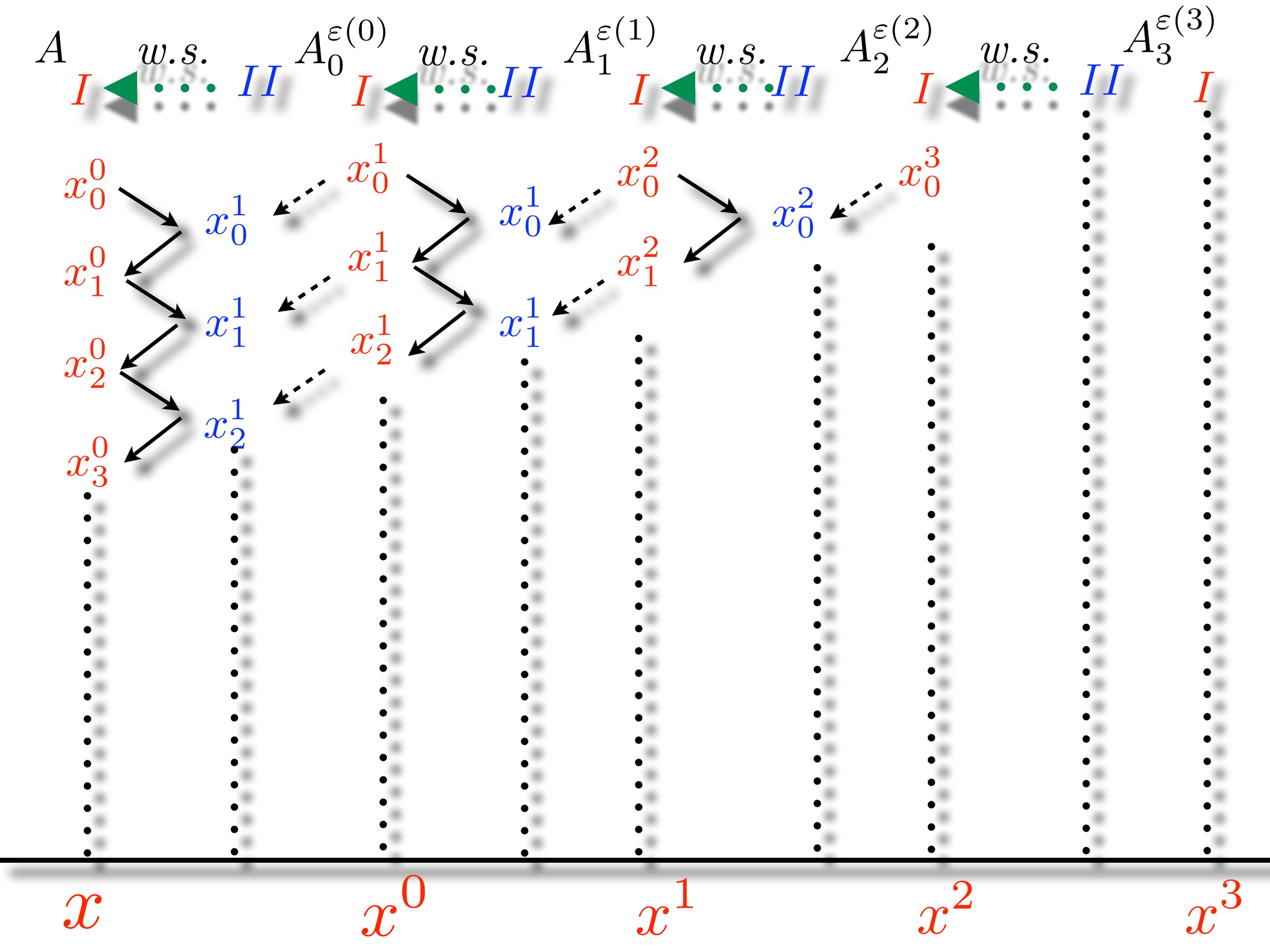
x_0^0 x_1^0 x_2^0 x_3^0 x_0^1 x_1^1 x_2^1 x_0^1 x_1^1 x_0^2 x_1^2 x_0^2 x_0^3

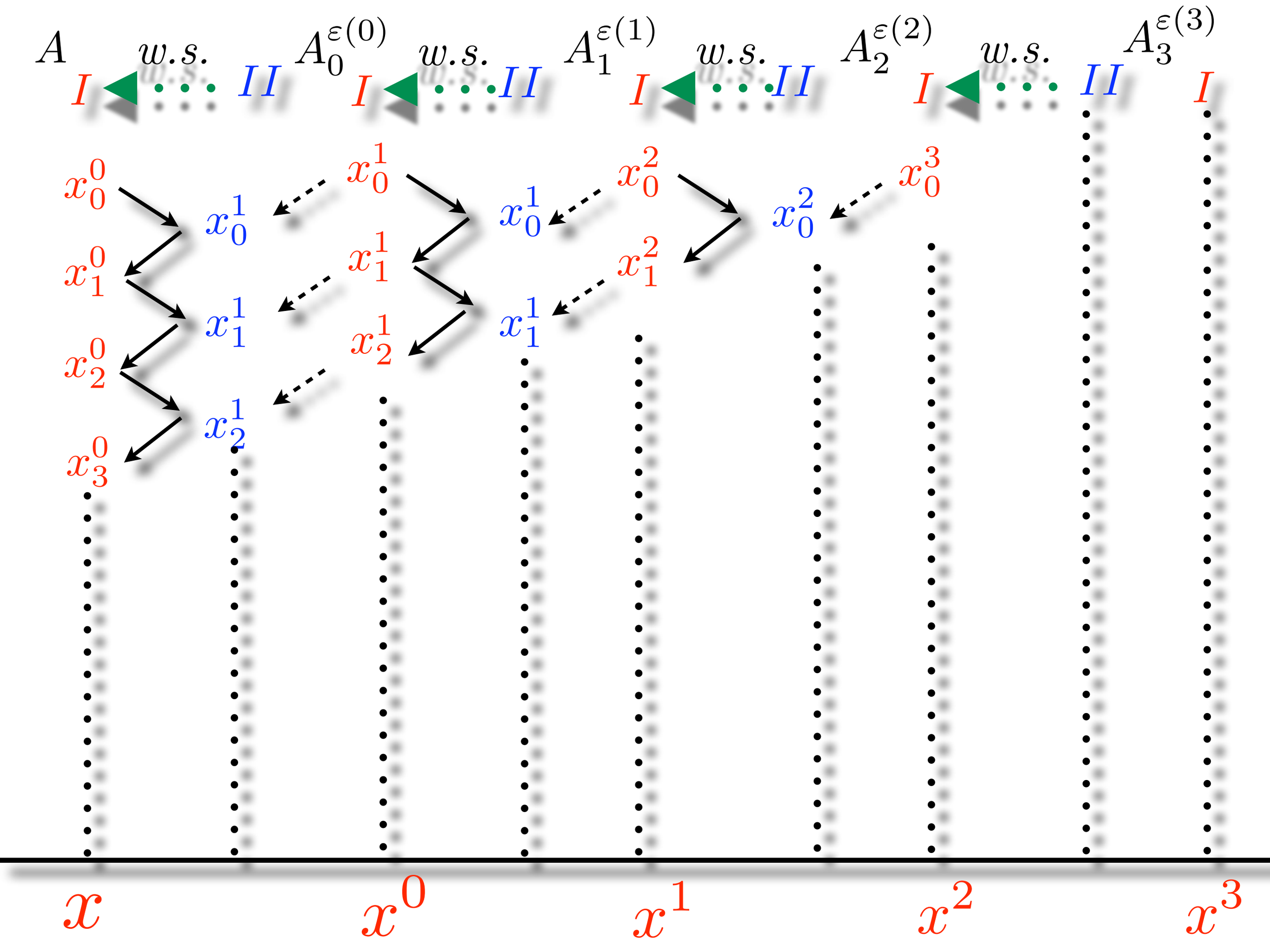
x

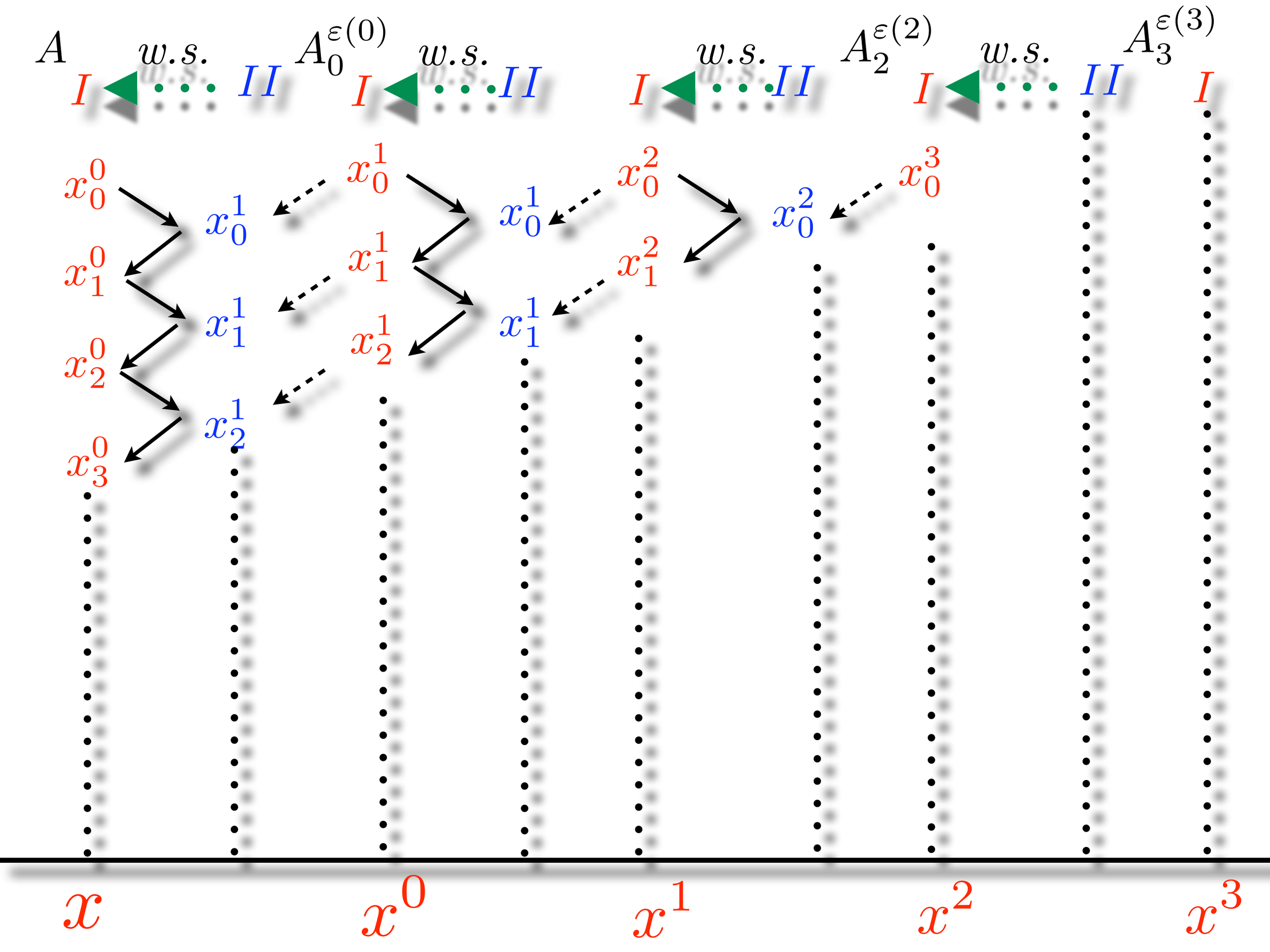

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0

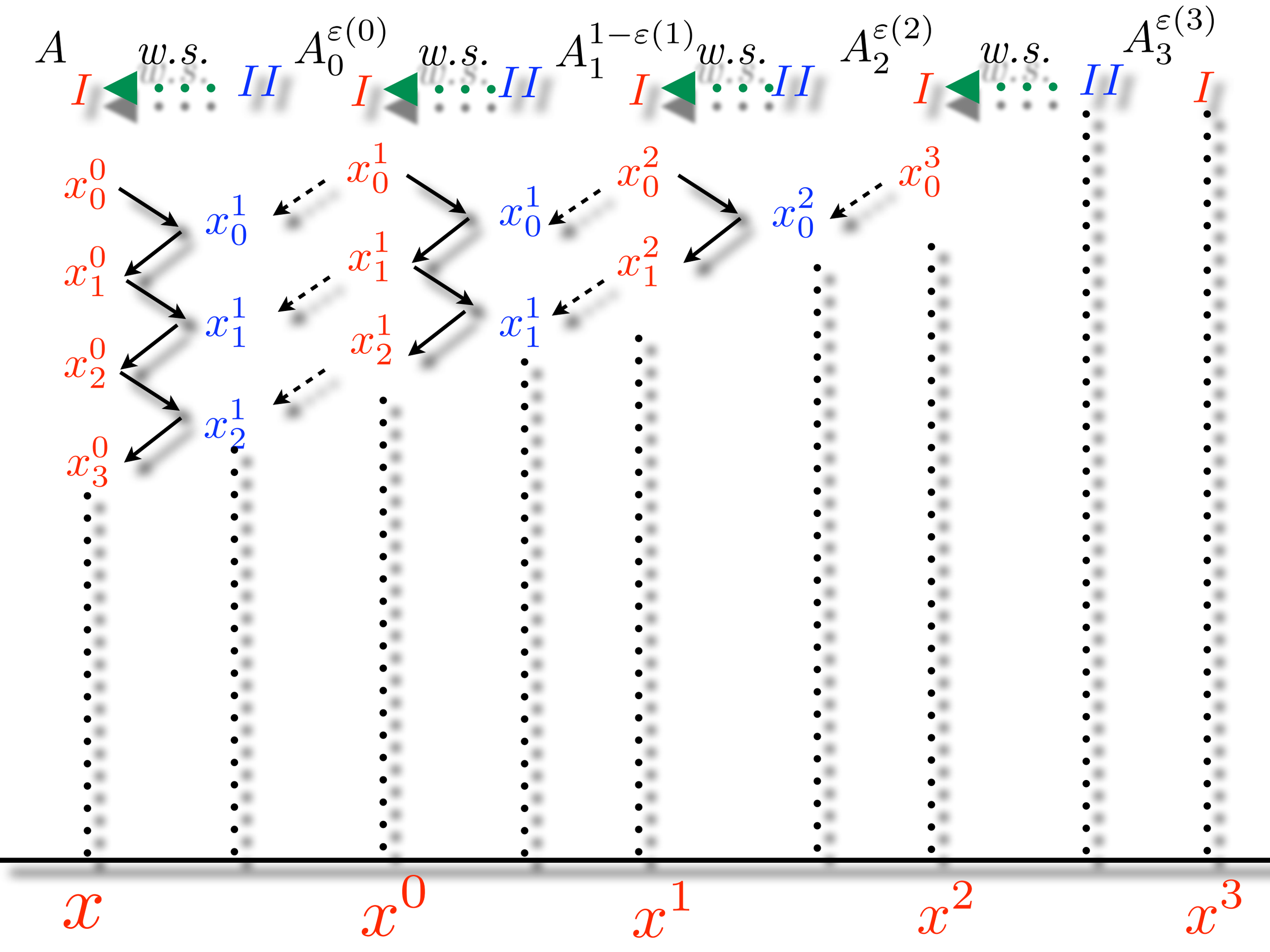

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1

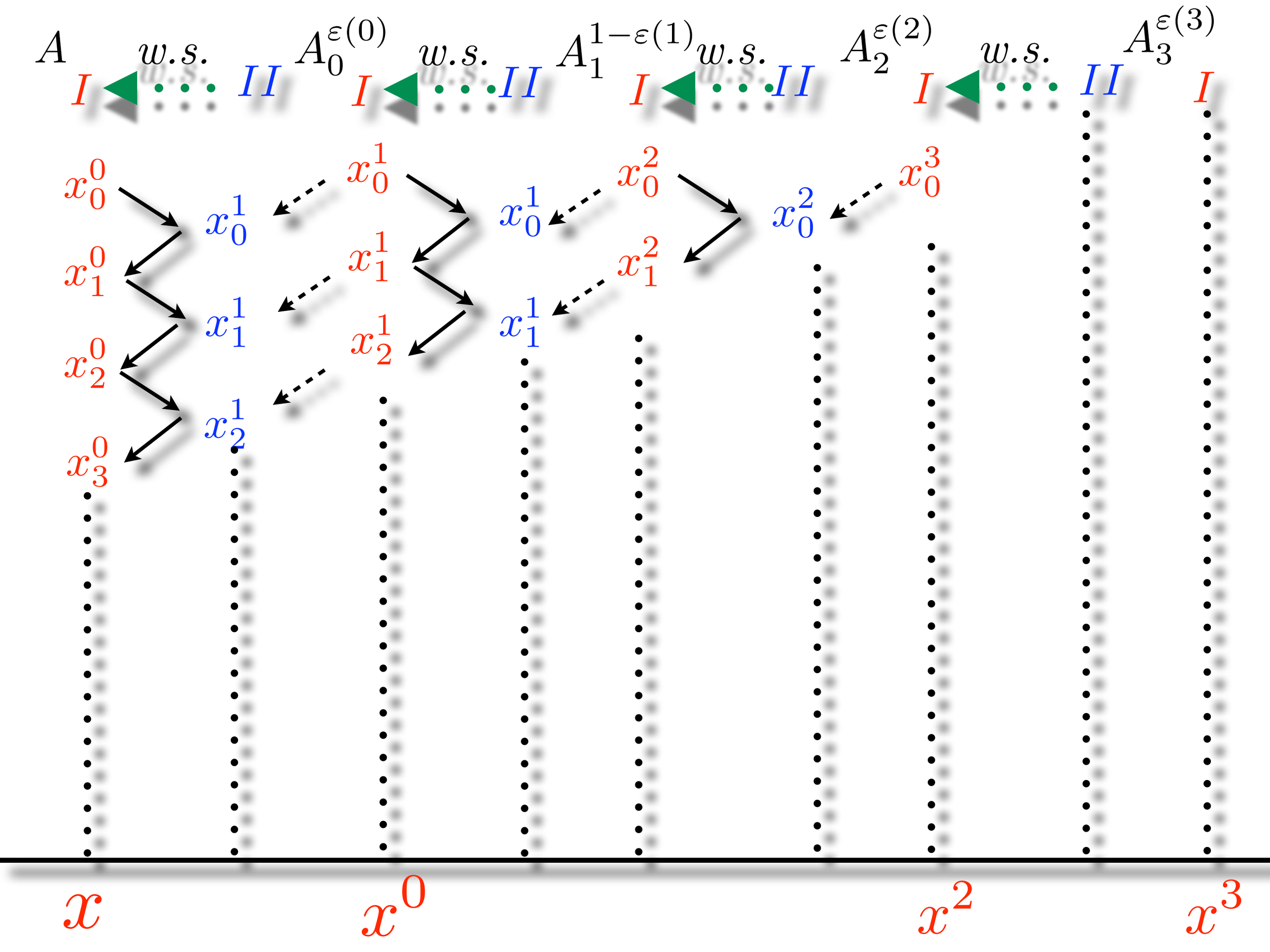

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2

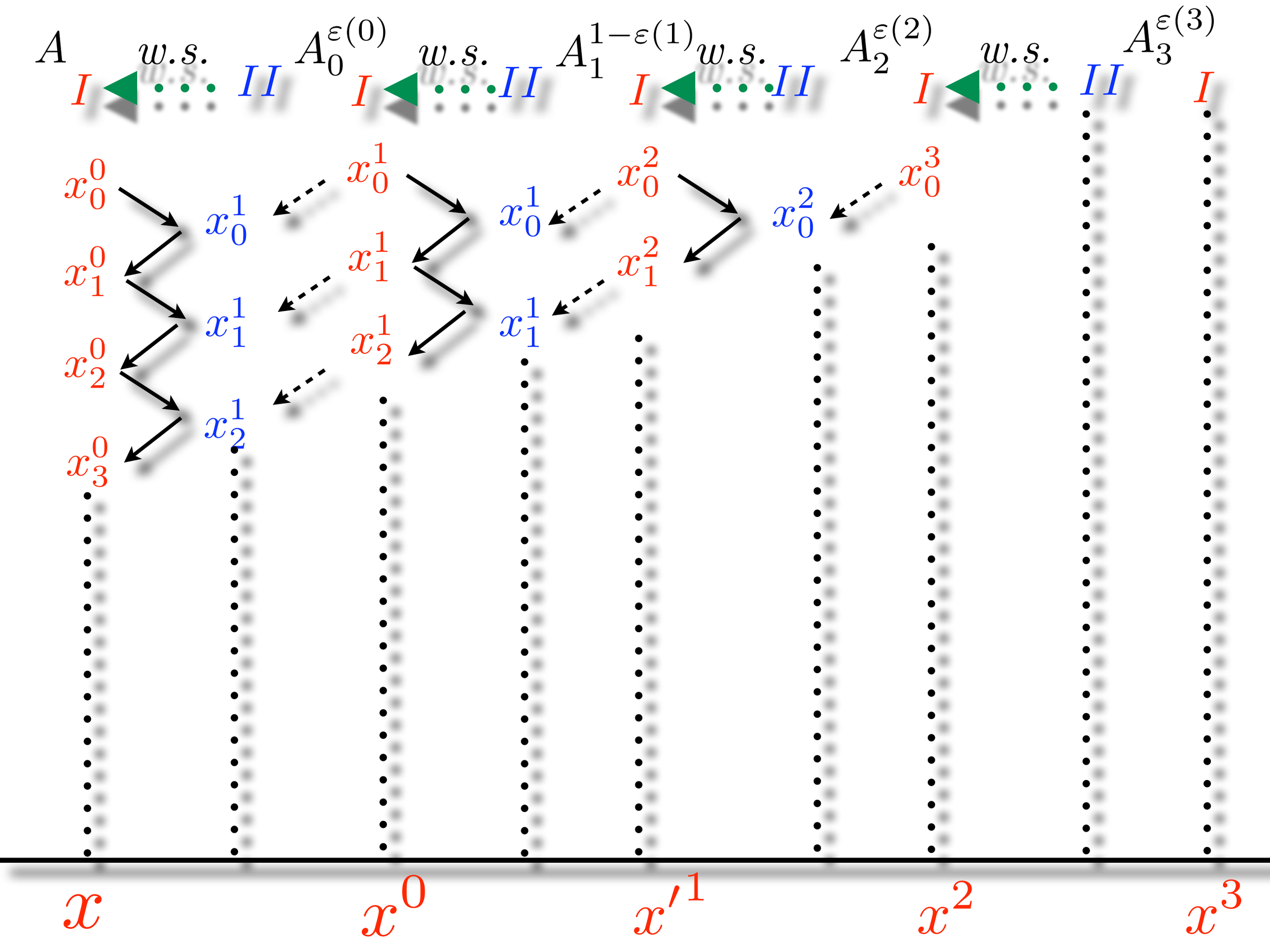

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3

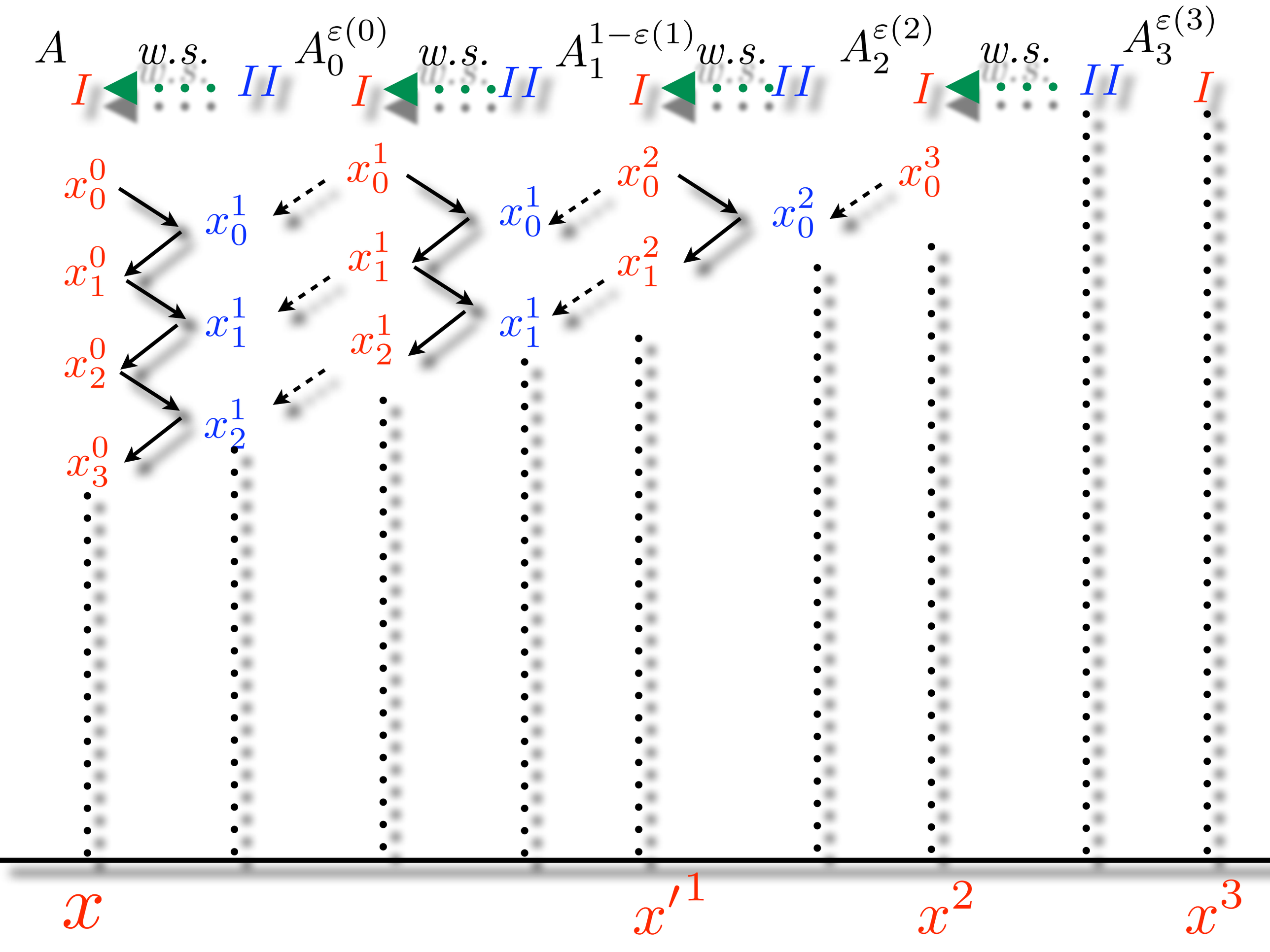

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3

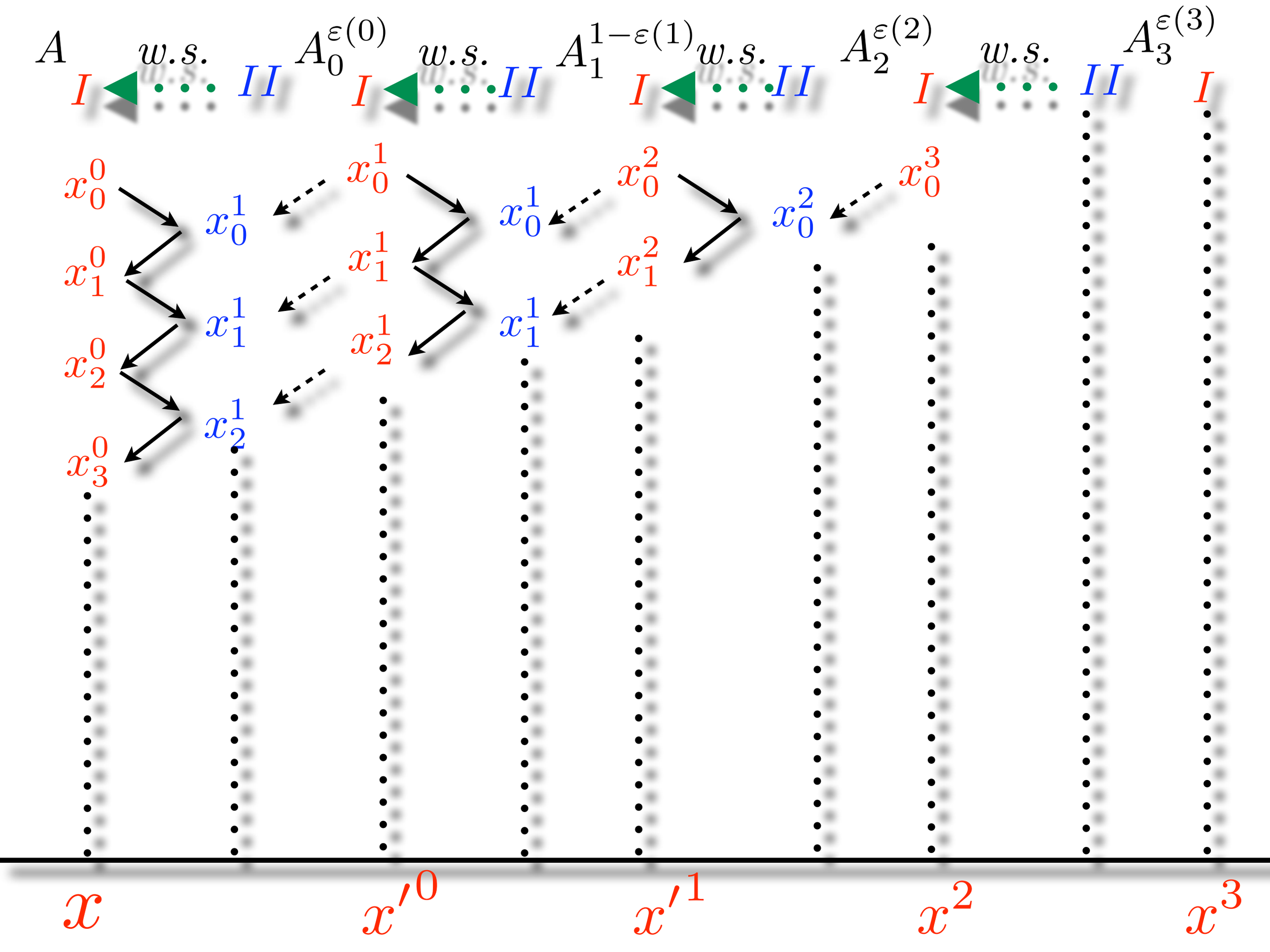

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x'^1
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x'^1
x^2
x^3

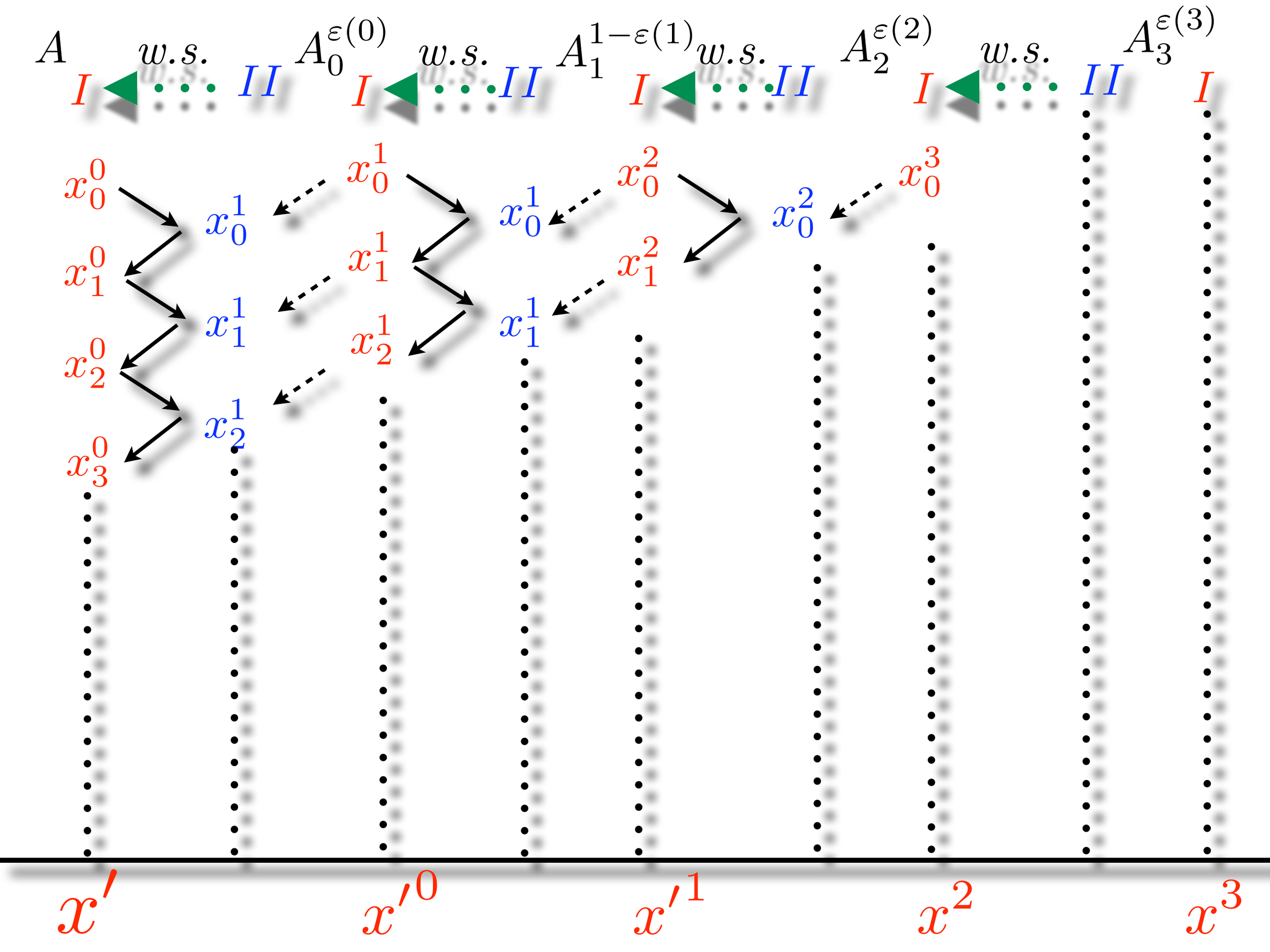

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x'^0
x'^1
x^2
x^3

A $w.s.$ $A_0^{\varepsilon(0)}$ $w.s.$ $A_1^{1-\varepsilon(1)}$ $w.s.$ $A_2^{\varepsilon(2)}$ $w.s.$ $A_3^{\varepsilon(3)}$

I II I II I II I II I

x_0^0 x_1^0 x_2^0 x_3^0 x_0^1 x_1^1 x_2^1 x_0^1 x_1^1 x_0^2 x_1^2 x_0^2 x_0^3

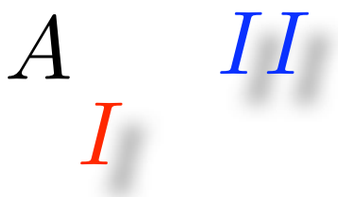
x'^0 x'^1 x^2 x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x'
x'^0
x'^1
x^2
x^3

A

A

I


A
II
I

A II $A_0^{\varepsilon(0)}$

I

A II $A_0^{\varepsilon(0)}$

I

x_0^0

A II $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$

I

x_0^0

A II $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$

I

x_0^0

x_1^0

A $I\!I$ $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$ $A_2^{\varepsilon(2)}$

I

x_0^0

x_1^0

A	II	$A_0^{\varepsilon(0)}$	$A_1^{\varepsilon(1)}$	$A_2^{\varepsilon(2)}$
I				
x_0^0				
x_1^0				
x_2^0				

A	II	$A_0^{\varepsilon(0)}$	$A_1^{\varepsilon(1)}$	$A_2^{\varepsilon(2)}$	$A_3^{\varepsilon(3)}$
I					
x_0^0					
x_1^0					
x_2^0					

A	II	$A_0^{\varepsilon(0)}$	$A_1^{\varepsilon(1)}$	$A_2^{\varepsilon(2)}$	$A_3^{\varepsilon(3)}$
I					
x_0^0					
x_1^0					
x_2^0					
x_3^0					

A	II	$A_0^{\varepsilon(0)}$	$A_1^{\varepsilon(1)}$	$A_2^{\varepsilon(2)}$	$A_3^{\varepsilon(3)}$
I					
x_0^0					
x_1^0					
x_2^0					
x_3^0					

A	II	$A_0^{\varepsilon(0)}$	$A_1^{\varepsilon(1)}$	$A_2^{\varepsilon(2)}$	$A_3^{\varepsilon(3)}$
I					
x_0^0					
x_1^0					
x_2^0					
x_3^0					

A II $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$ $A_2^{\varepsilon(2)}$ $A_3^{\varepsilon(3)}$

I

x_0^0

x_1^0

x_2^0

x_3^0

x

A II $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$ $A_2^{\varepsilon(2)}$ $A_3^{\varepsilon(3)}$

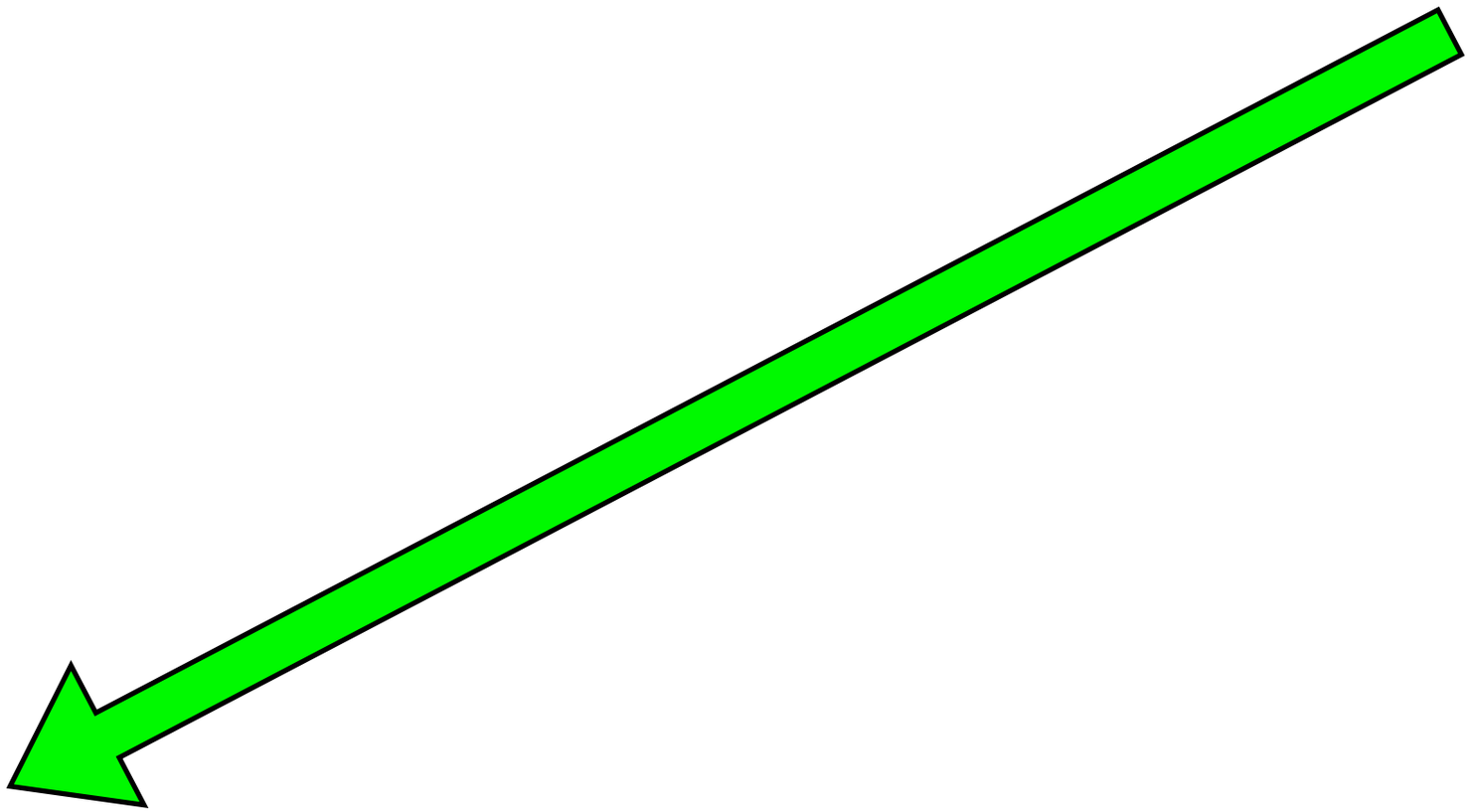
I

x_0^0

x_1^0

x_2^0

x_3^0



x

A II $A_0^{\varepsilon(0)}$ $A_1^{\varepsilon(1)}$ $A_2^{\varepsilon(2)}$ $A_3^{\varepsilon(3)}$

I

x_0^0

x_1^0

x_2^0

x_3^0

Lipschitz function

x

$I_{\parallel}$

$II_{\parallel}$

I $\varepsilon(0)$

II

I $\varepsilon(0)$

II

x_0^0

I $\varepsilon(0)$ $\varepsilon(1)$

II

x_0^0

I $\varepsilon(0)$ $\varepsilon(1)$

II

x_0^0

x_1^0

I	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$
II			
x_0^0			
x_1^0			

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$
$II_{\prime\prime}$			
x_0^0			
x_1^0			
x_2^0			

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				

$I_{\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				
x_3^0				

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				
x_3^0				

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				
x_3^0				

I_f $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

II_f

x_0^0

x_1^0

x_2^0

x_3^0

x

I $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

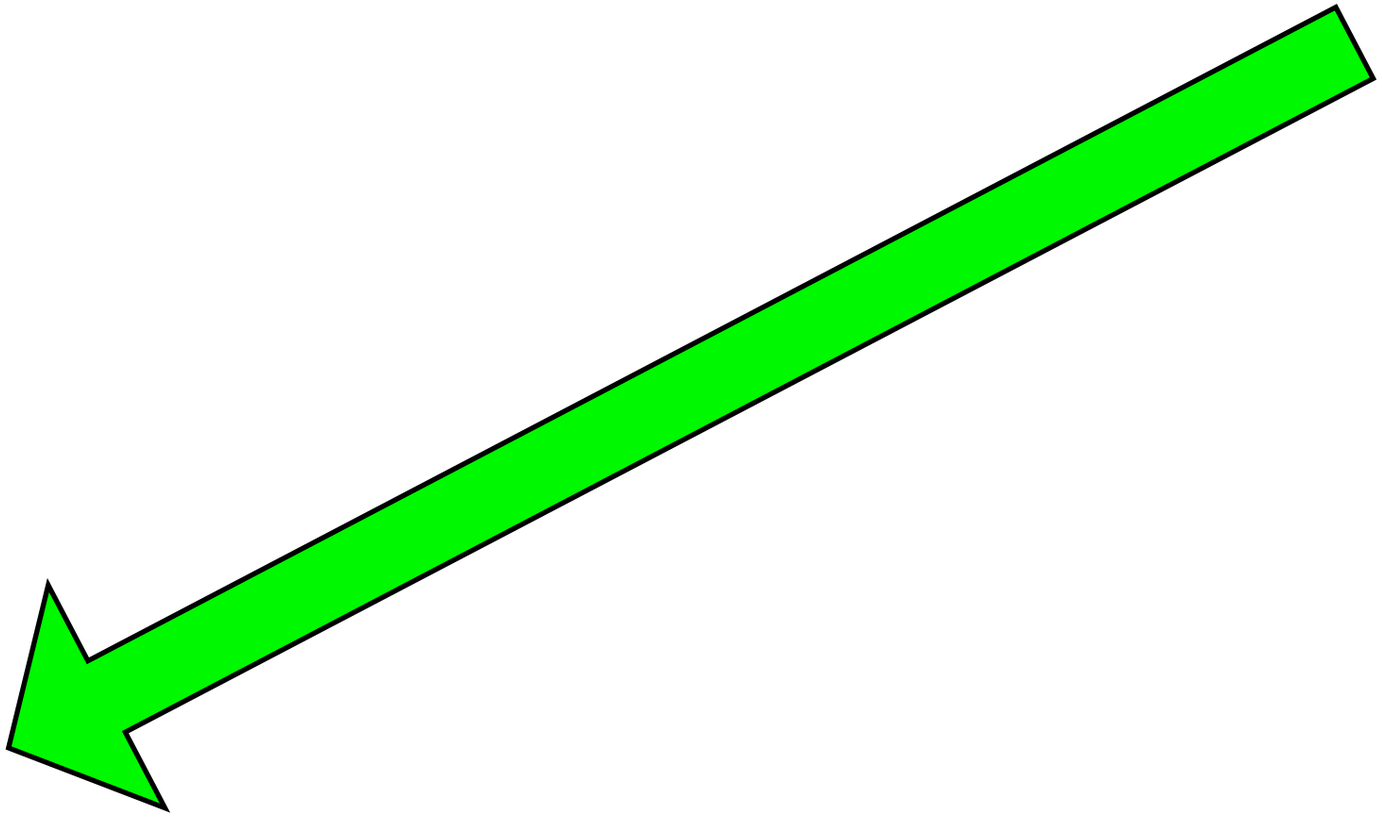
II

x_0^0

x_1^0

x_2^0

x_3^0



x

I $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

II

x_0^0

x_1^0

x_2^0

x_3^0

Lipschitz function

x

$I_{\parallel}$

$II_{\parallel}$

I $\varepsilon(0)$

II

I $\varepsilon(0)$

II

x_0^0

I $\varepsilon(0)$ $\varepsilon(1)$

II

x_0^0

I $\varepsilon(0)$ $\varepsilon(1)$

II

x_0^0

x_1^0

I	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$
II			
x_0^0			
x_1^0			

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$
$II_{\prime\prime}$			
x_0^0			
x_1^0			
x_2^0			

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				

$I_{\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				
x_3^0				

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				
x_3^0				

$I_{\prime\prime}$	$\varepsilon(0)$	$\varepsilon(1)$	$\varepsilon(2)$	$\varepsilon(3)$
$II_{\prime\prime}$				
x_0^0				
x_1^0				
x_2^0				
x_3^0				

I_f $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

II_f

x_0^0

x_1^0

x_2^0

x_3^0

x

I $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

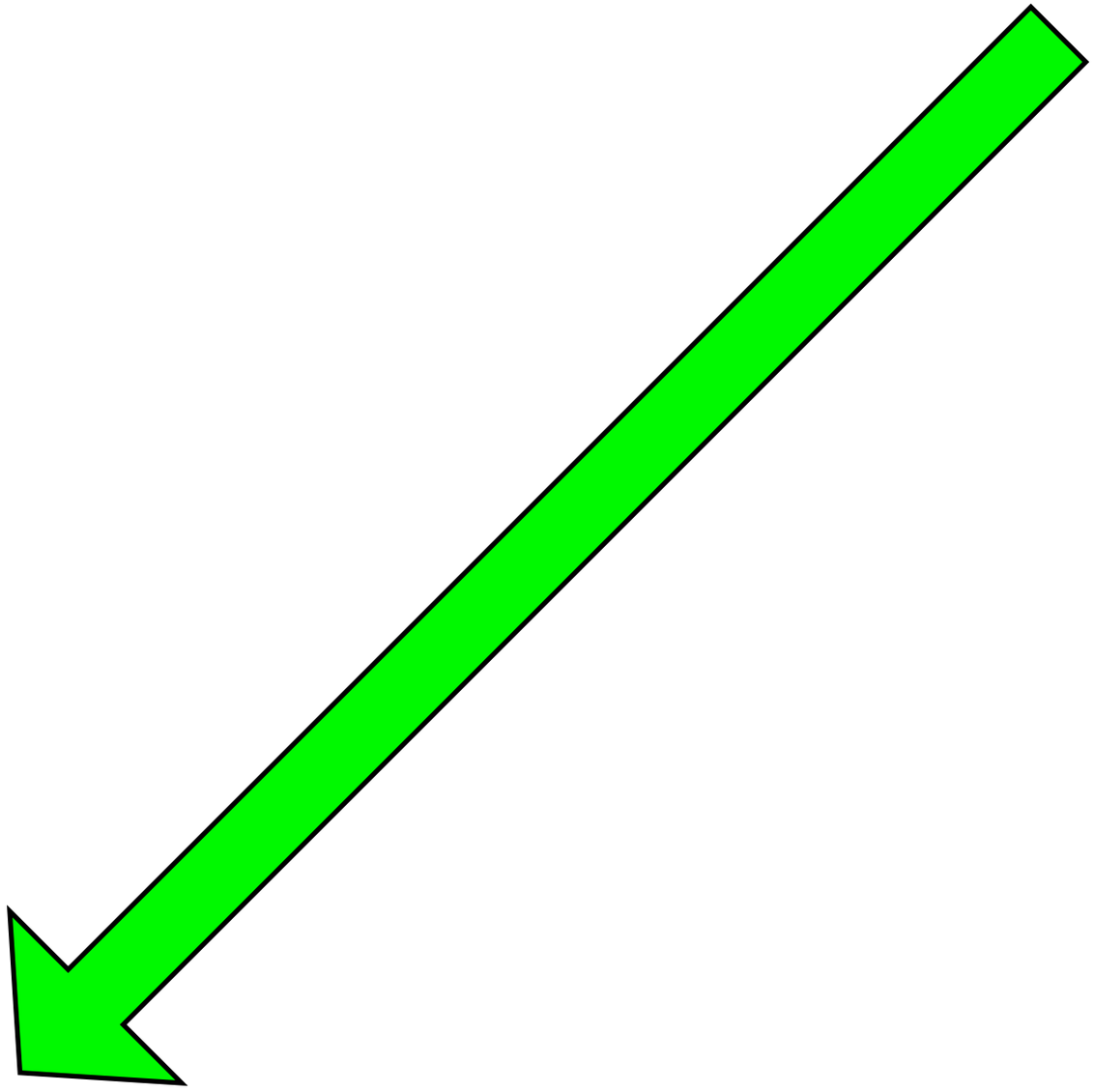
II

x_0^0

x_1^0

x_2^0

x_3^0



x

I $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

II

x_0^0

x_1^0

x_2^0

x_3^0

Lipschitz

x

I $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

II

x_0^0

x_1^0

x_2^0

x_3^0

Lipschitz

$$B = \{\varepsilon : \quad x \in A\}$$

x

I $\varepsilon(0)$ $\varepsilon(1)$ $\varepsilon(2)$ $\varepsilon(3)$

II

x_0^0

x_1^0

x_2^0

x_3^0

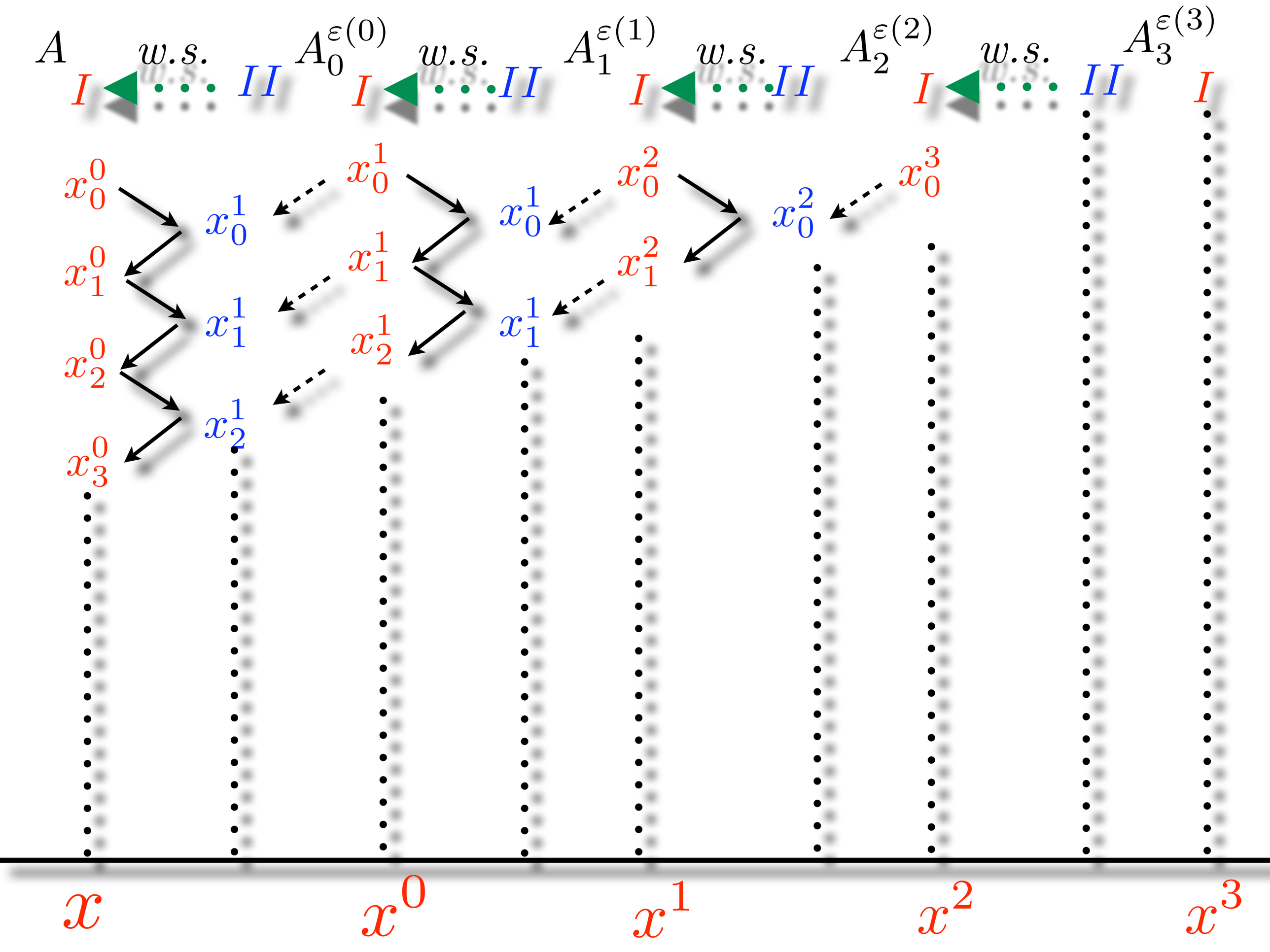
$\vdots$

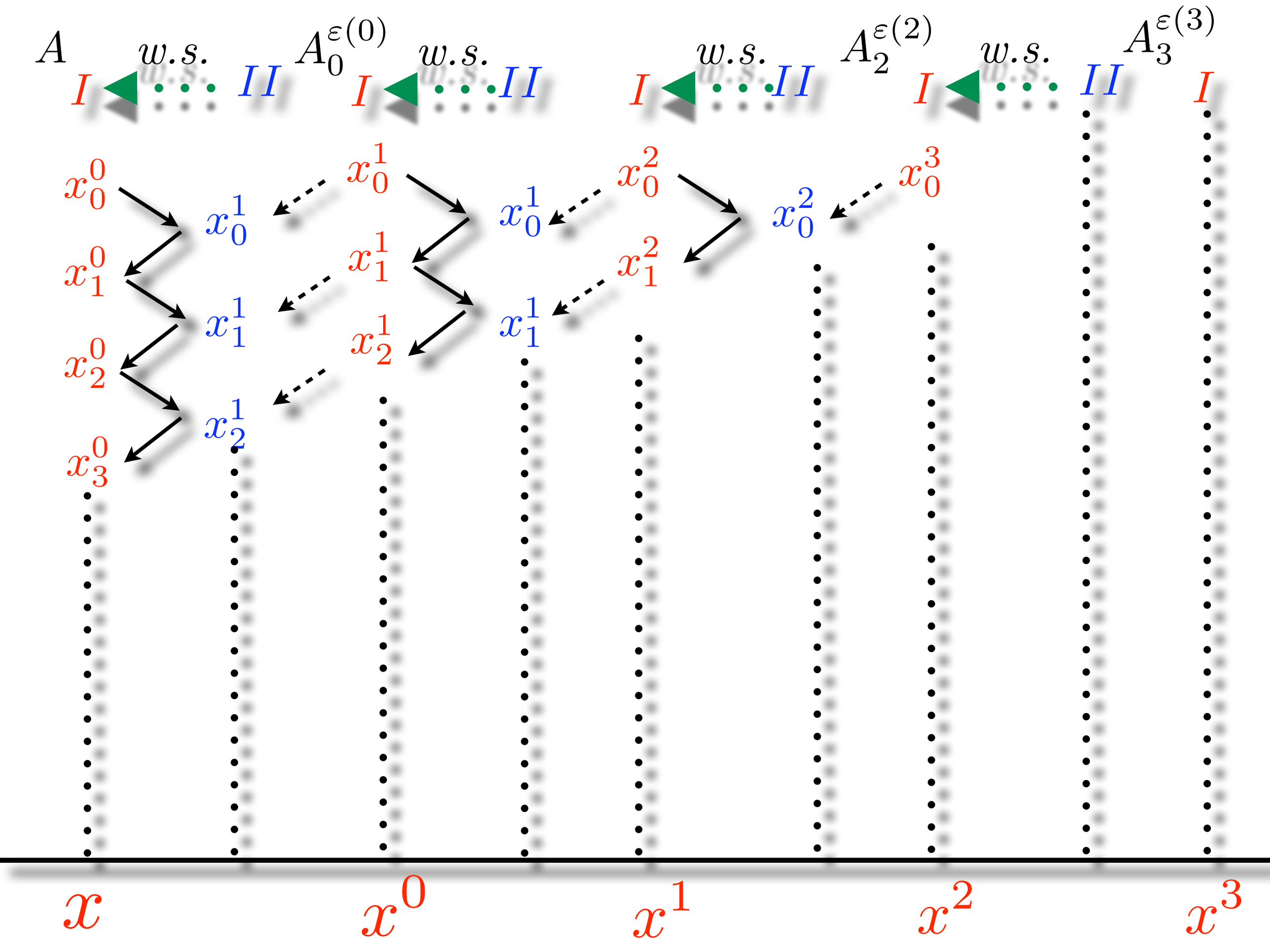
Lipschitz

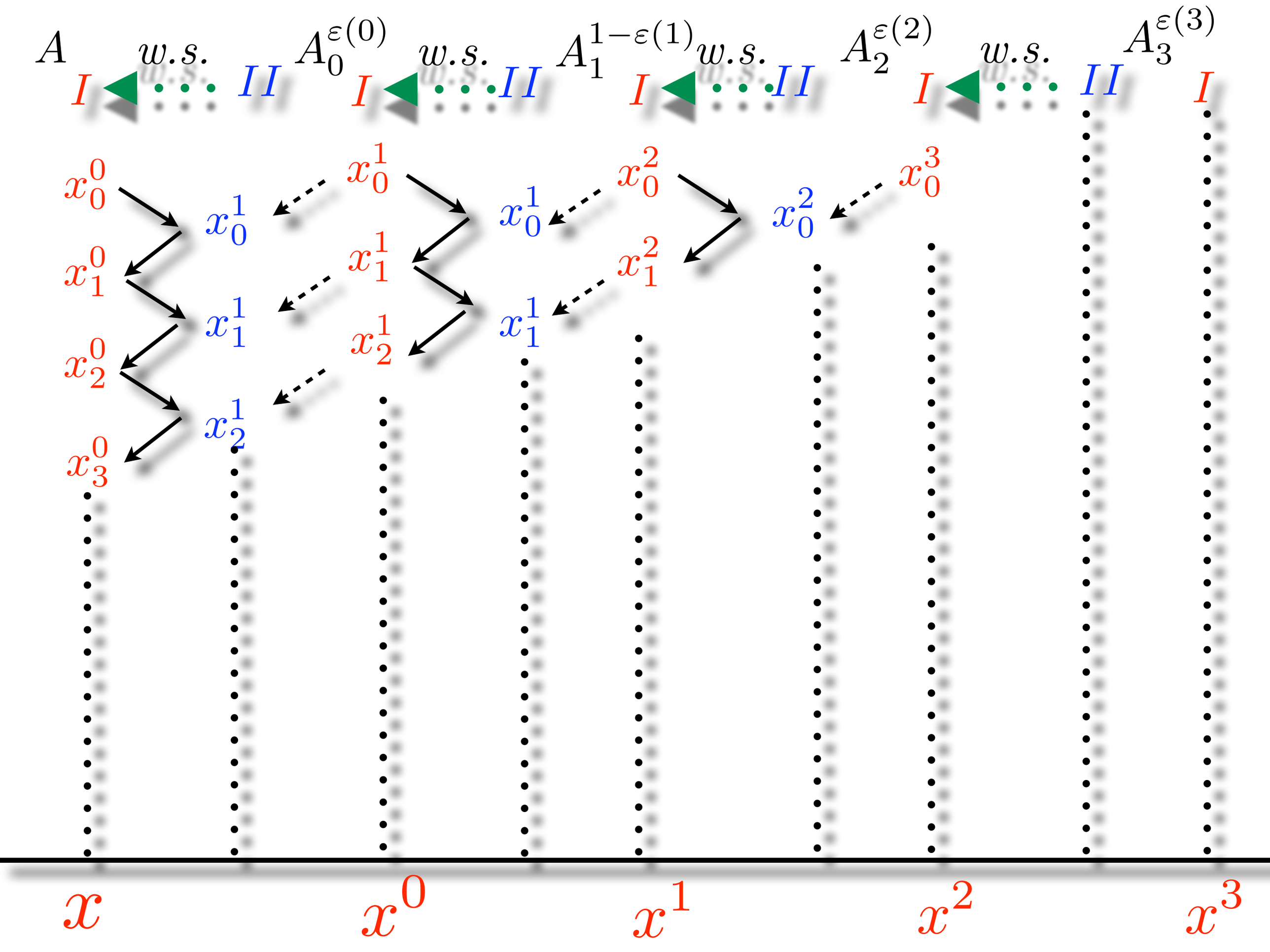
$$B = \{\varepsilon : \quad x \in A\}$$

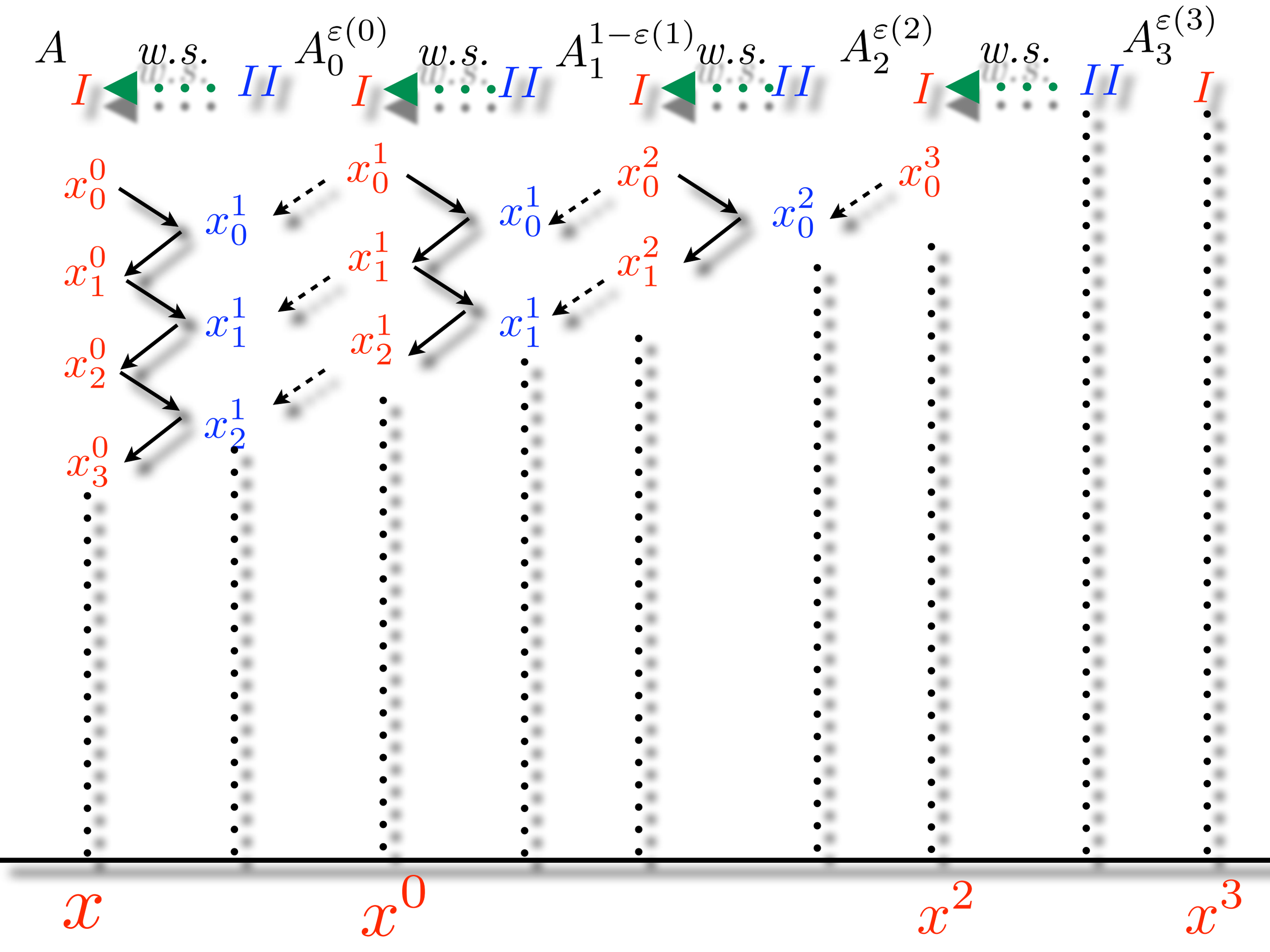
(B is a flip set !)

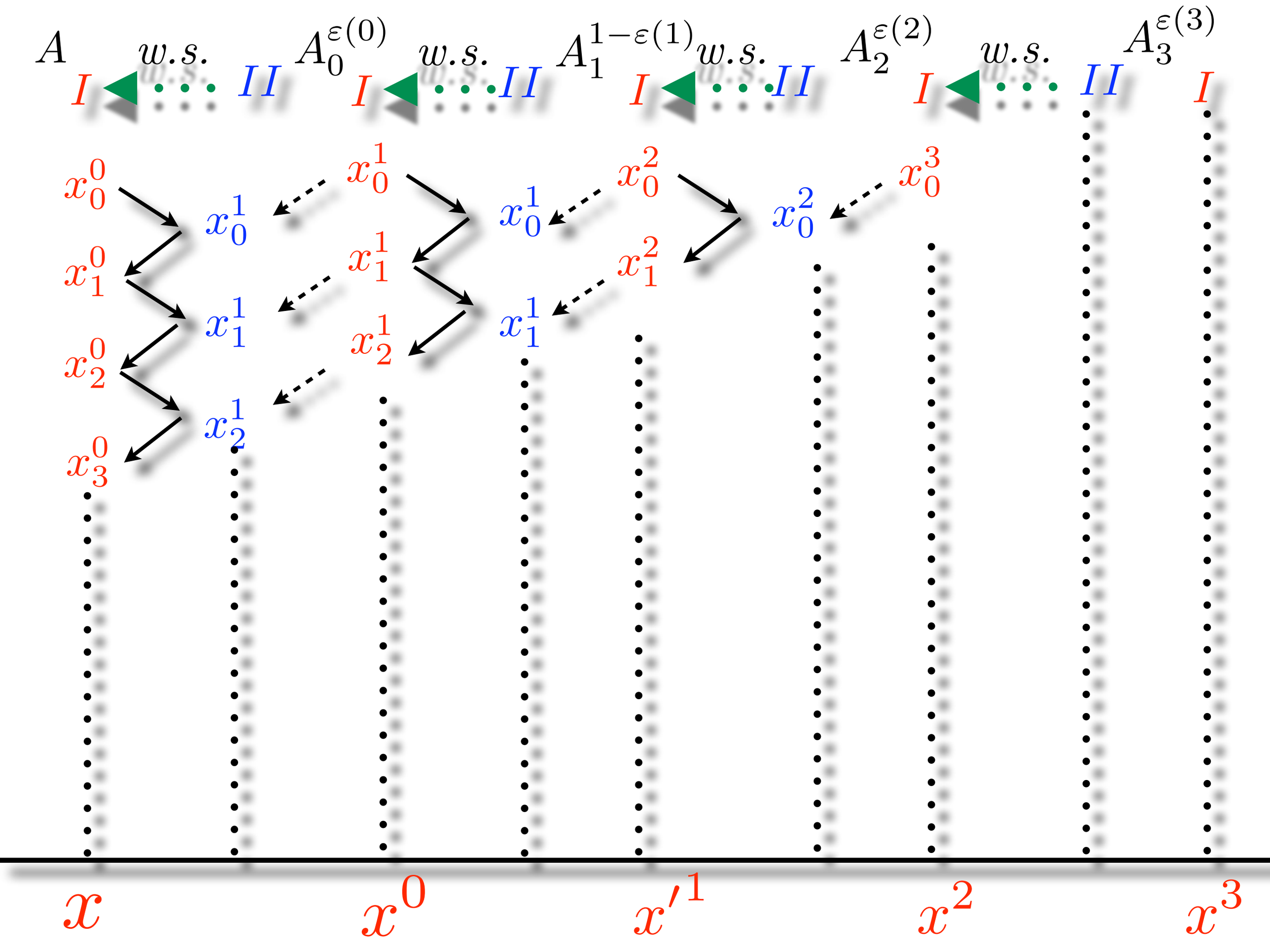
x

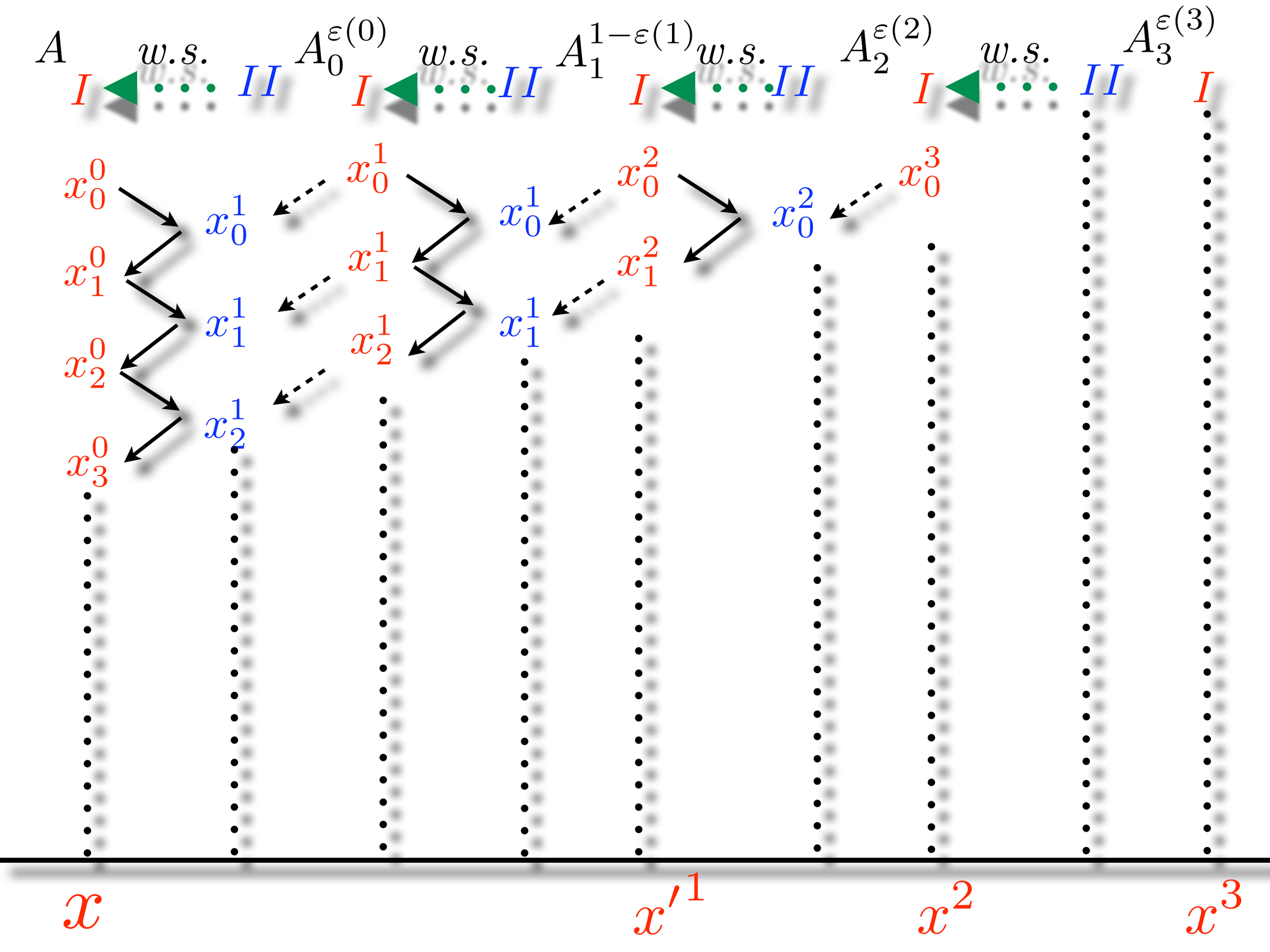

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3

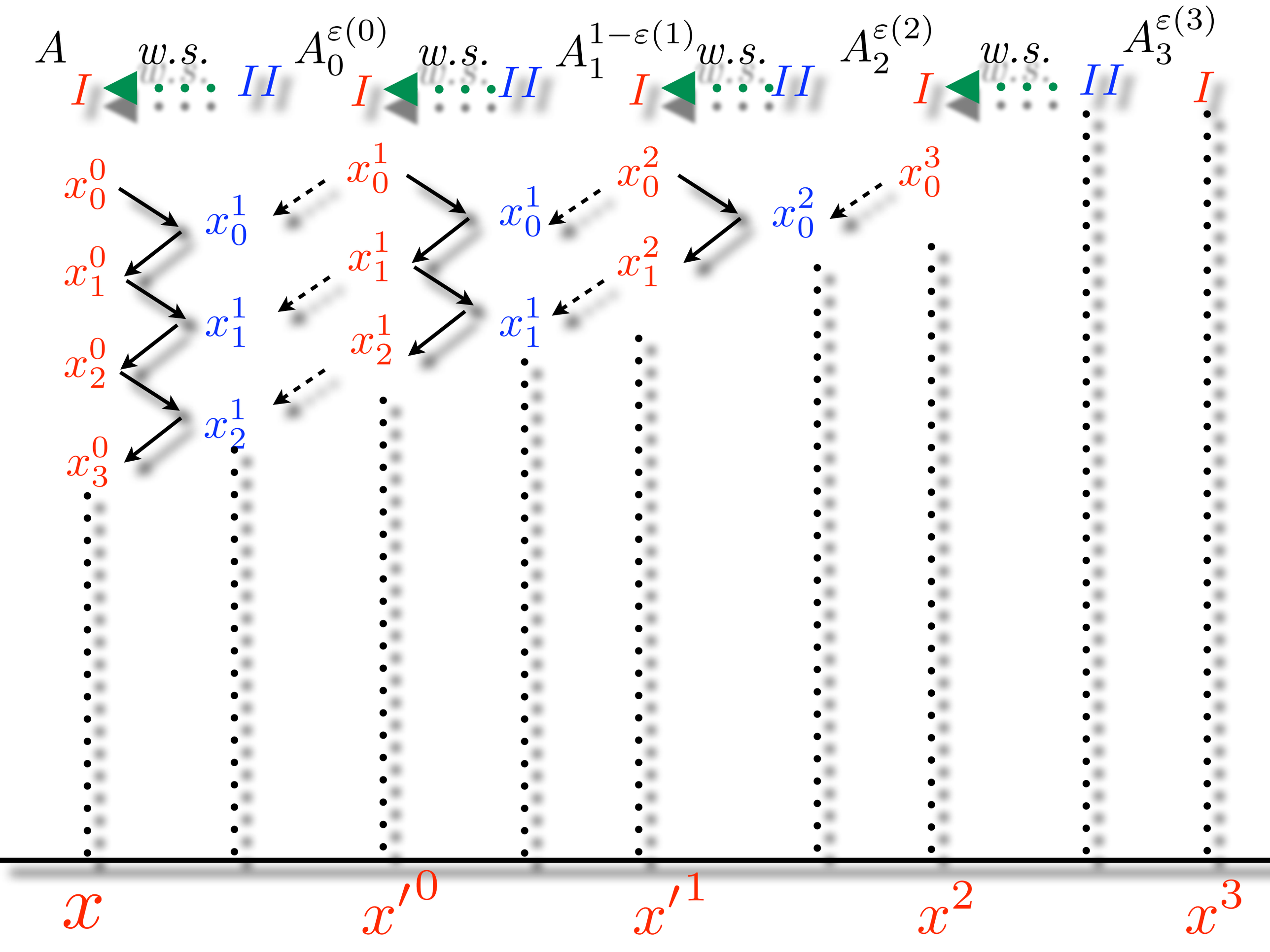

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^1
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x^0
x'^1
x^2
x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x'^1
x^2
x^3

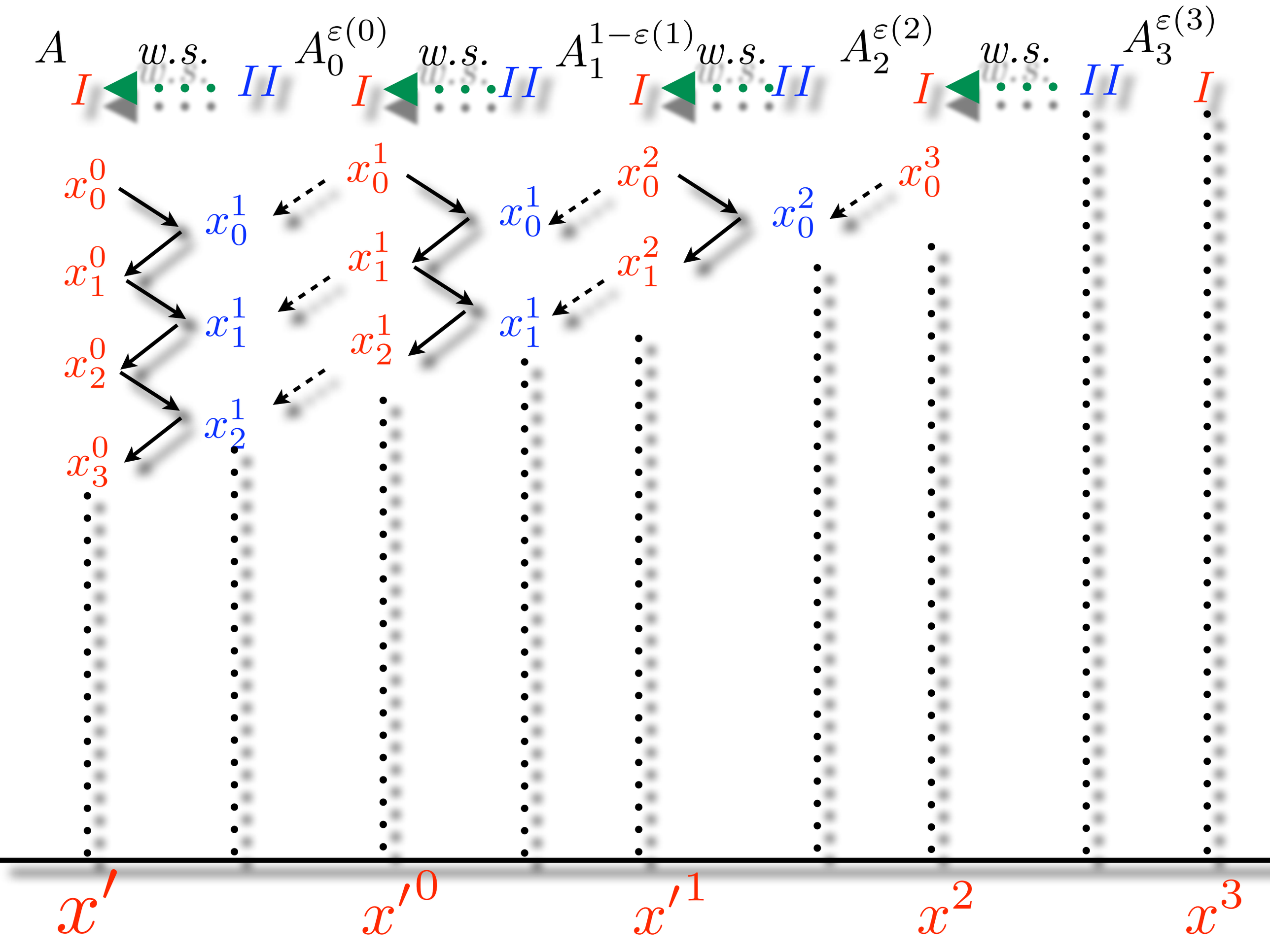

A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x
x'^0
x'^1
x^2
x^3

A $w.s.$ $A_0^{\varepsilon(0)}$ $w.s.$ $A_1^{1-\varepsilon(1)}$ $w.s.$ $A_2^{\varepsilon(2)}$ $w.s.$ $A_3^{\varepsilon(3)}$

I II I II I II I II I

x_0^0 x_1^0 x_2^0 x_3^0 x_0^1 x_1^1 x_2^1 x_0^1 x_1^1 x_0^2 x_1^2 x_0^2 x_0^3

x'^0 x'^1 x^2 x^3


A
$w.s.$
I
II
$A_0^{\varepsilon(0)}$
$w.s.$
I
II
$A_1^{1-\varepsilon(1)}$
$w.s.$
I
II
$A_2^{\varepsilon(2)}$
$w.s.$
I
II
$A_3^{\varepsilon(3)}$
I
x_0^0
x_1^0
x_2^0
x_3^0
x_0^1
x_1^1
x_2^1
x_0^1
x_1^1
x_0^2
x_1^2
x_0^2
x_0^3
x'
x'^0
x'^1
x^2
x^3

$$\mathcal{BM}(B)$$

I wins $\iff x \in B$

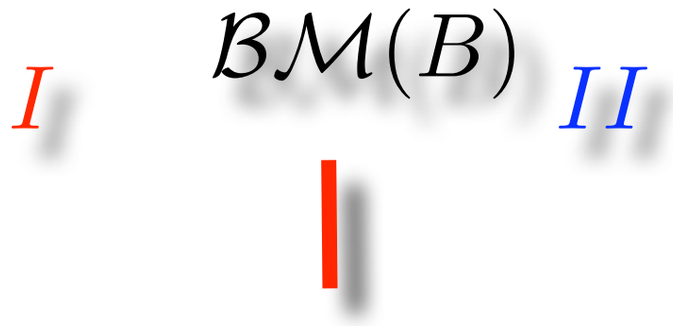
I $\mathcal{BM}(B)$

I wins $\iff x \in B$

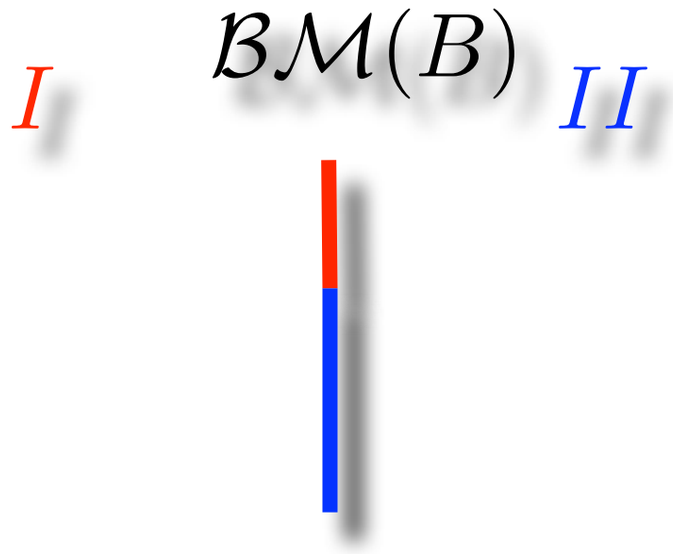
$$\mathcal{BM}(B)$$

I II

I wins $\iff x \in B$



I wins $\iff x \in B$



I wins $\iff x \in B$

$\mathcal{BM}(B)$

I II



I wins $\iff x \in B$

$\mathcal{BM}(B)$

I II



I wins $\iff x \in B$

$\mathcal{BM}(B)$

I II



I wins $\iff x \in B$

$\mathcal{BM}(B)$

I II



I wins $\iff x \in B$

$\mathcal{BM}(B)$

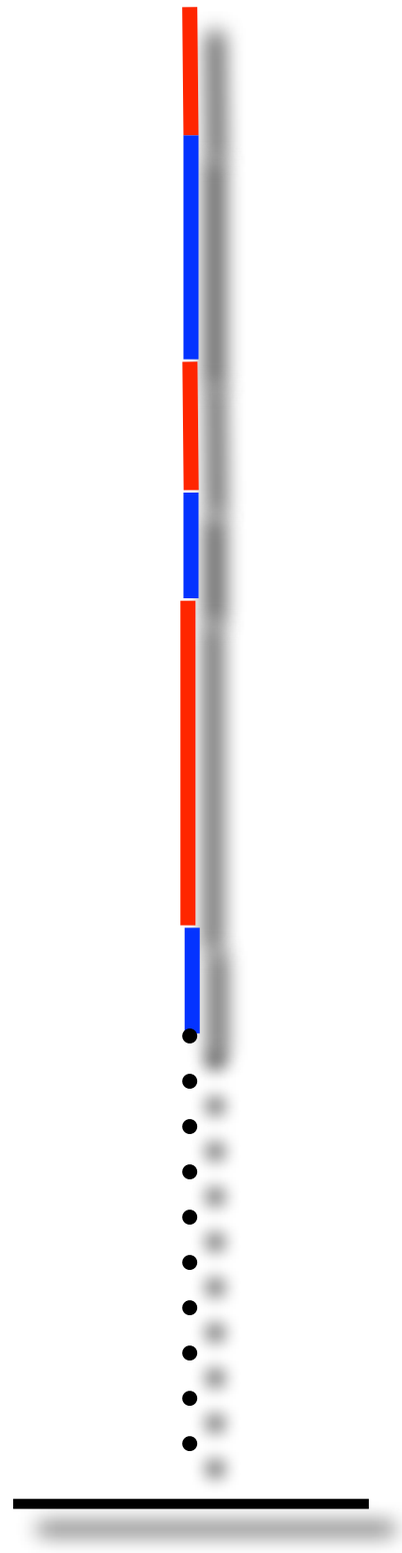
I II



I wins $\iff x \in B$

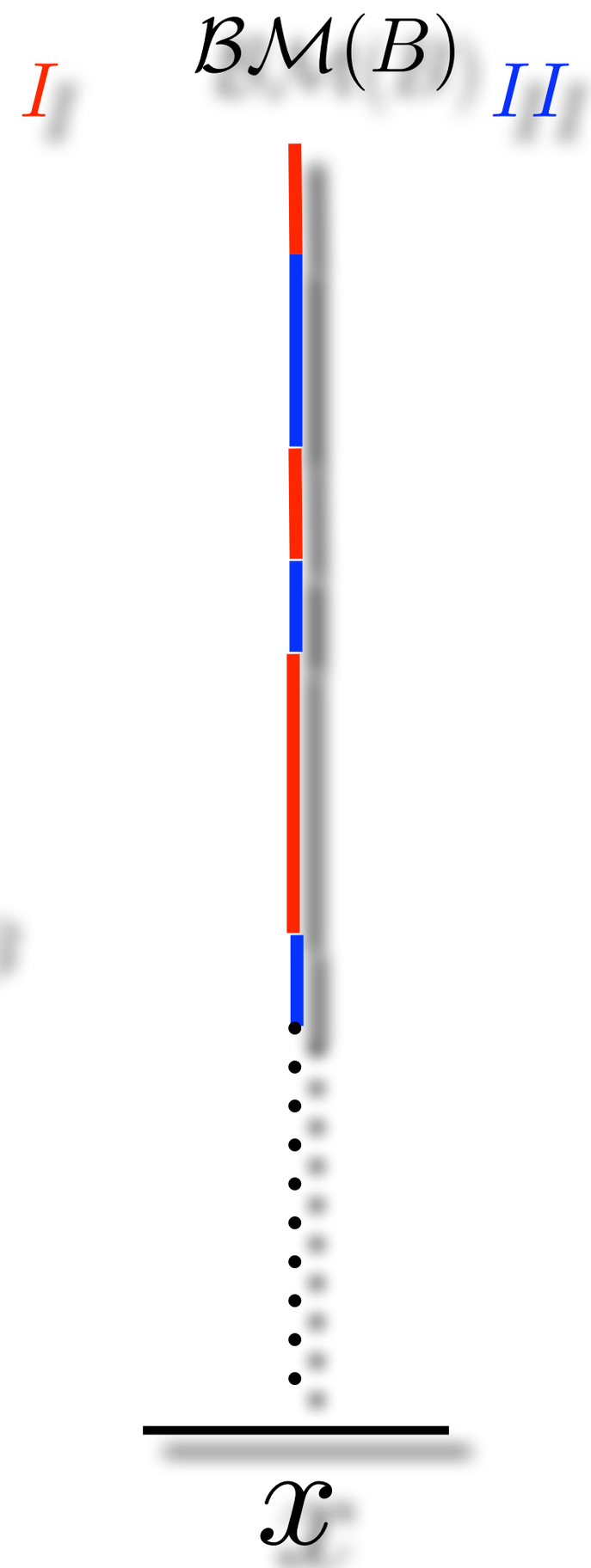
$\mathcal{BM}(B)$

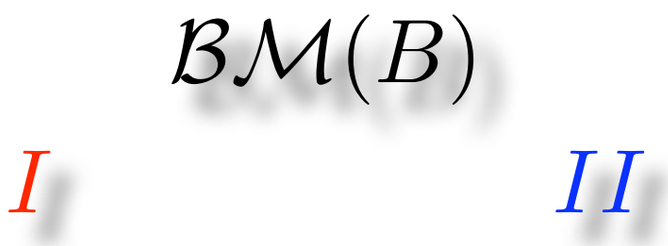
I II

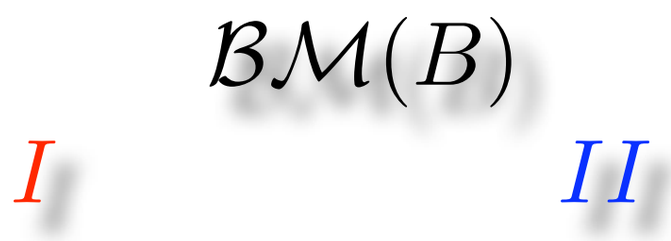


I wins $\iff x \in B$

I wins $\iff x \in B$

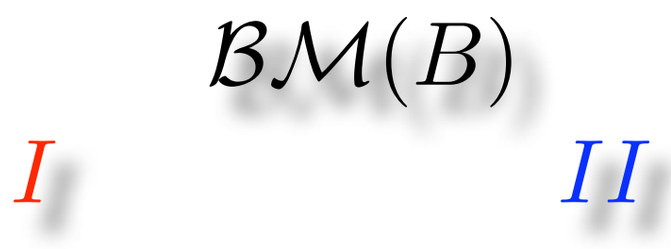






$$I \text{ wins} \iff x \in A$$



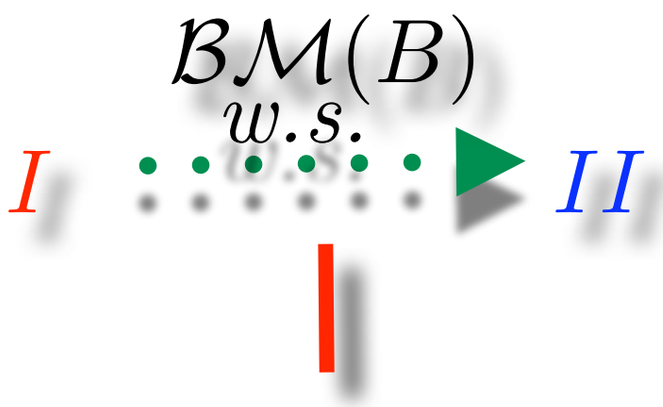


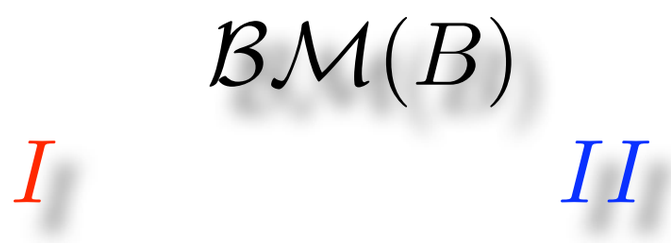
$$I \text{ wins} \iff x \in A$$



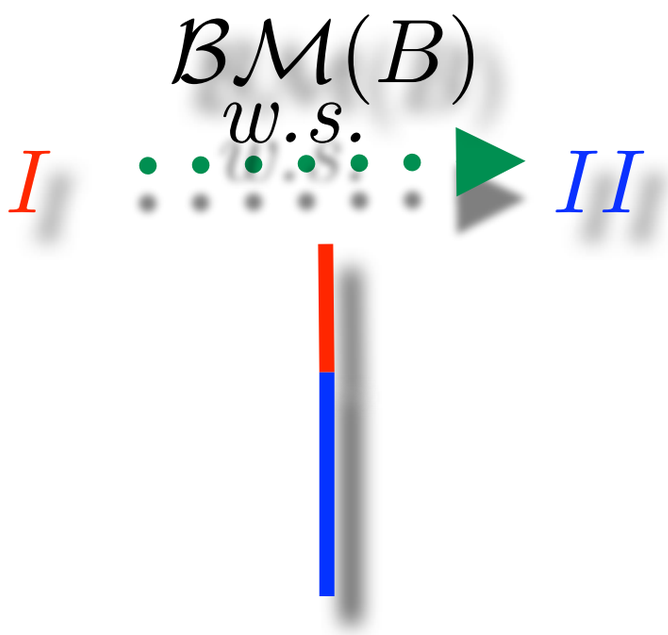


$$I \text{ wins} \iff x \in A$$





$$I \text{ wins} \iff x \in A$$





$$I \text{ wins} \iff x \in A$$



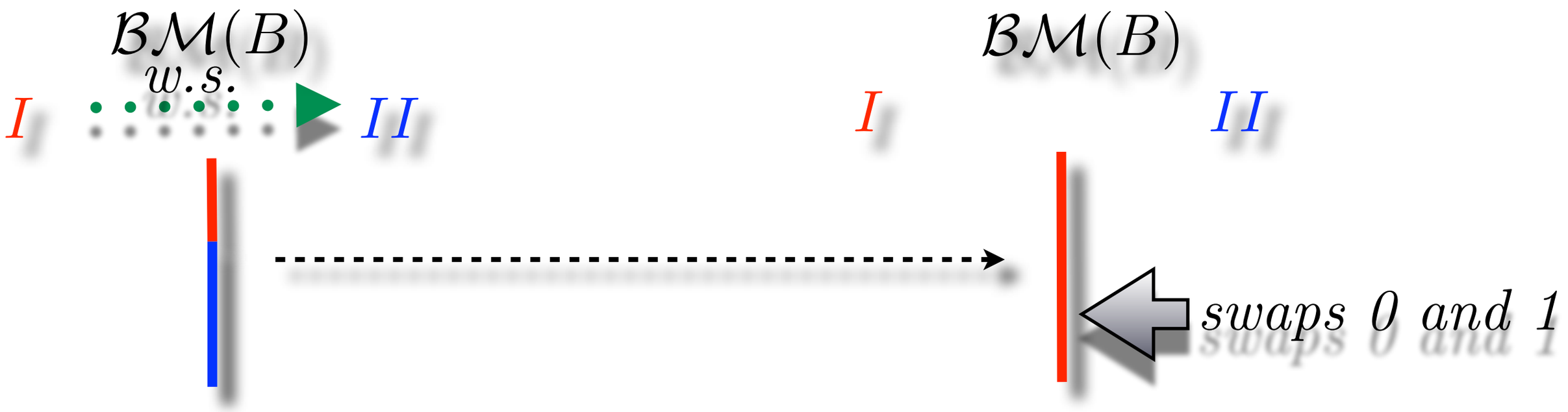
$$I \text{ wins} \iff x \in A$$



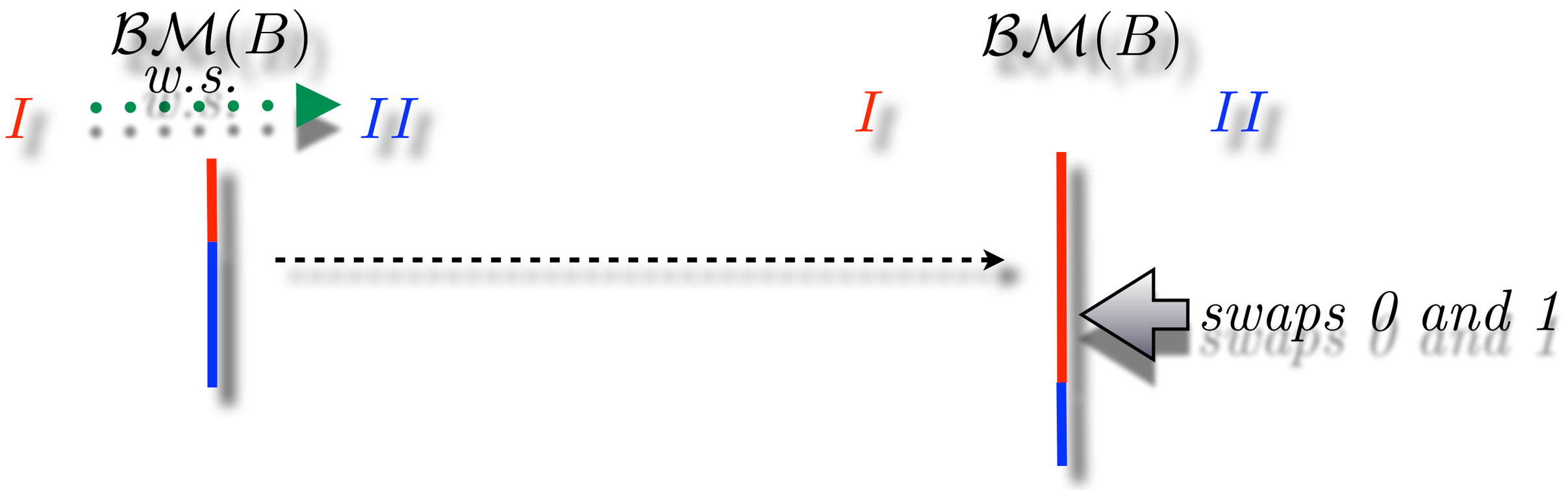
$$I \text{ wins} \iff x \in A$$



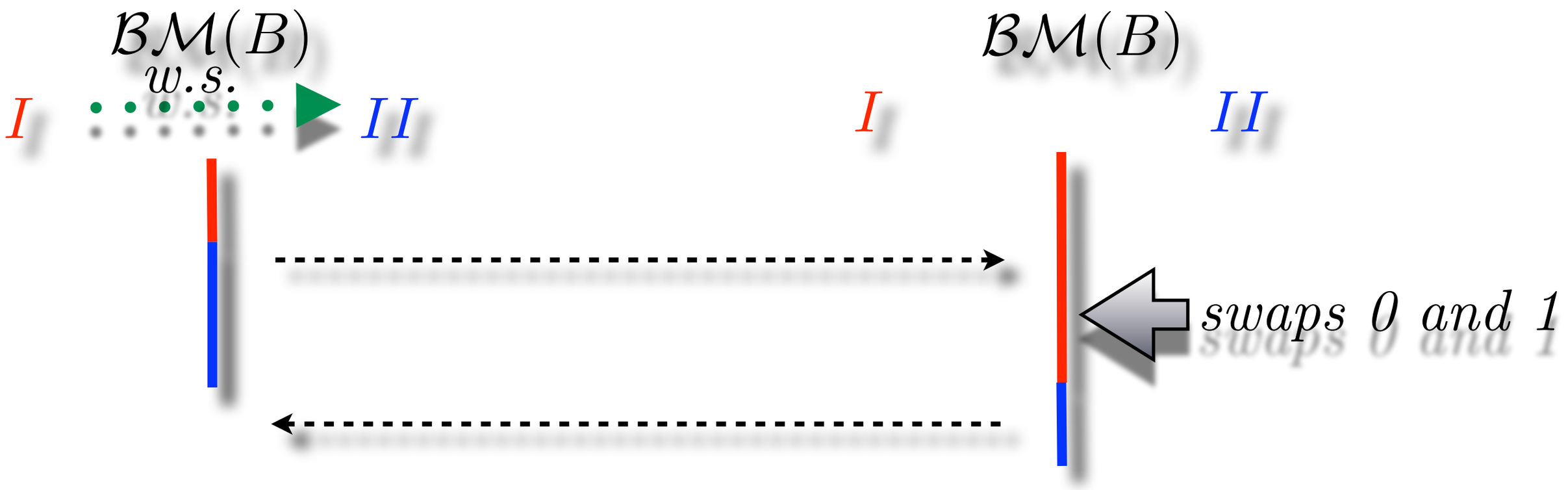
$$I \text{ wins} \iff x \in A$$



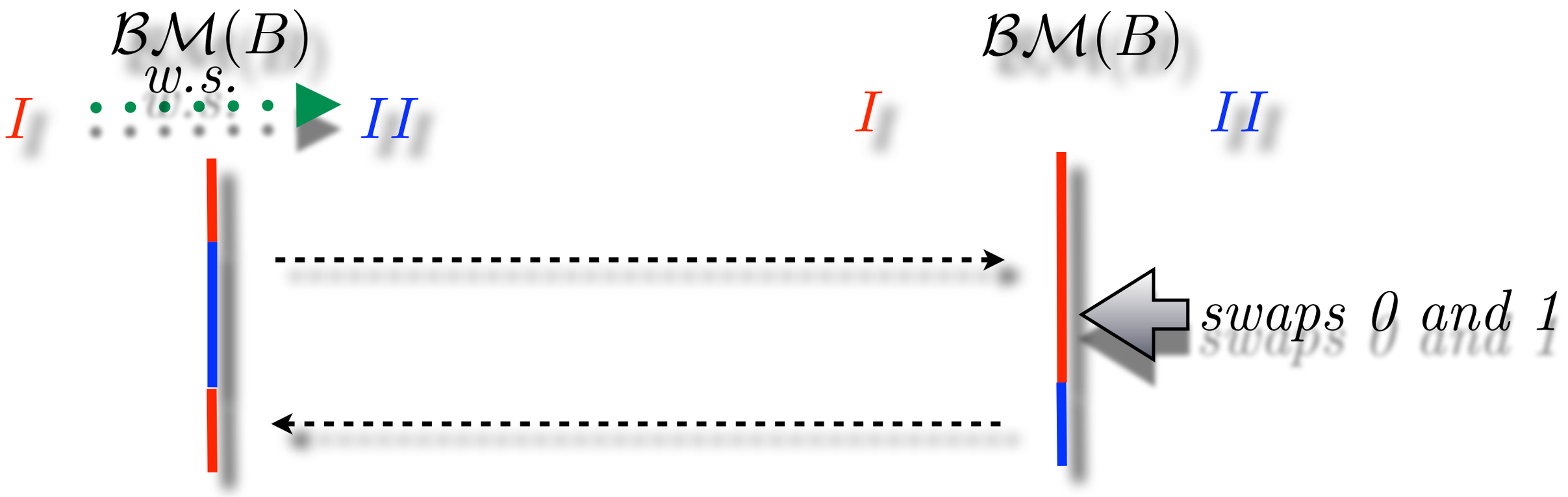
$$I \text{ wins} \iff x \in A$$



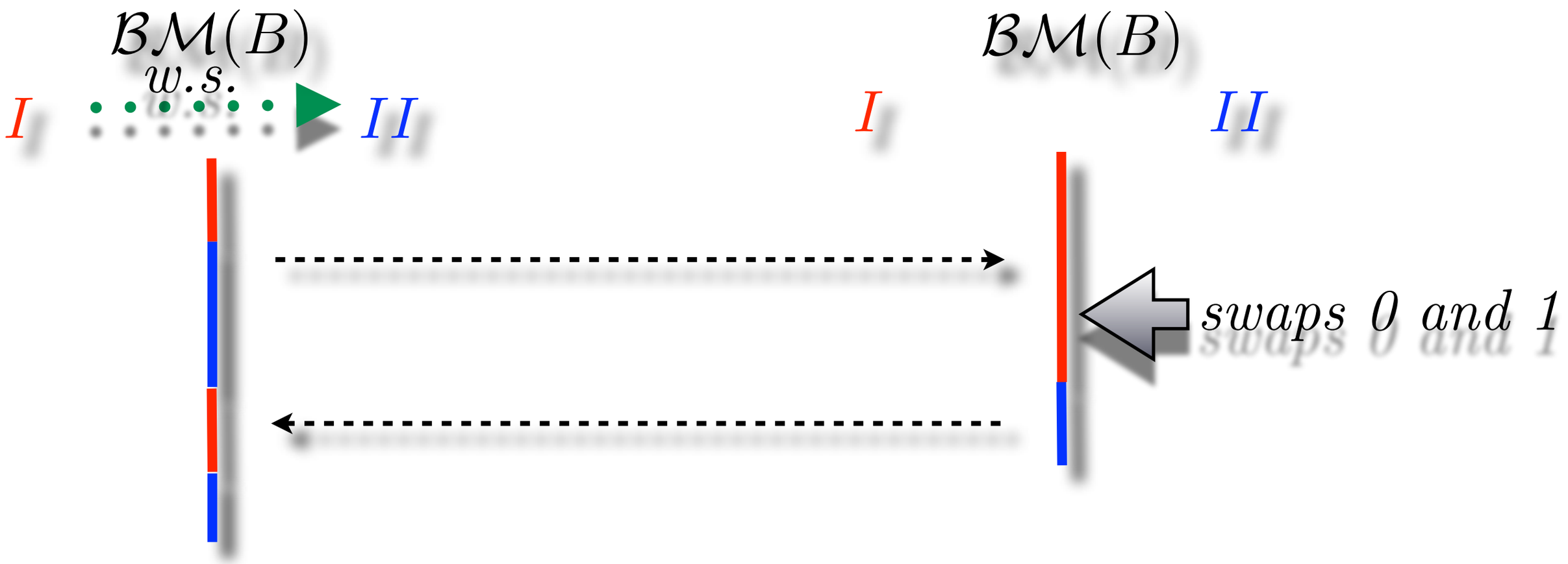
$$I \text{ wins} \iff x \in A$$



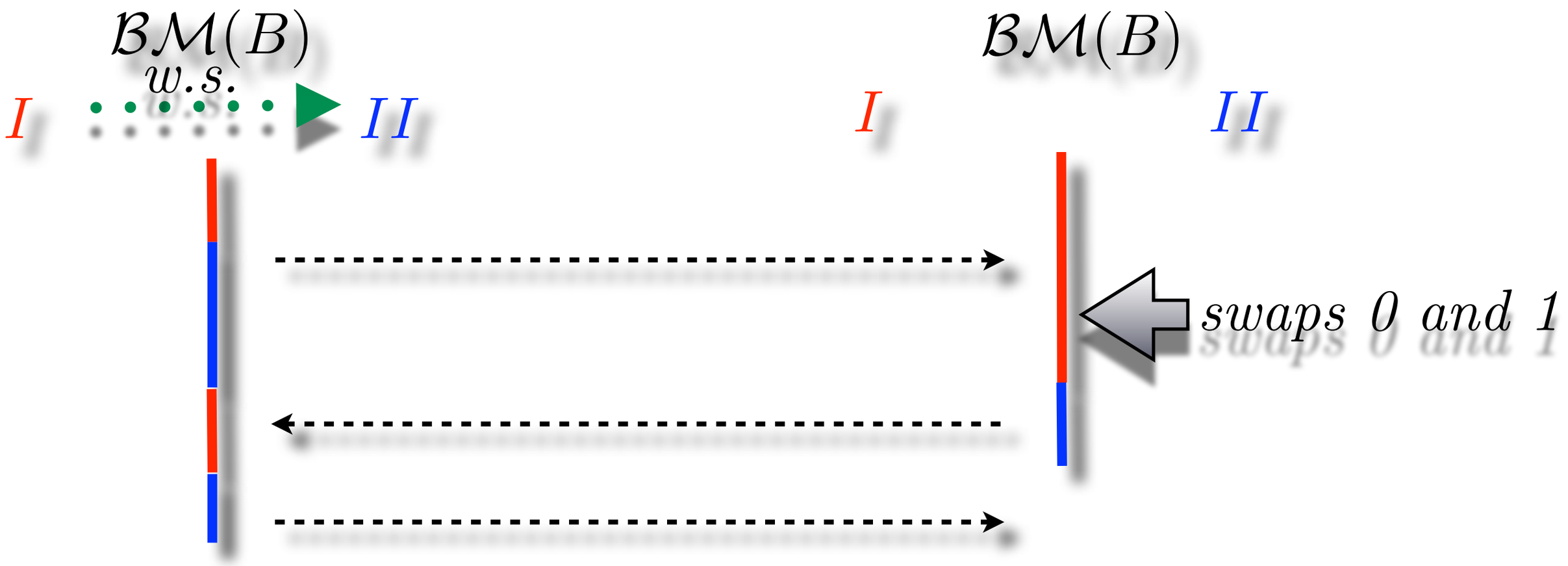
$$I \text{ wins} \iff x \in A$$



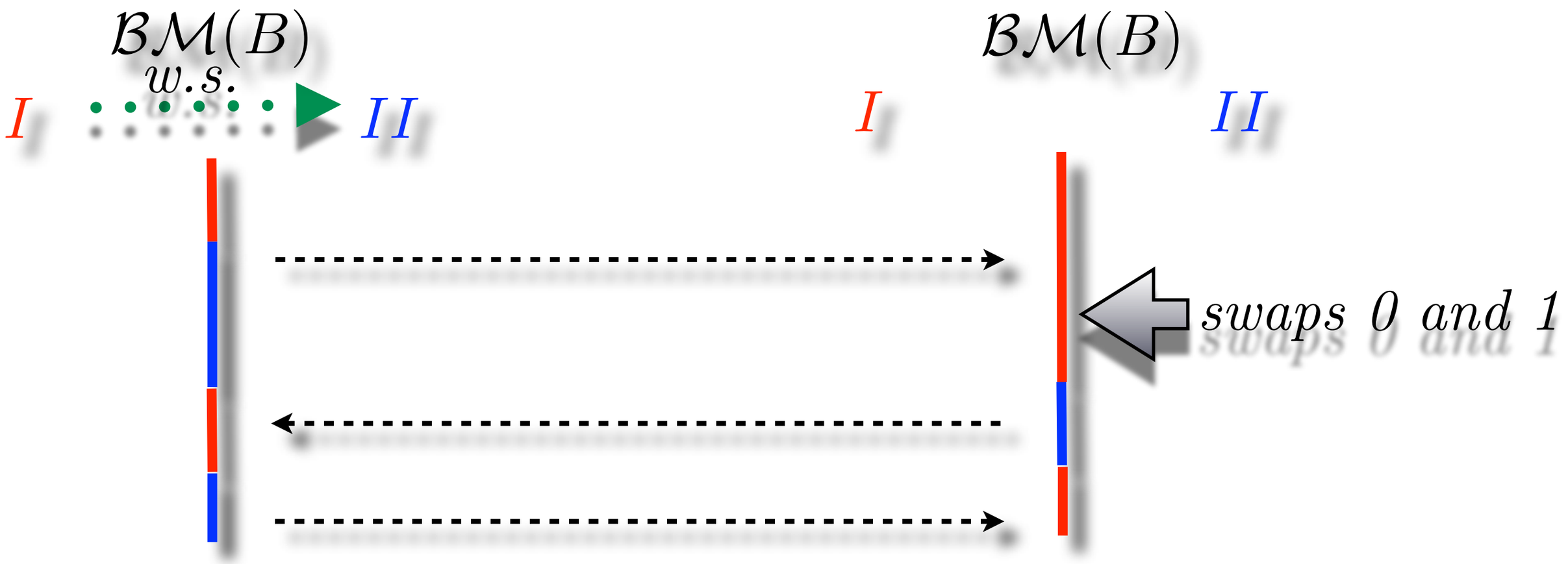
$$I \text{ wins} \iff x \in A$$



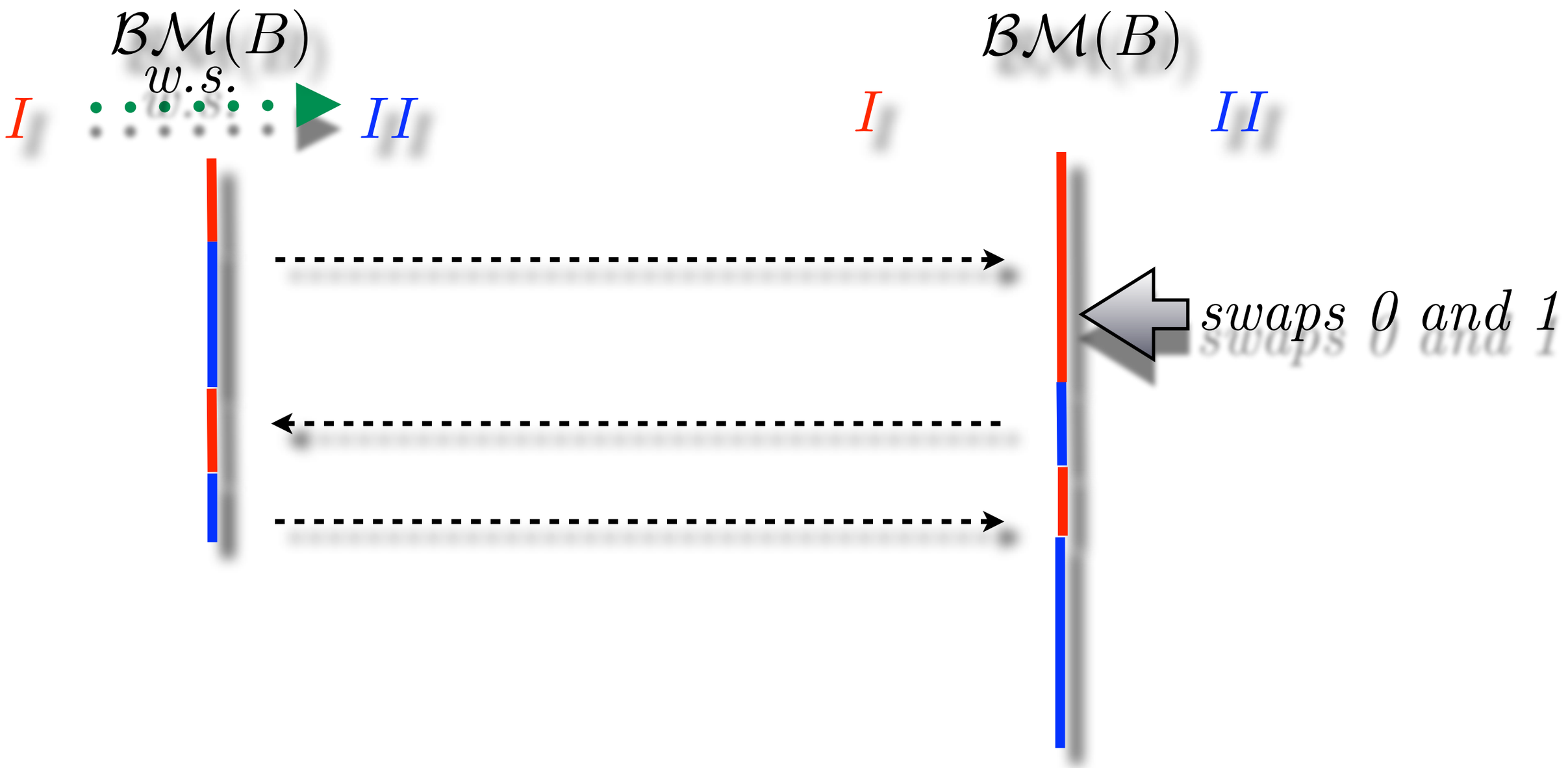
$$I \text{ wins} \iff x \in A$$



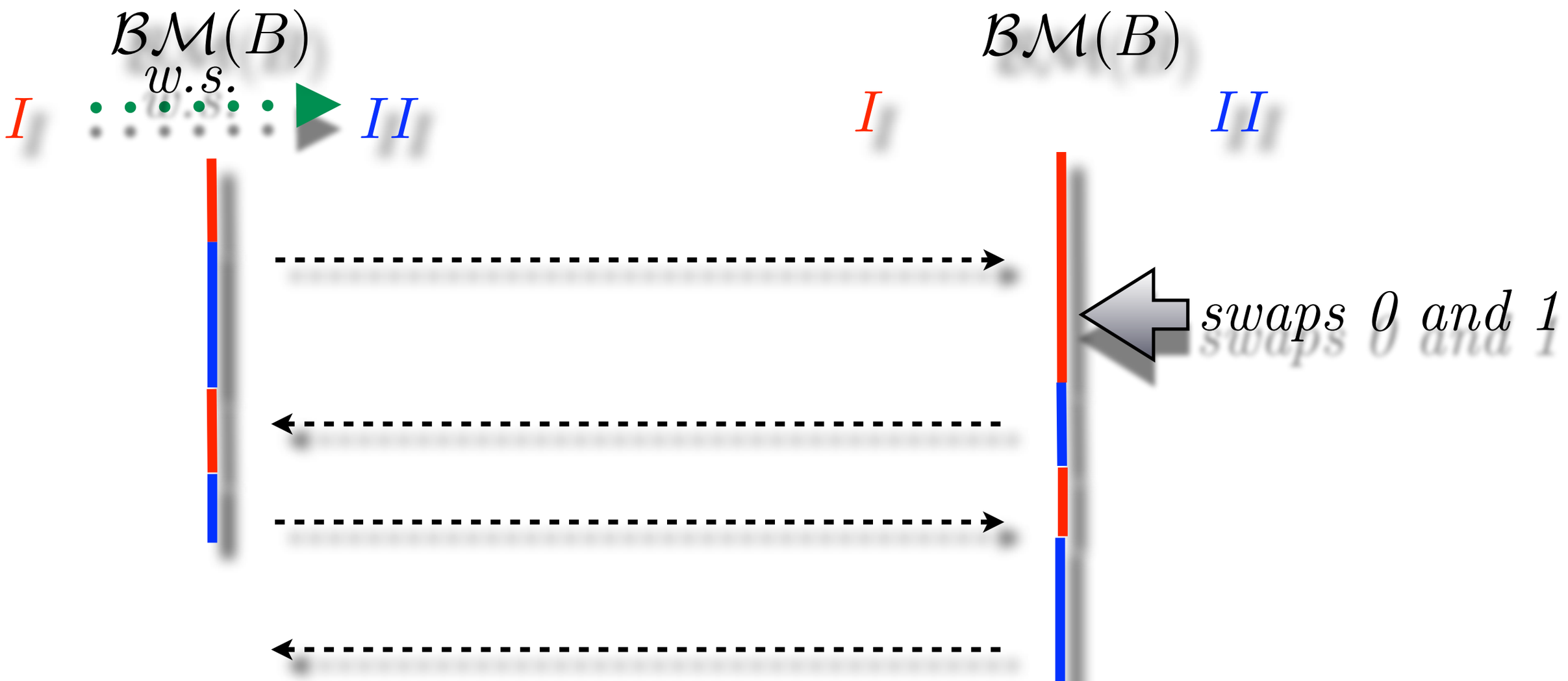
$$I \text{ wins} \iff x \in A$$



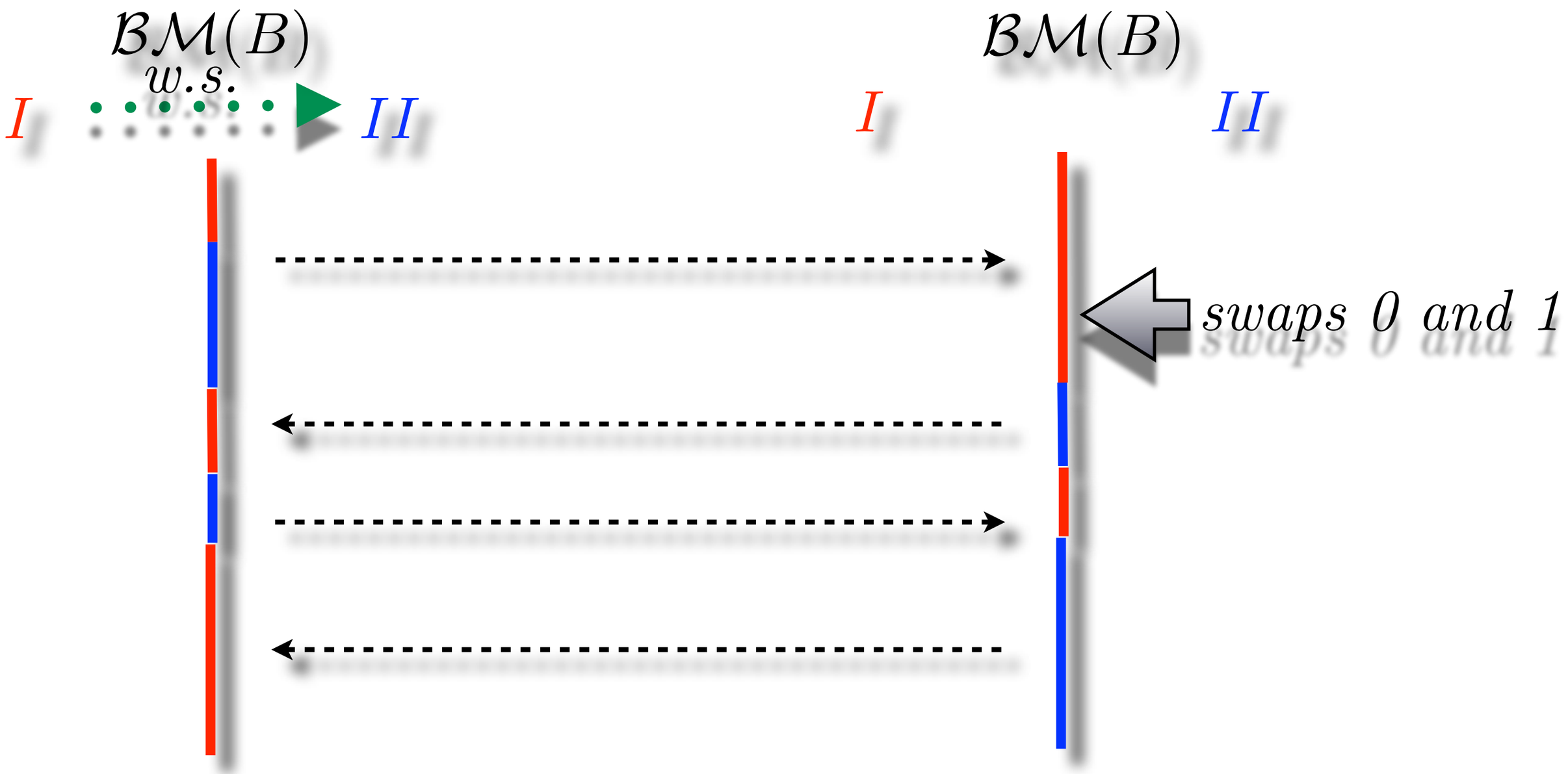
$$I \text{ wins} \iff x \in A$$



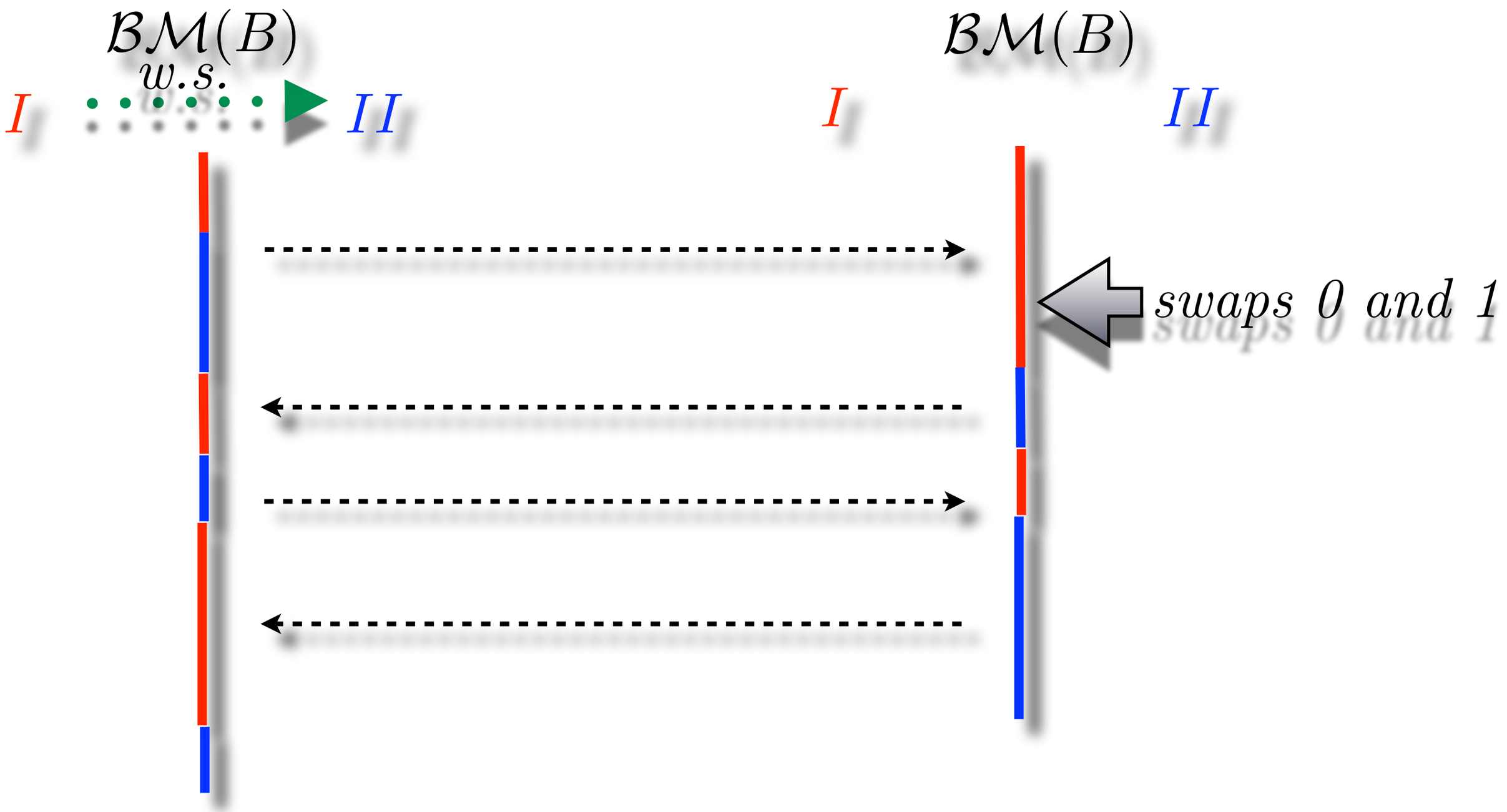
I wins $\iff x \in A$



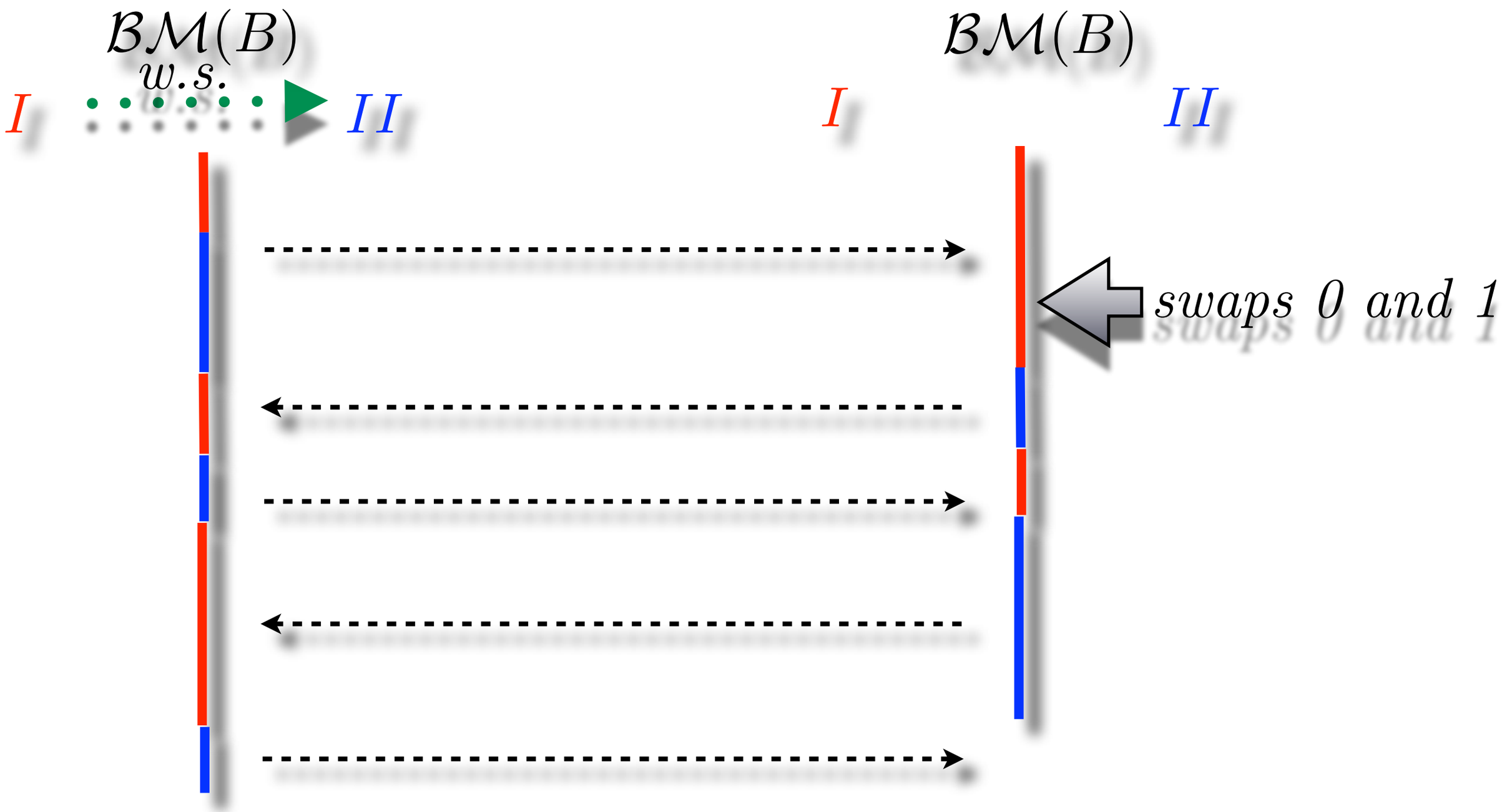
I wins $\iff x \in A$



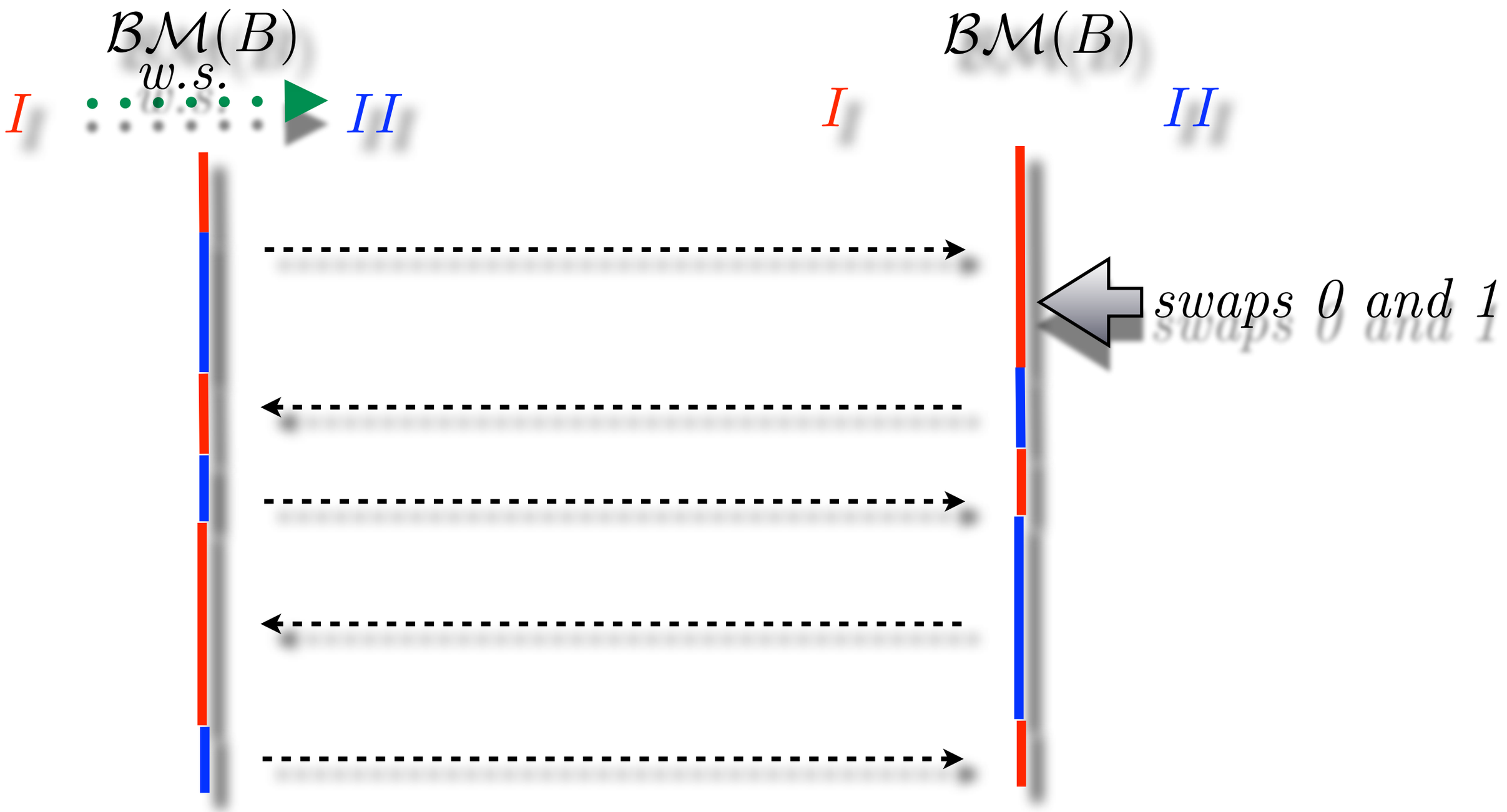
I wins $\iff x \in A$



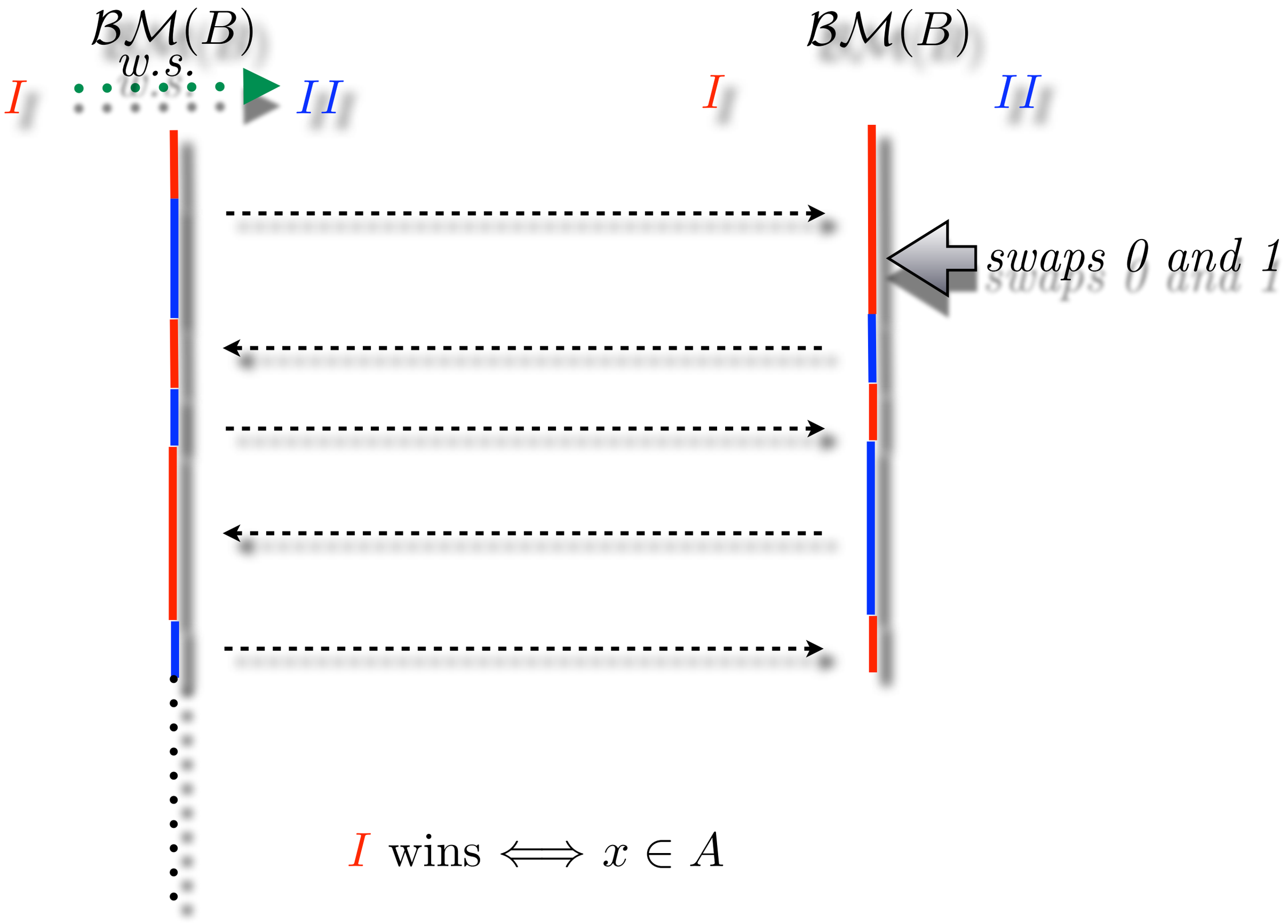
I wins $\iff x \in A$



I wins $\iff x \in A$



I wins $\iff x \in A$



I wins $\iff x \in A$

$\mathcal{BM}(B)$

$w.s.$

I II

$\mathcal{BM}(B)$

I II

$swaps\ 0\ and\ 1$

I wins $\iff x \in A$

$\mathcal{BM}(B)$ w.s.

I II

$\mathcal{BM}(B)$

I II

swaps 0 and 1

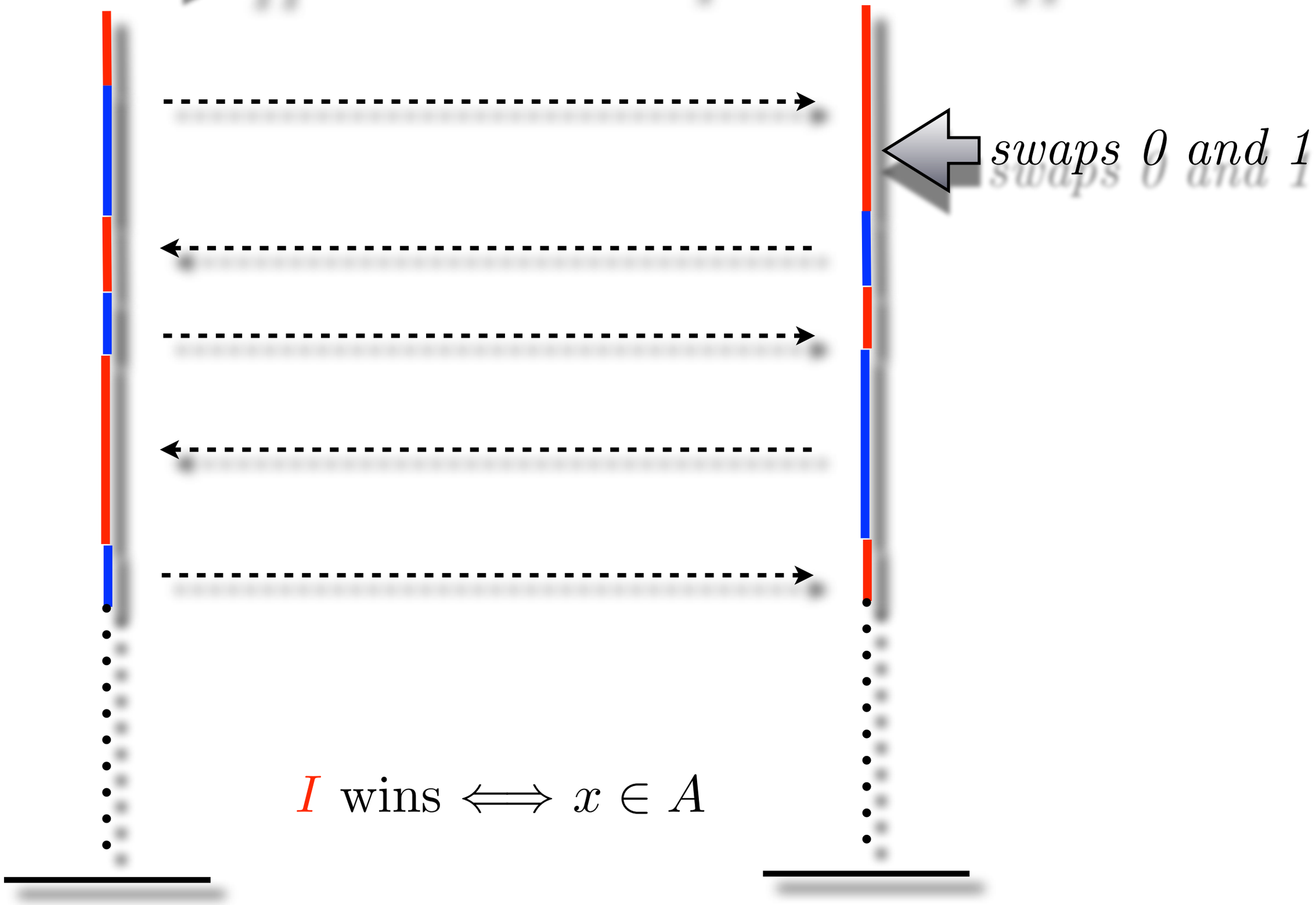
I wins $\iff x \in A$

$\mathcal{BM}(B)$ w.s.

I II

$\mathcal{BM}(B)$

I II



I wins $\iff x \in A$

$\mathcal{BM}(B)$

w.s.

I II

$\mathcal{BM}(B)$

I II

swaps 0 and 1

I wins $\iff x \in A$

x

$\mathcal{BM}(B)$

$w.s.$

I II

$\mathcal{BM}(B)$

I II

swaps 0 and 1

I wins $\iff x \in A$

x

x'

$\mathcal{BM}(B)$

$w.s.$

I II

$\mathcal{BM}(B)$

I II

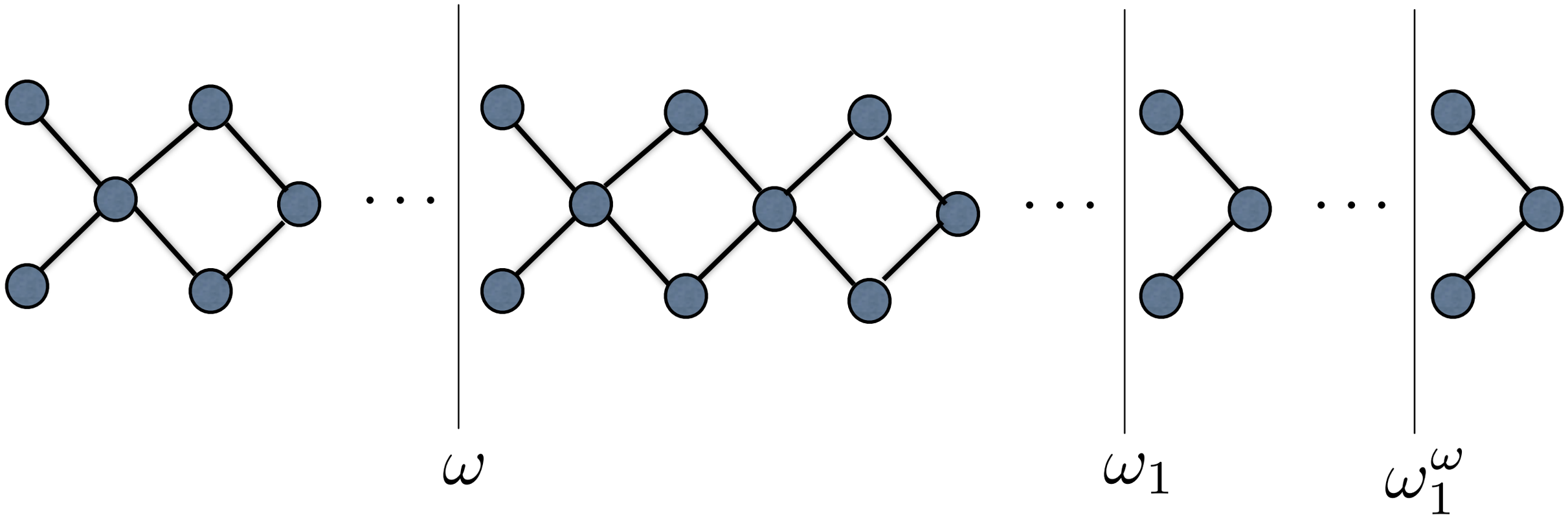
$swaps\ 0\ and\ 1$

$I\ has\ a\ w.s.$

I wins $\iff x \in A$

x

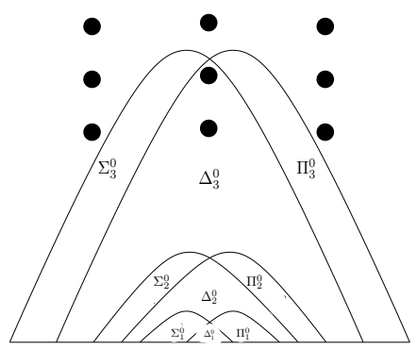
x'


ω
ω_1
ω_1^ω

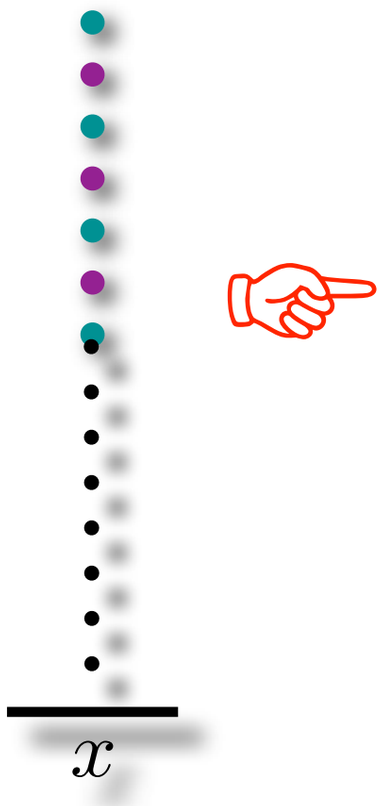


An example: evaluation game for 1st order logic

non determined game

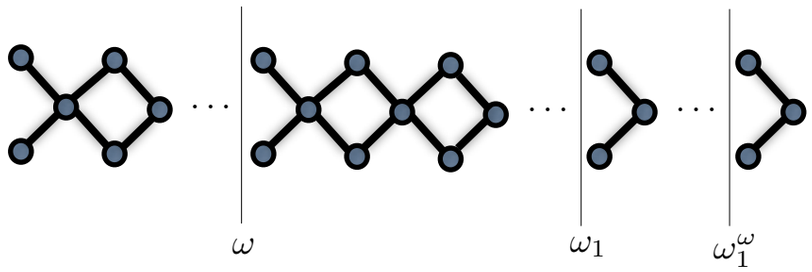


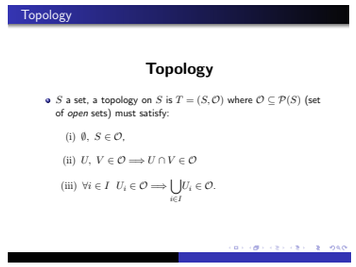
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

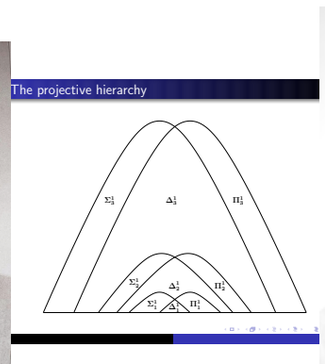
Games that construct mappings



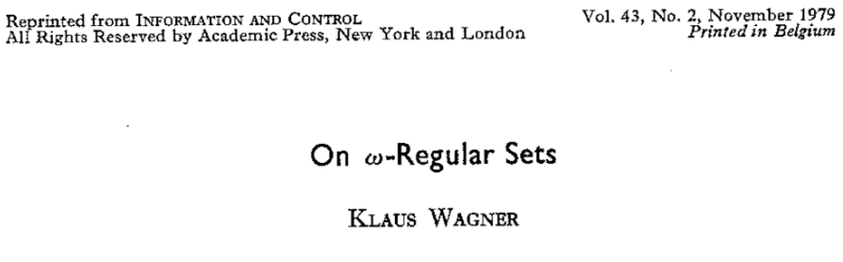










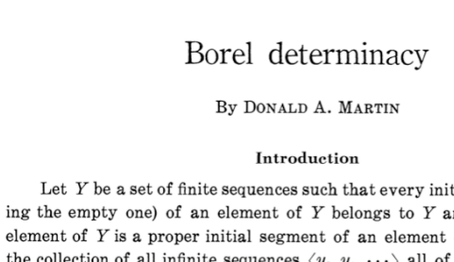


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
All Rights Reserved by Academic Press, New York and London
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
All Rights Reserved by Academic Press, New York and London
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

Prefix-topology

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

Prefix-topology

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

Prefix-topology

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
All Rights Reserved by Academic Press, New York and London
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER

Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
All Rights Reserved by Academic Press, New York and London
Printed in Belgium

In 1986, Pierre Simonnet realized that the
Wagner Hierarchy
was the trace of omega-regular sets on the
Wadge Hierarchy

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
Printed in Belgium
All Rights Reserved by Academic Press, New York and London

In 1986, Pierre Simonnet realized that the Wagner Hierarchy was the trace of omega-regular sets on the Wadge Hierarchy

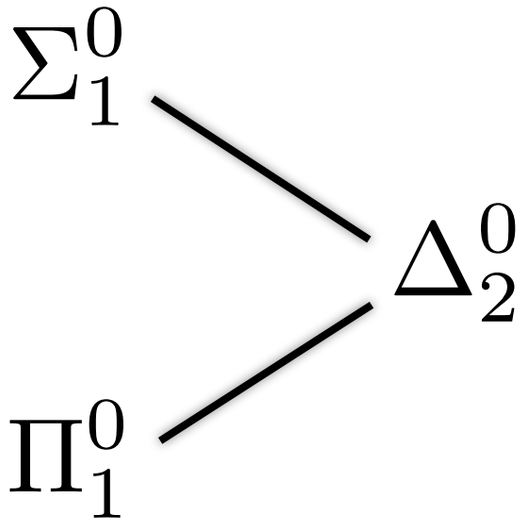
The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

The Borel Hierarchy

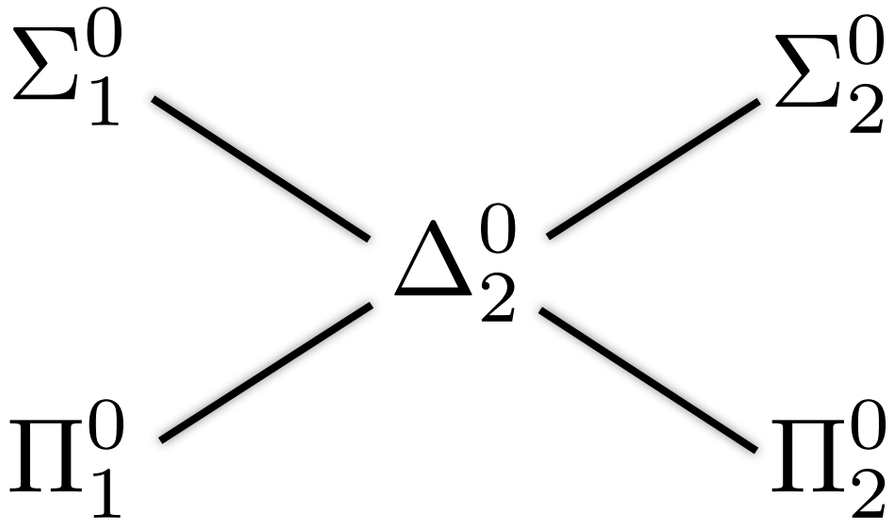
Σ_1^0

Π_1^0

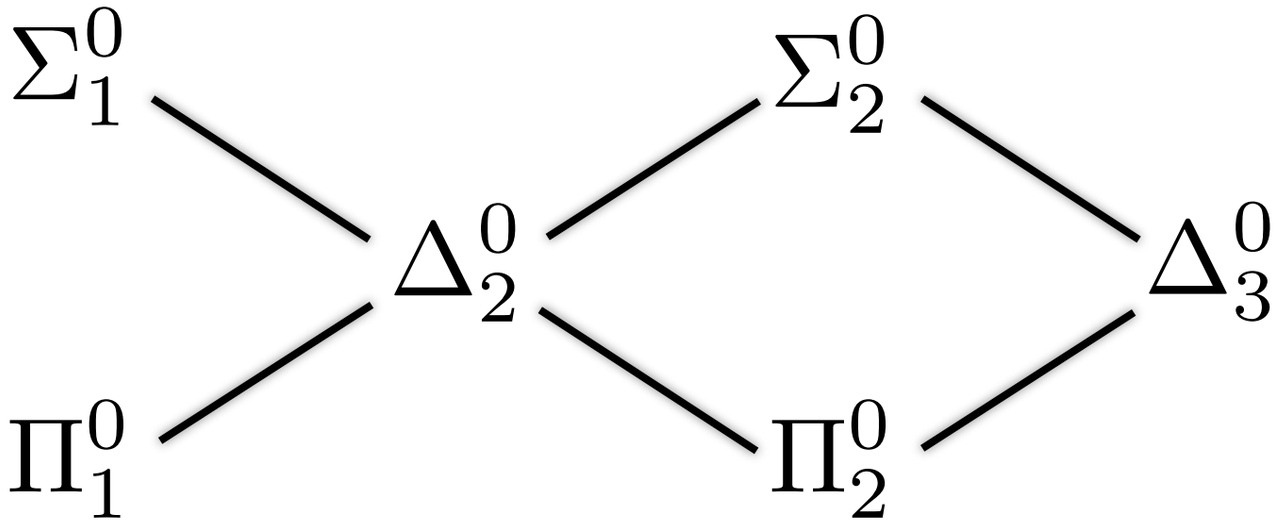
The Borel Hierarchy



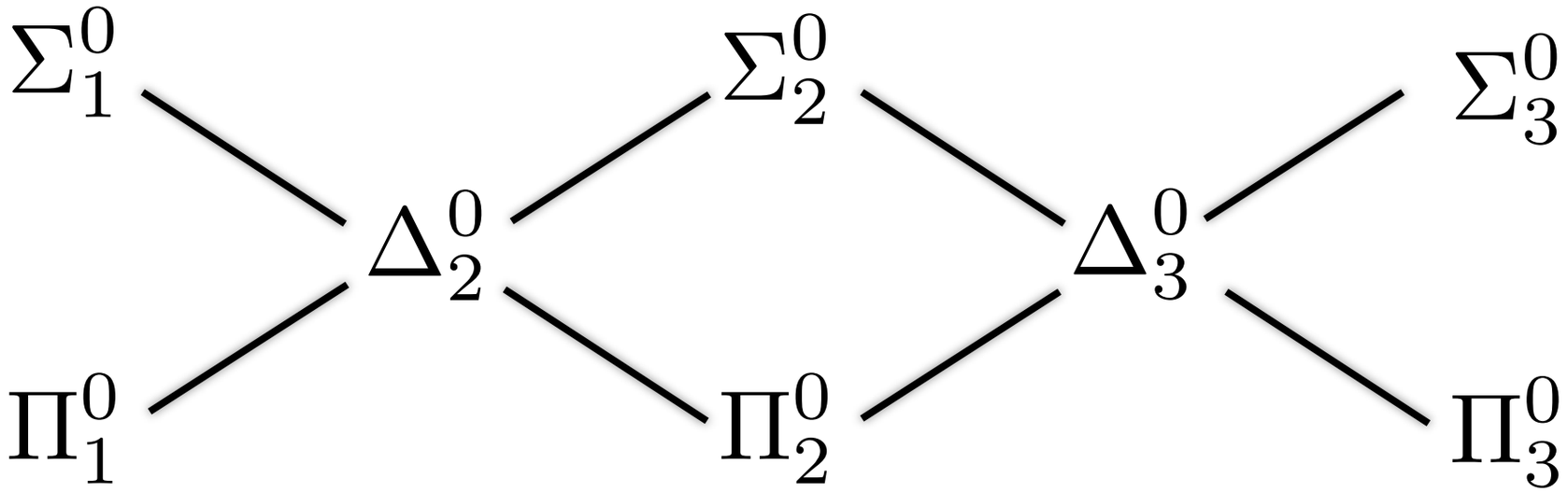
The Borel Hierarchy



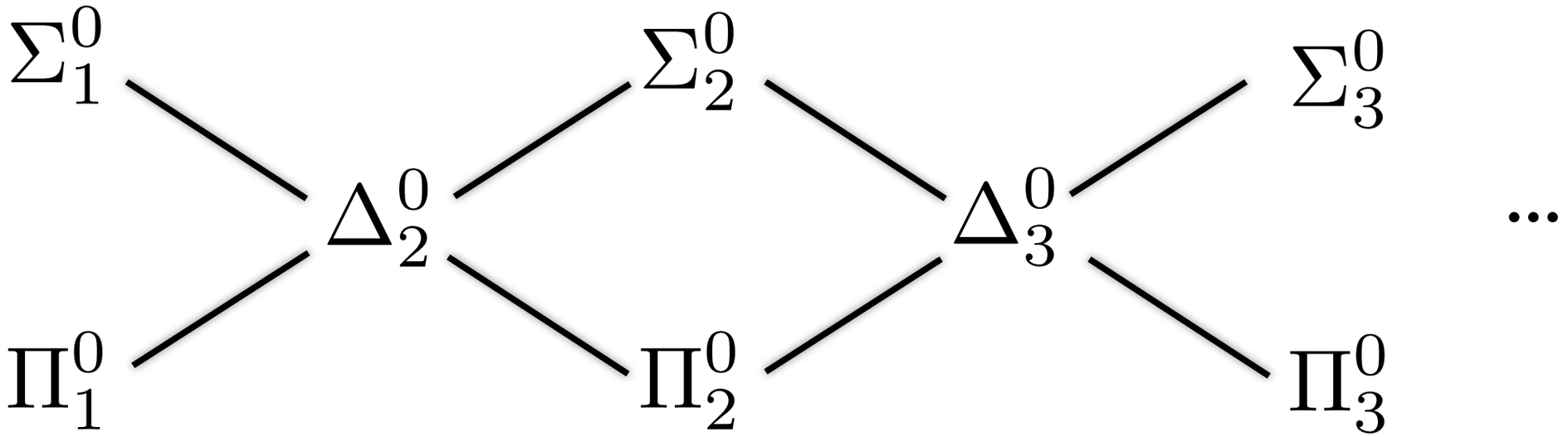
The Borel Hierarchy



The Borel Hierarchy



The Borel Hierarchy

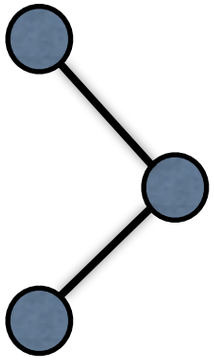


The Wadge Hierarchy

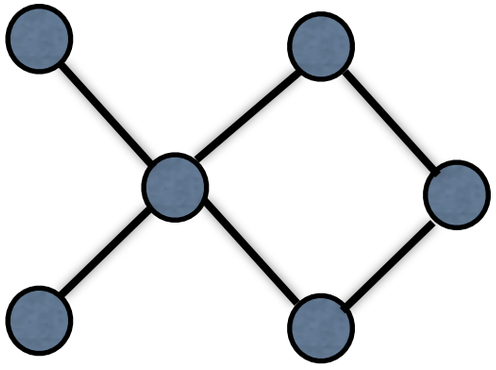
The Wadge Hierarchy



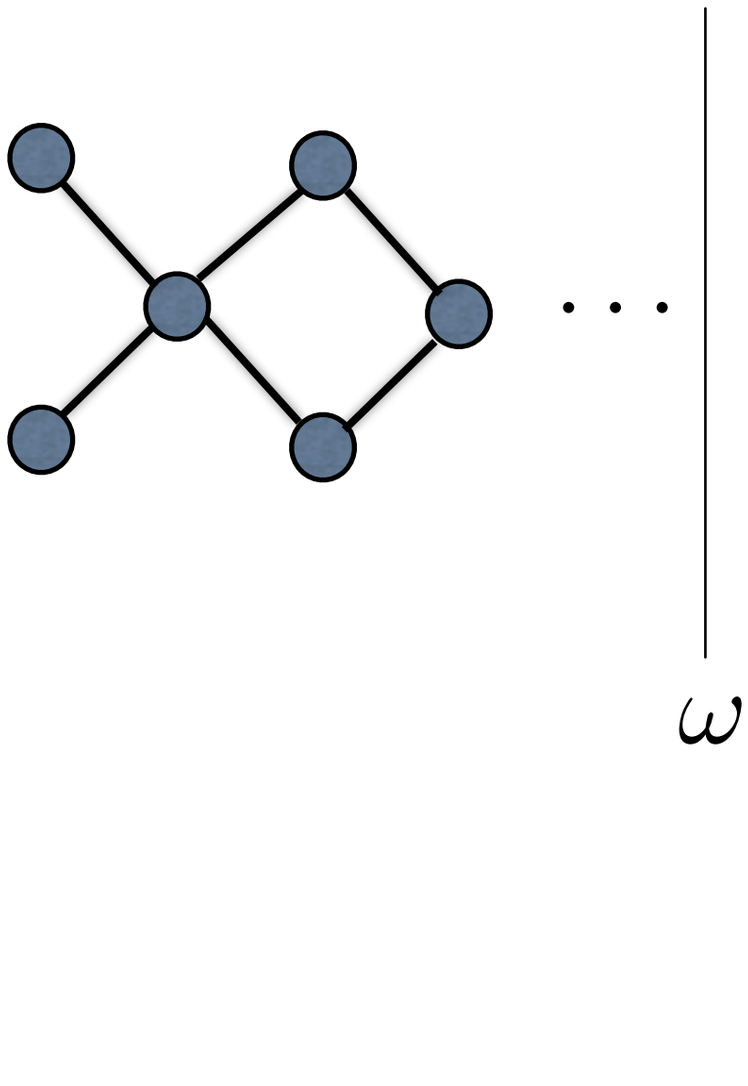
The Wadge Hierarchy



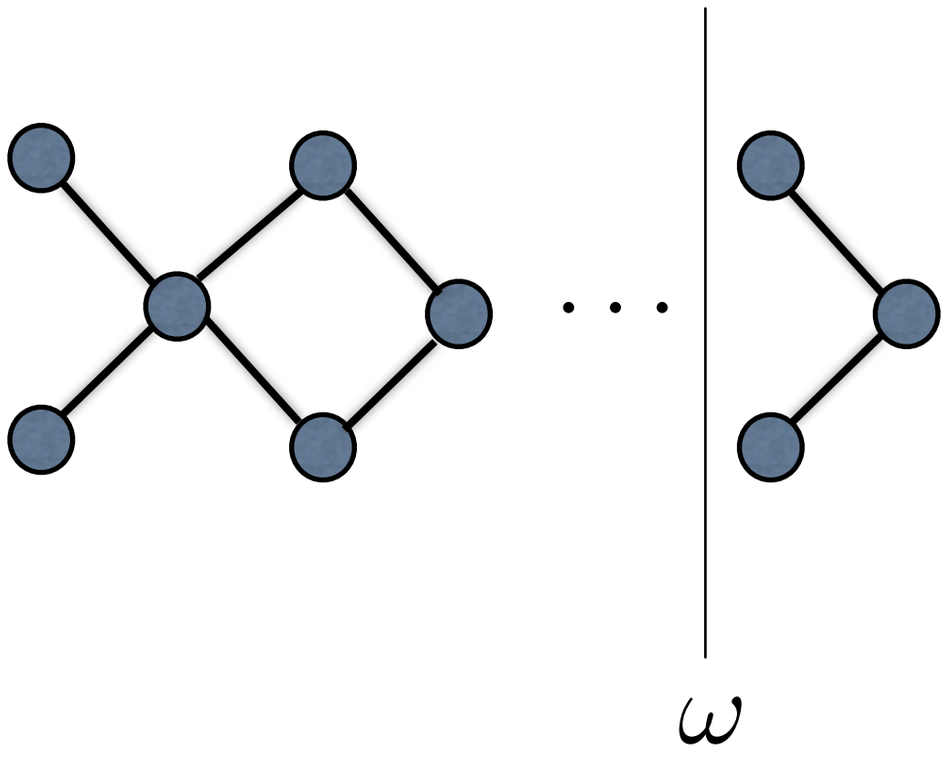
The Wadge Hierarchy



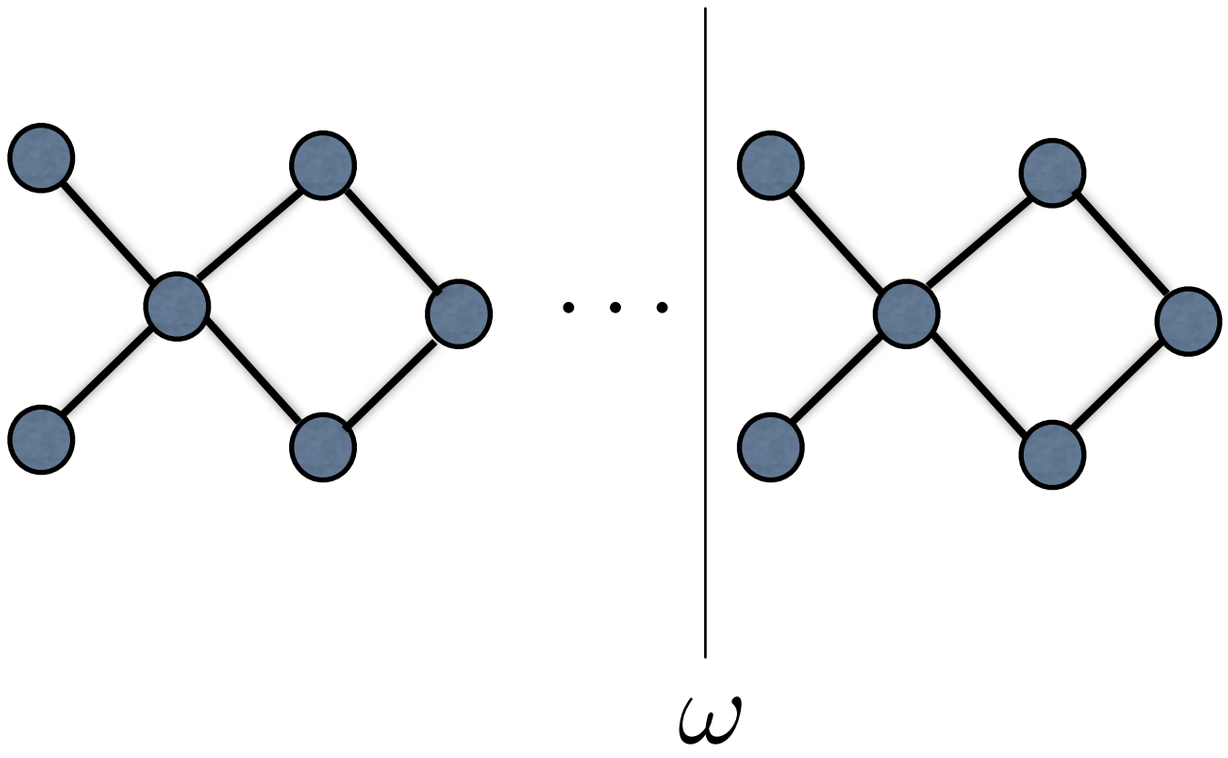
The Wadge Hierarchy



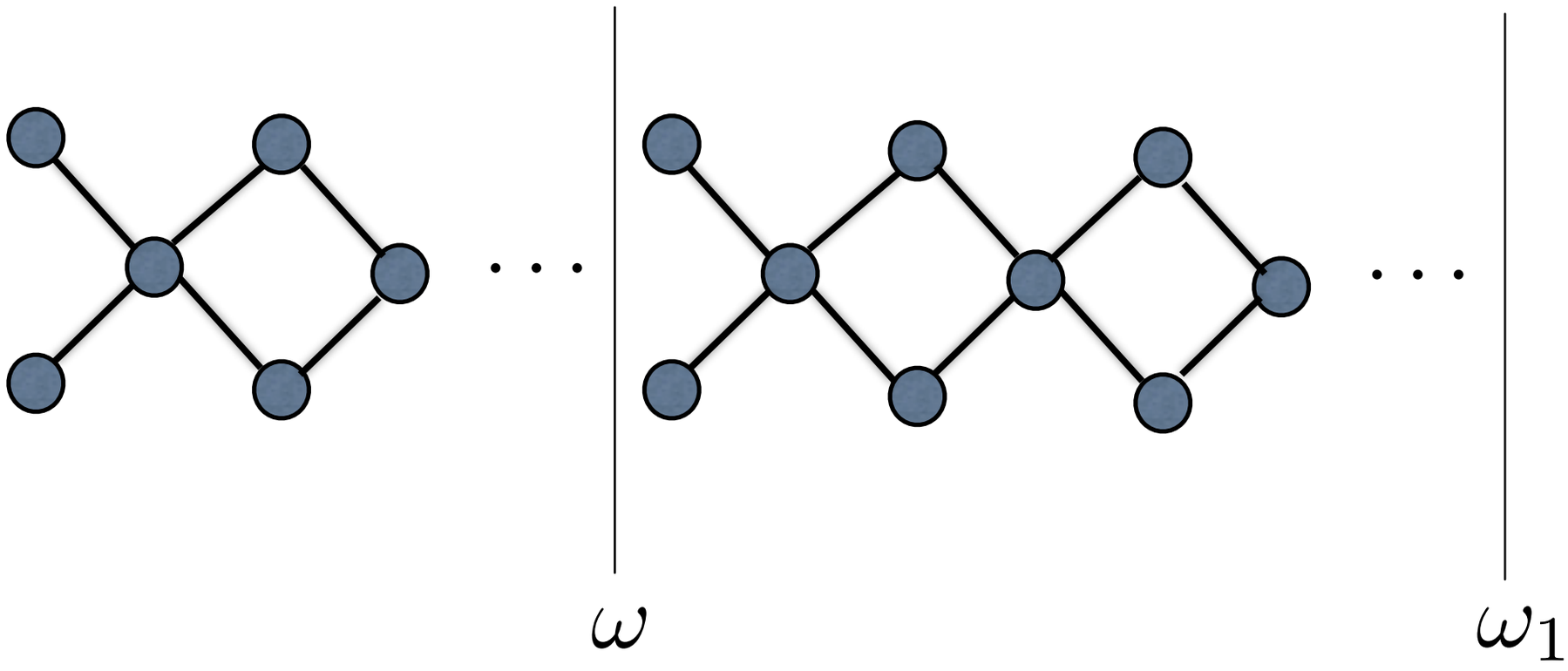
The Wadge Hierarchy



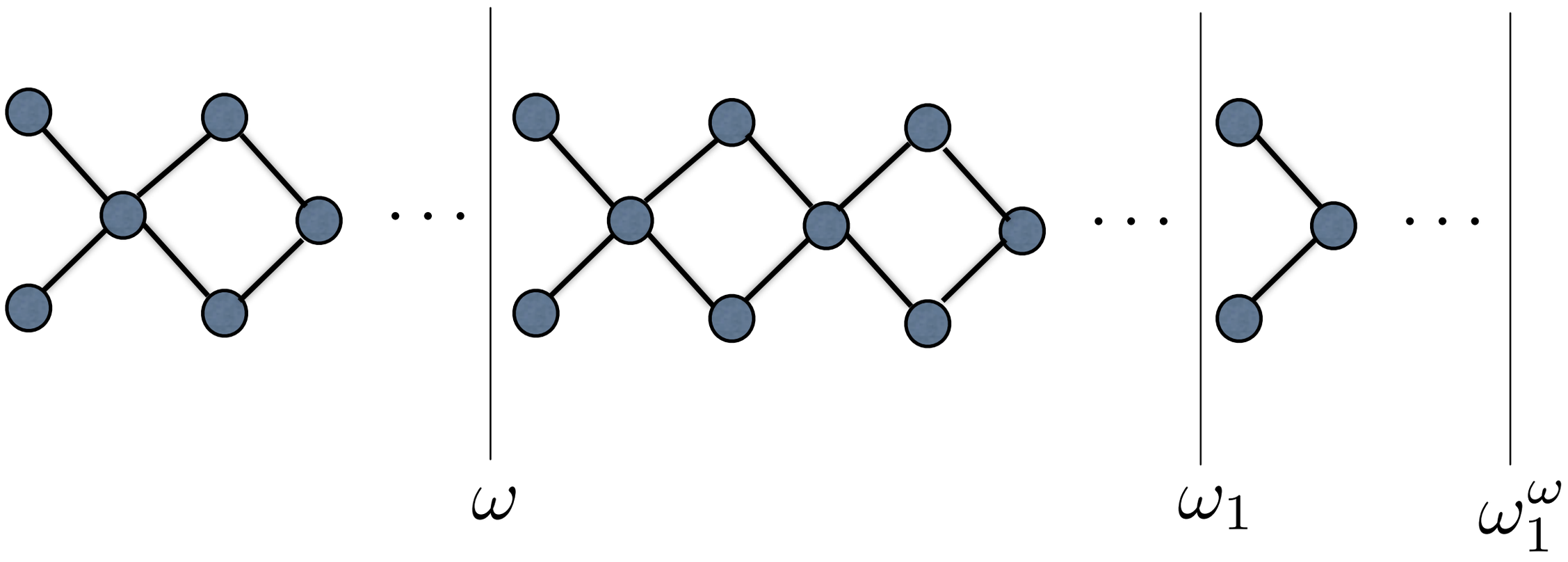
The Wadge Hierarchy



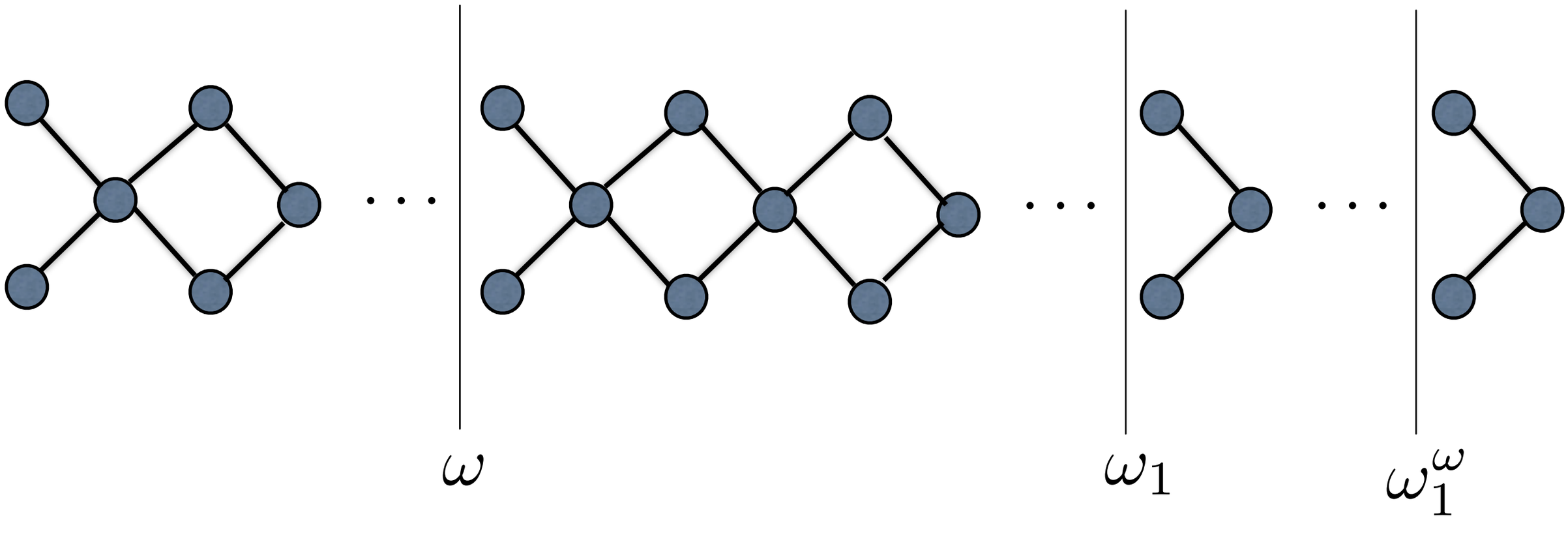
The Wadge Hierarchy



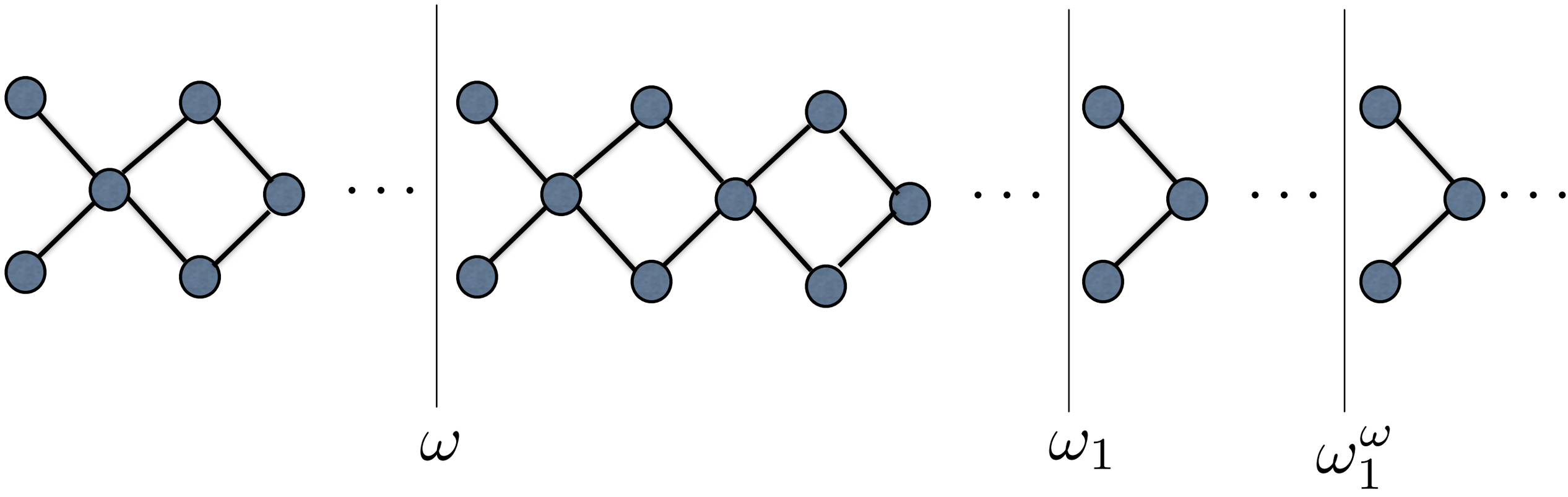
The Wadge Hierarchy



The Wadge Hierarchy

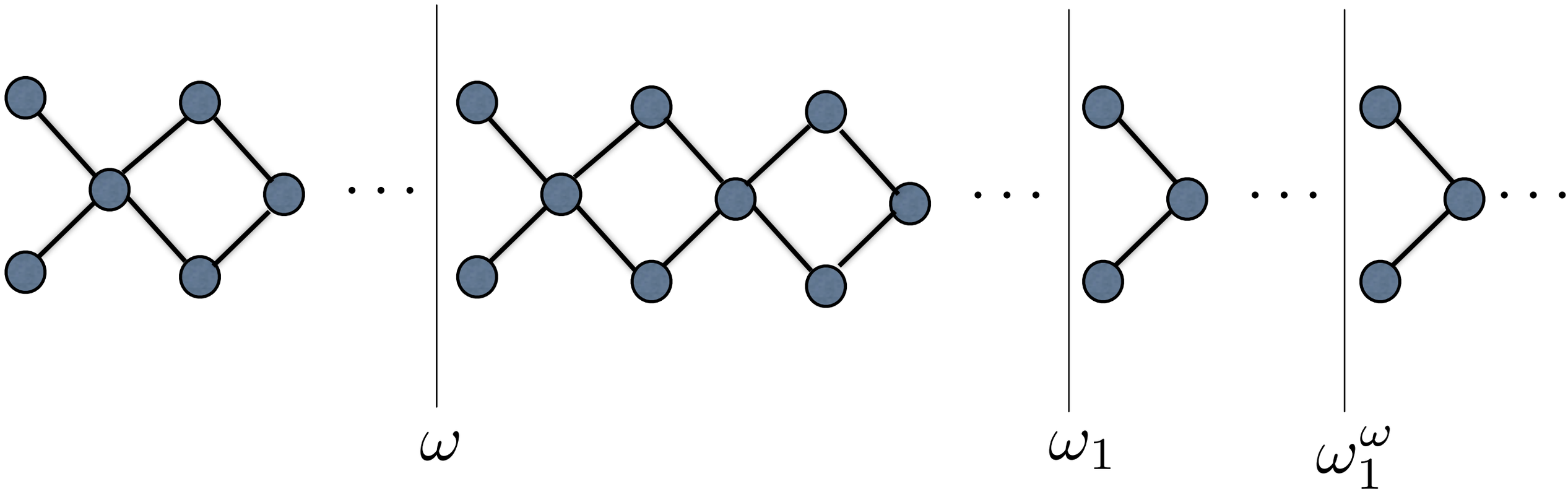


The Wadge Hierarchy



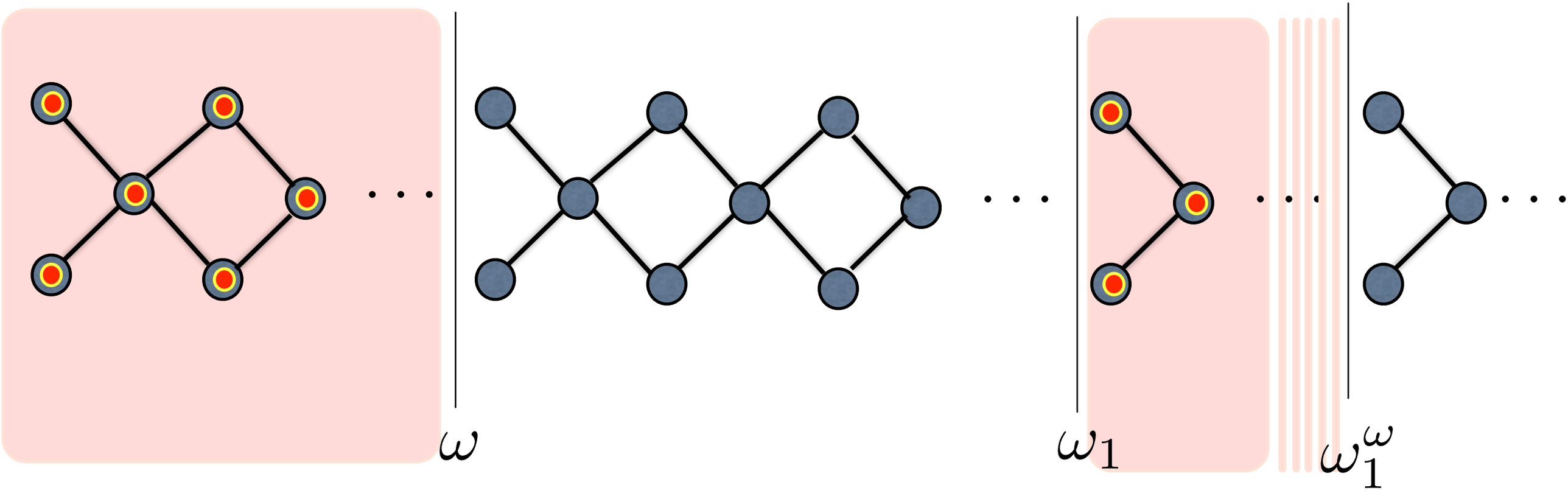
The Wadge Hierarchy

Where are the omega regular languages ?



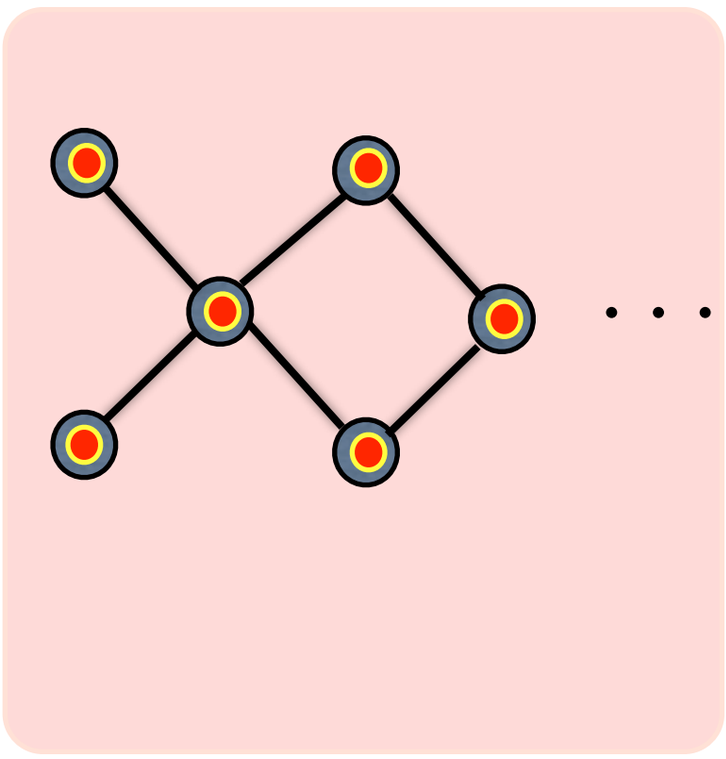
The Wadge Hierarchy

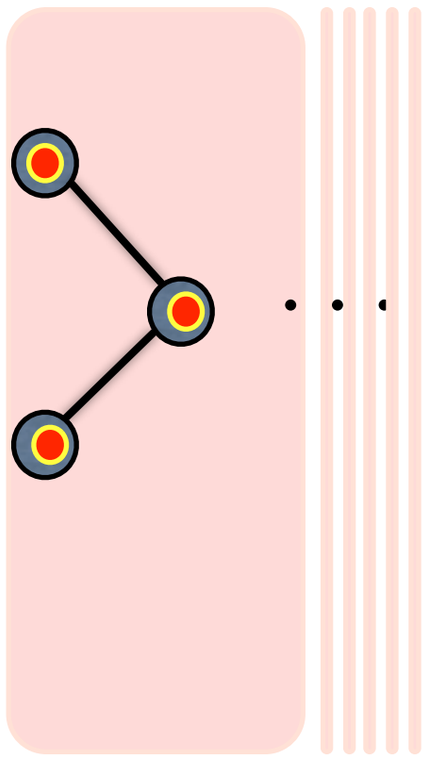
Where are the omega regular languages ?



The Wadge Hierarchy

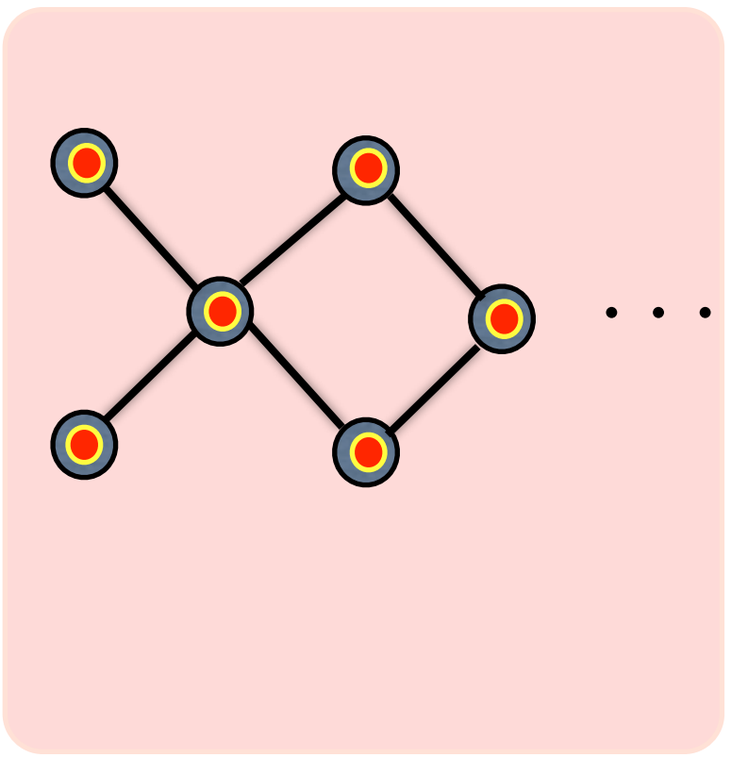
Where are the omega regular languages ?

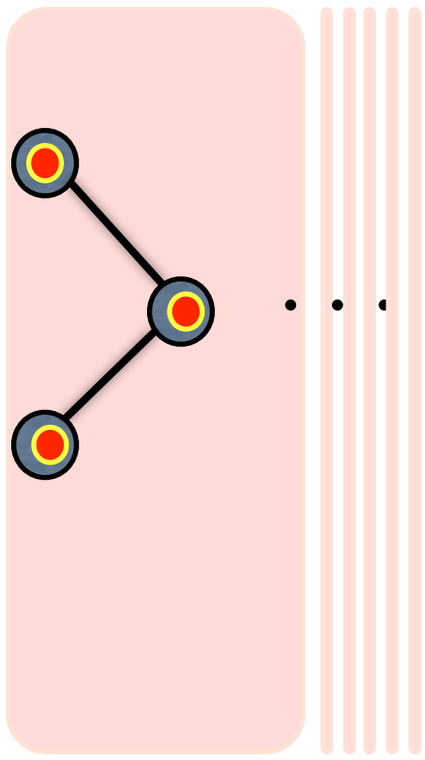




The Wadge Hierarchy

Where are the omega regular languages ?

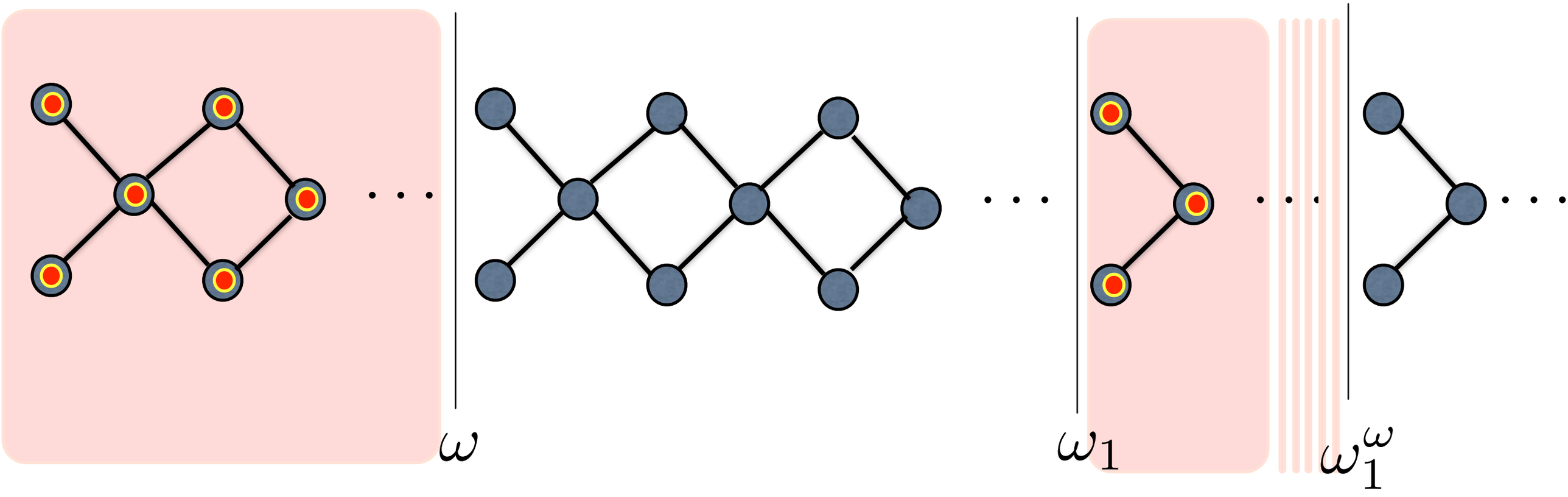




ω

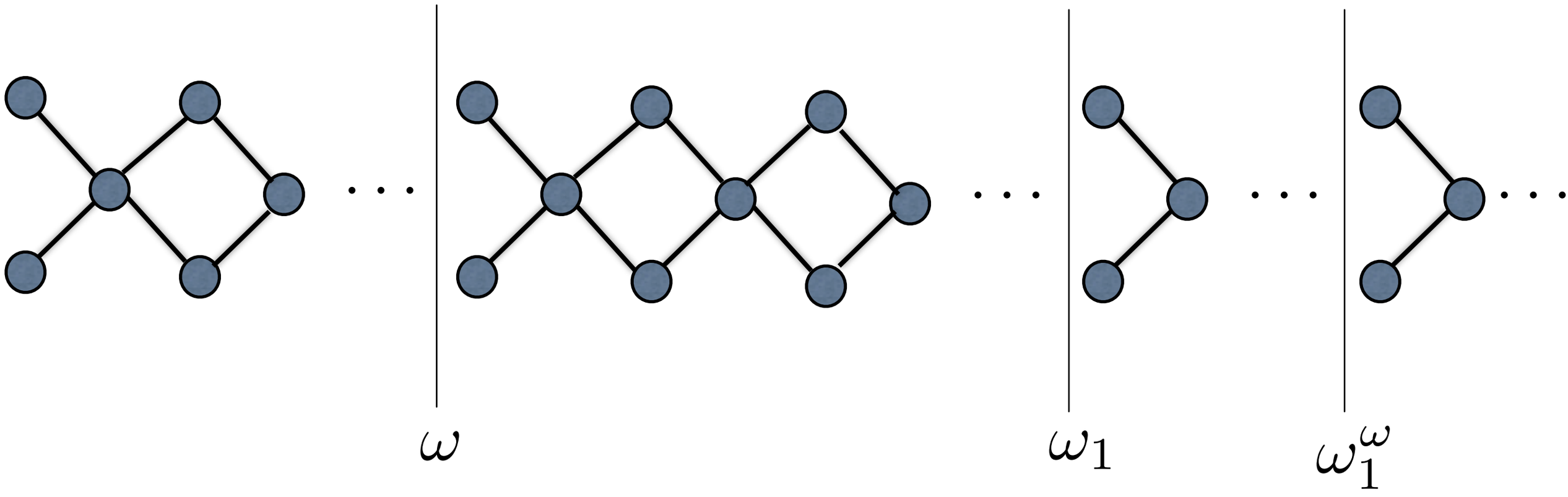
The Wadge Hierarchy

Where are the omega regular languages ?

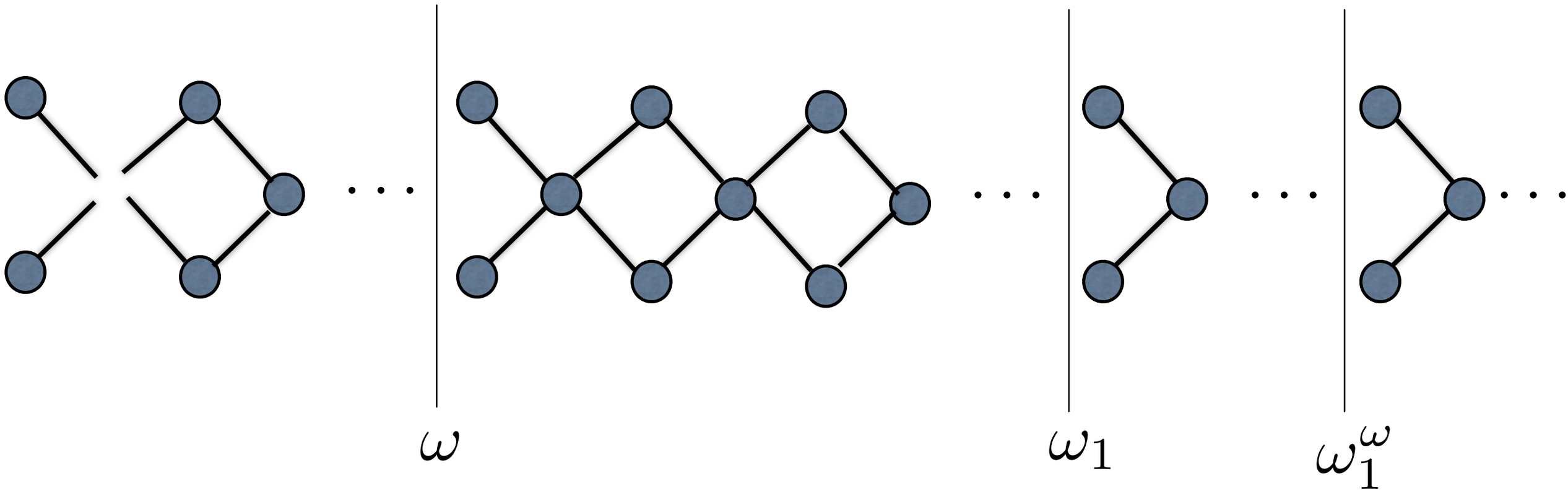


ω

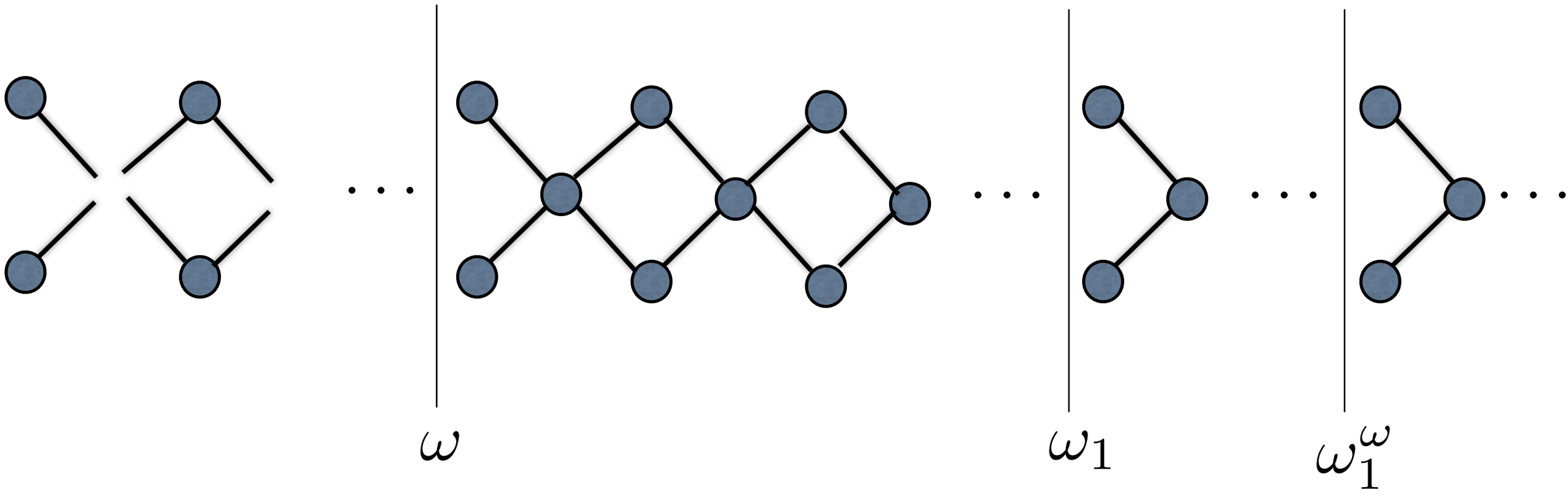
The Wadge Hierarchy



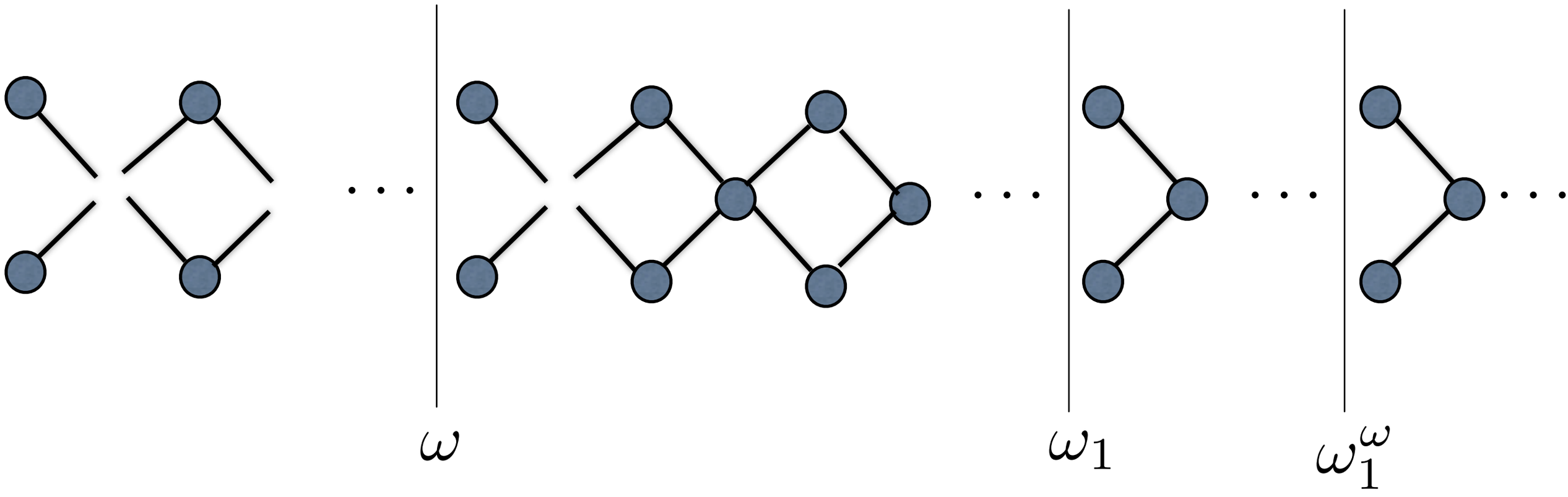
The Wadge Hierarchy



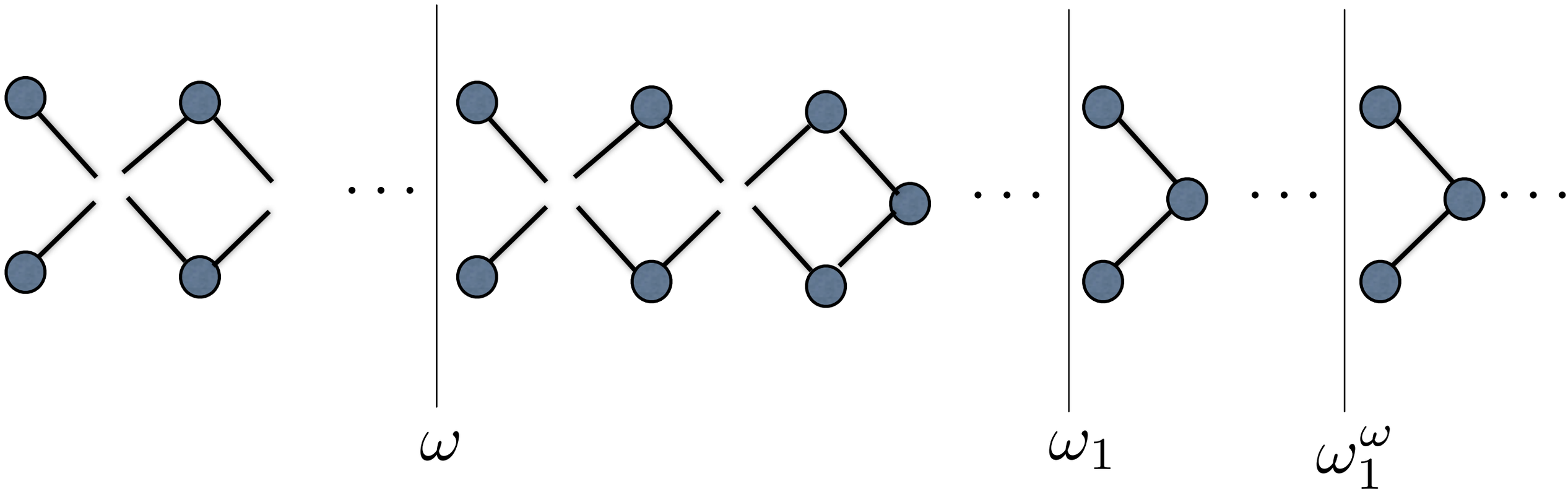
The Wadge Hierarchy



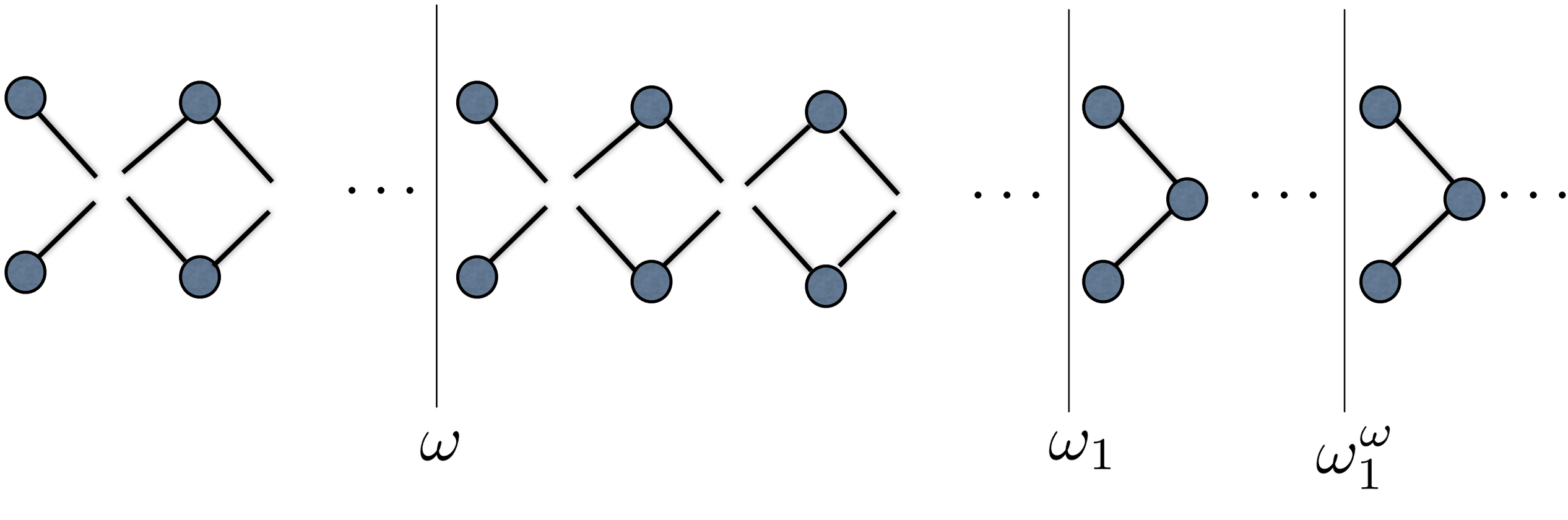
The Wadge Hierarchy



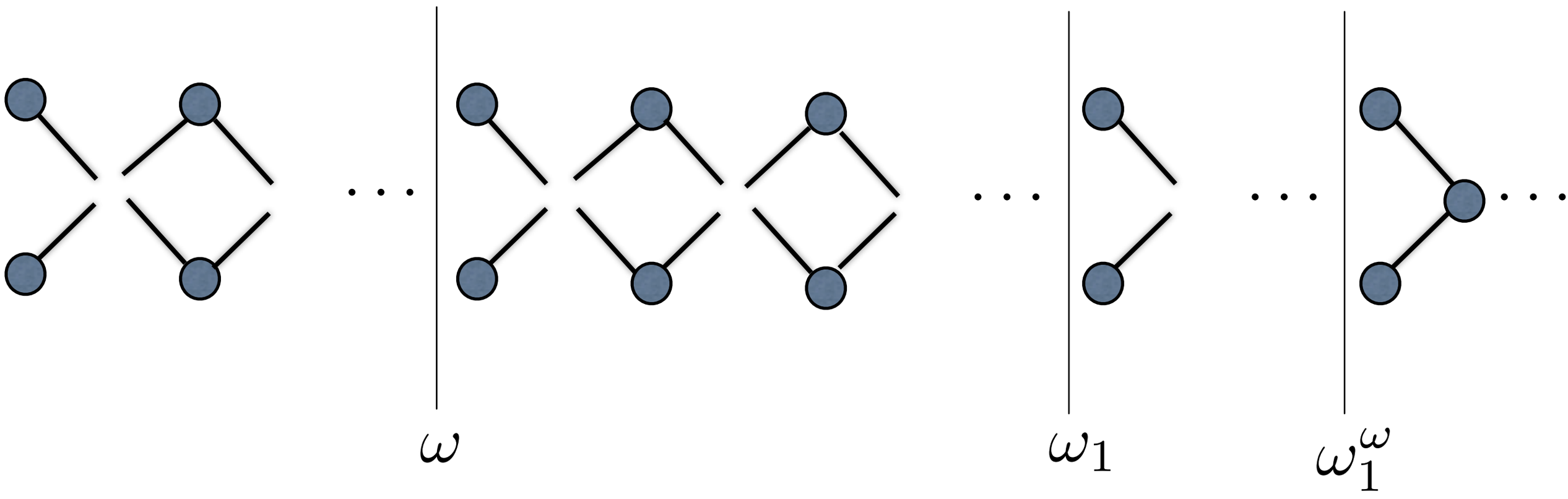
The Wadge Hierarchy



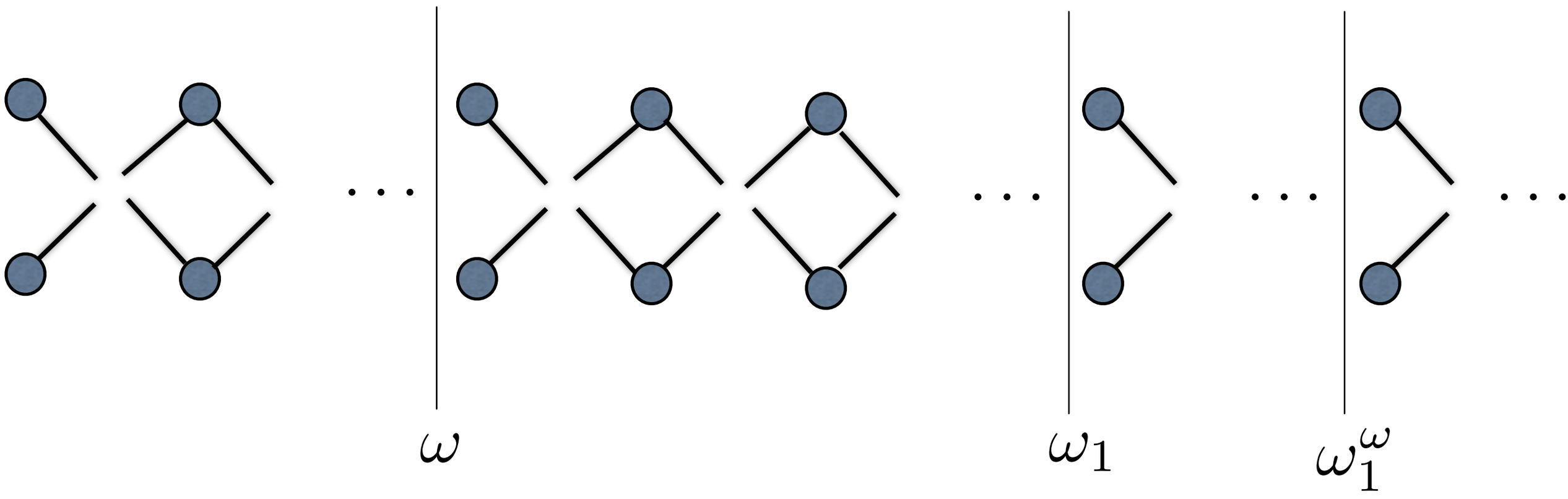
The Wadge Hierarchy



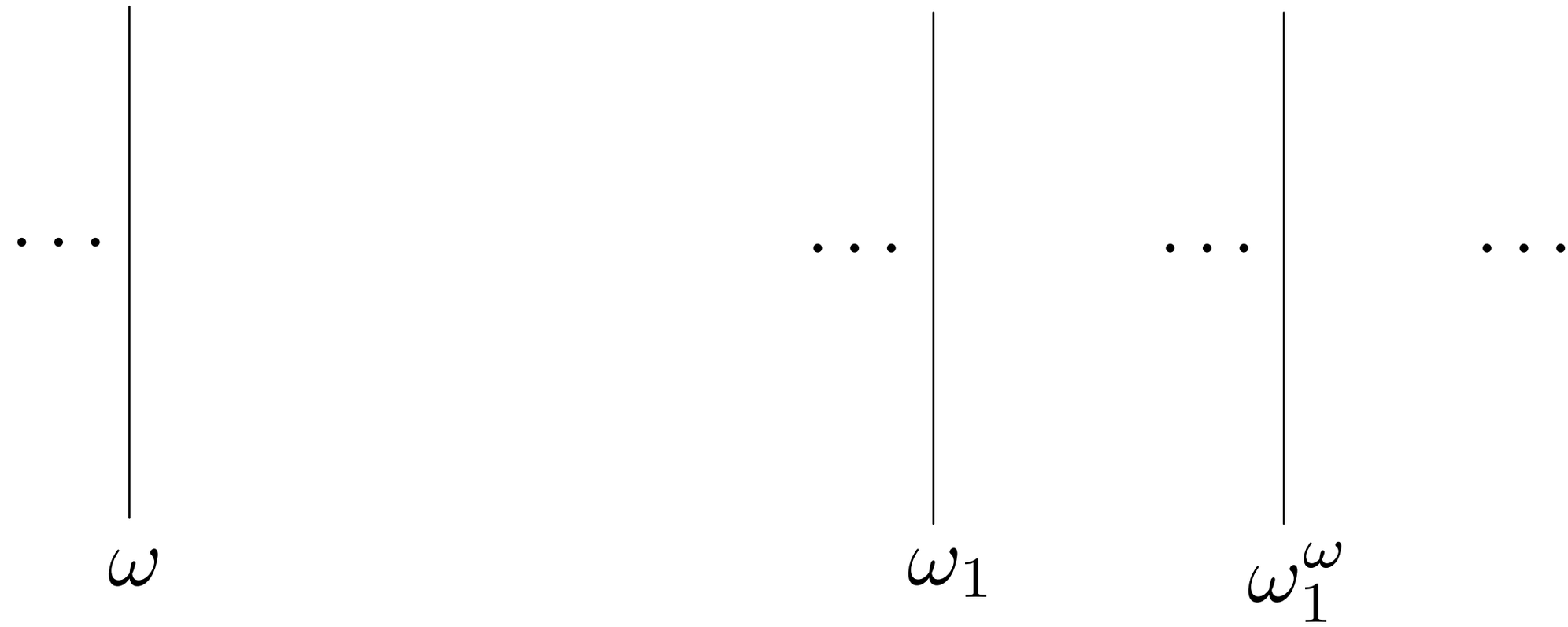
The Wadge Hierarchy



The Wadge Hierarchy



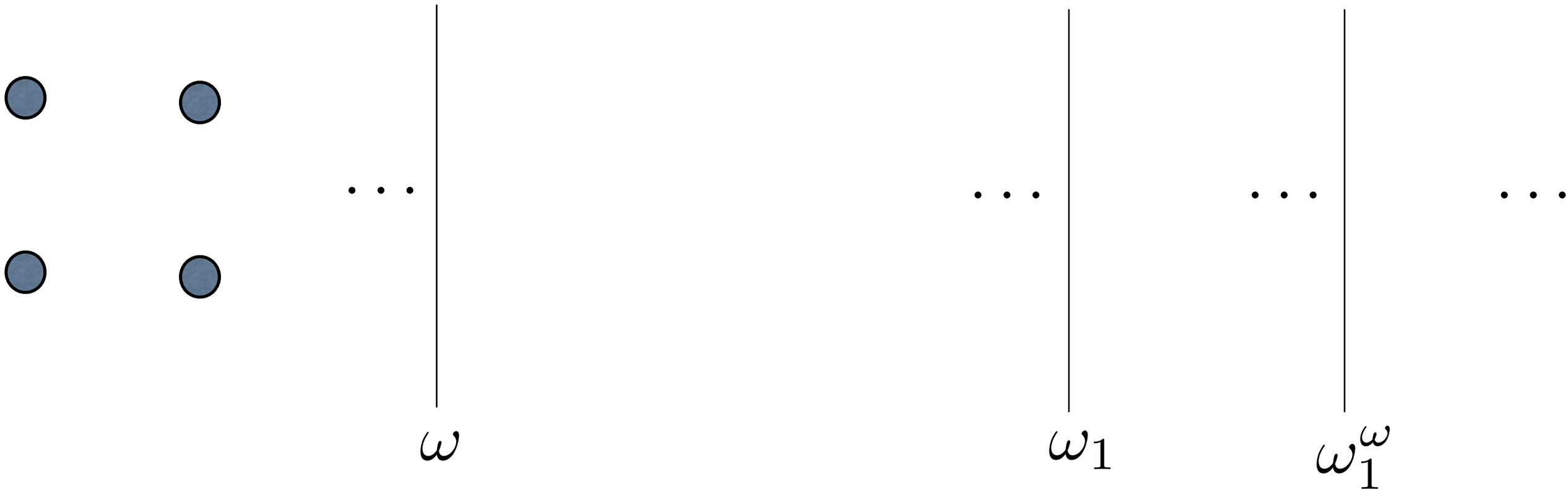
The Wadge Hierarchy



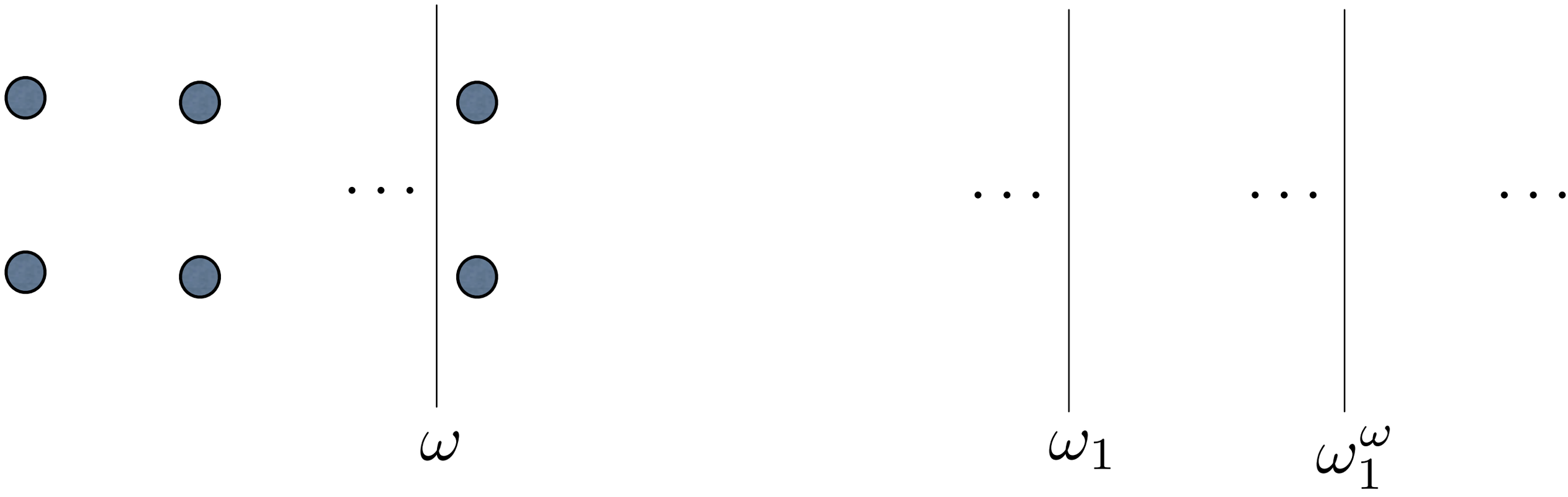
The Wadge Hierarchy



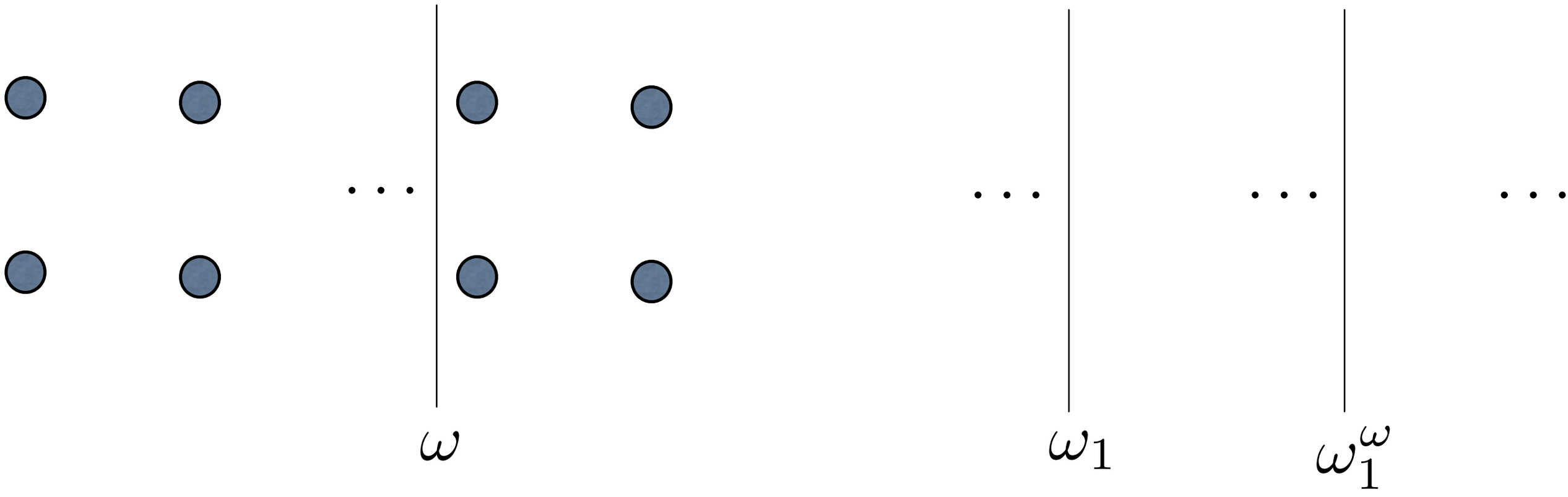
The Wadge Hierarchy



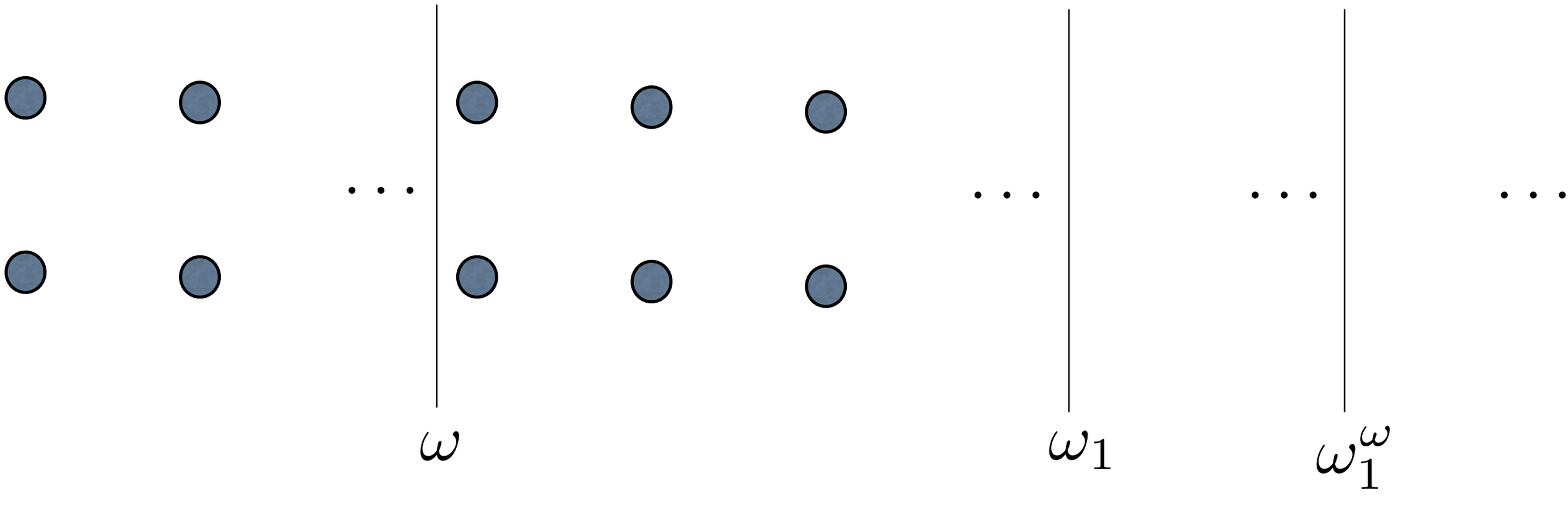
The Wadge Hierarchy



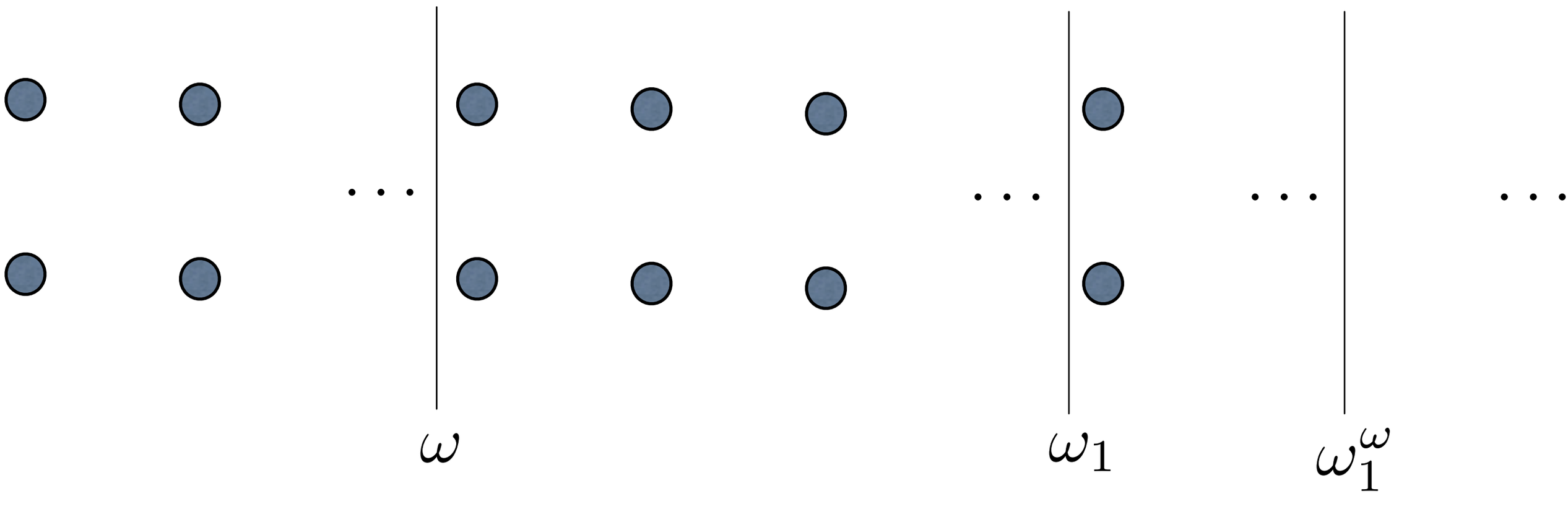
The Wadge Hierarchy



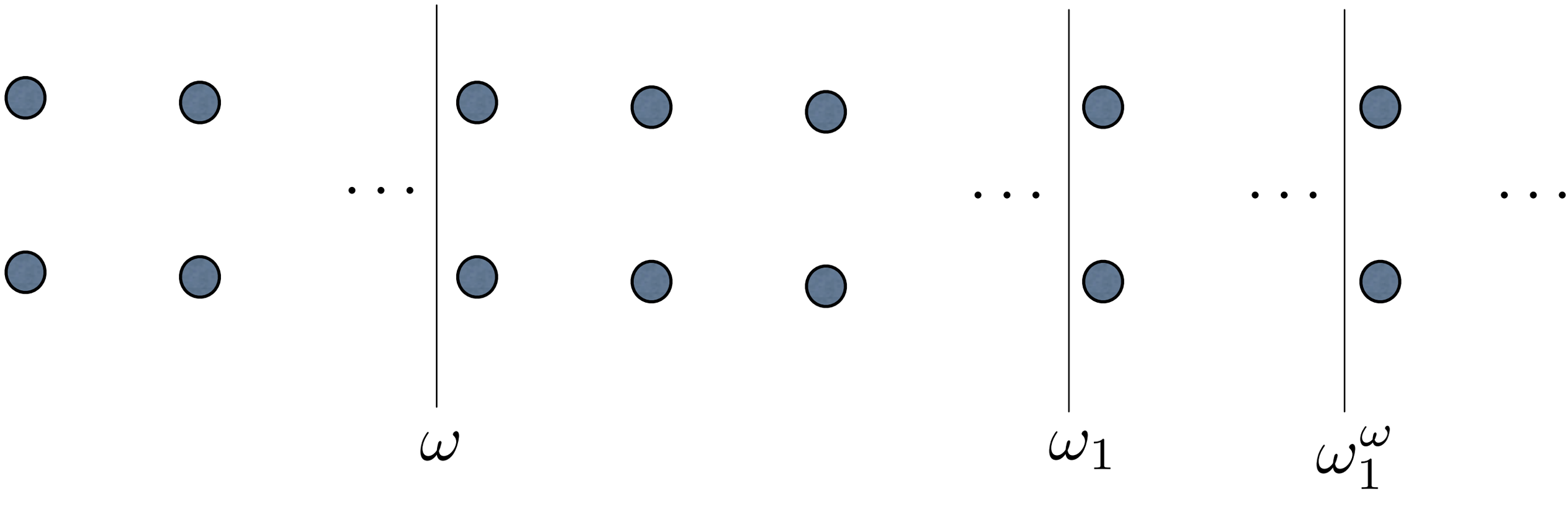
The Wadge Hierarchy



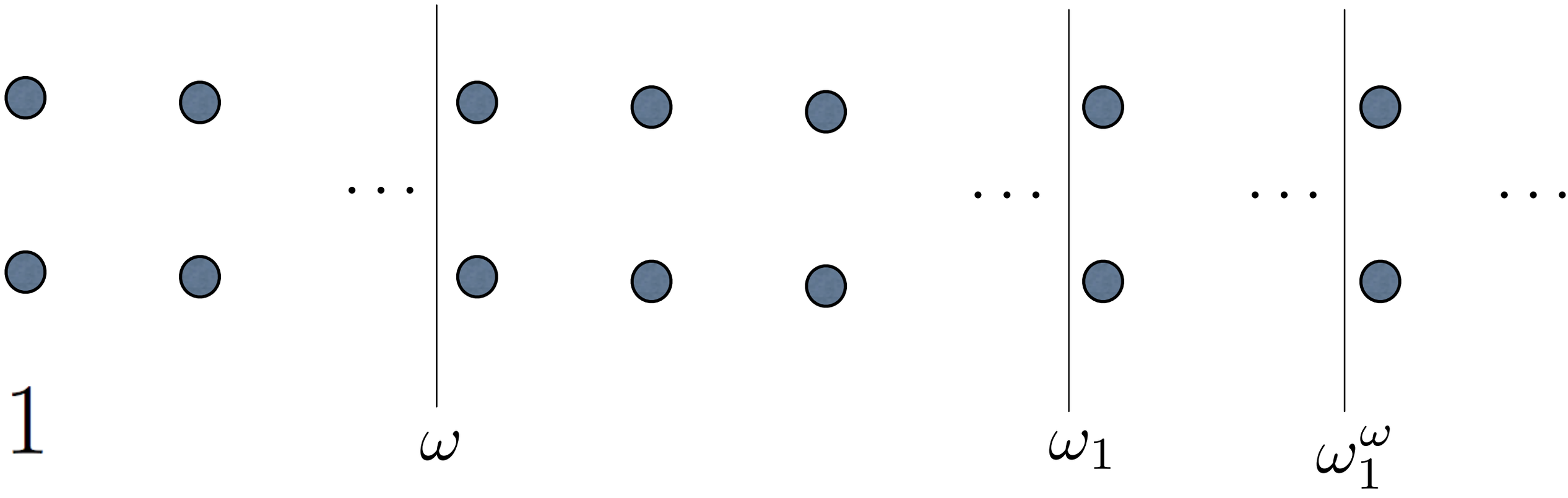
The Wadge Hierarchy



The Wadge Hierarchy



The Wadge Hierarchy



- $\{1\}$ $\rightsquigarrow$ $+, \cdot$ $\rightsquigarrow \omega$
- $\{1, \omega\}$ $\rightsquigarrow$ $+, \cdot$ $\rightsquigarrow \omega^\omega$

- ω-words $\rightsquigarrow$ *det.* $=$ *non det.*

  For $0 < \xi < \omega^\omega$ with

  - Position in the Wagner Hierarchy $\rightsquigarrow$

$$\xi = \omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot l_0$$

(with $n_0 < n_1 < \ldots < n_k < \omega$ and $\forall i \leq k \;\; 0 < l_i \in \omega$).

- Position in the Wadge Hierarchy $\rightsquigarrow$

$$\omega_1^{n_k} \cdot l_k + \omega_1^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega_1^{n_0} \cdot l_0$$

(with $n_0 < n_1 < \ldots < n_k < \omega$ and $\forall i \leq k \;\; 0 < l_i \in \omega$).

- ω-words $\rightsquigarrow$ *det.* $=$ *non det.*

  For $0 < \xi < \omega^\omega$ with

  - Position in the Wagner Hierarchy $\rightsquigarrow$

$$\xi = \boxed{\omega^{n_k}} \bullet l_k \oplus \boxed{\omega^{n_{k-1}}} \bullet l_{k-1} \oplus \ldots \oplus \boxed{\omega^{n_0}} \bullet l_0$$

(with $n_0 < n_1 < \ldots < n_k < \omega$ and $\forall i \leq k \;\; 0 < l_i \in \omega$).

- Position in the Wadge Hierarchy $\rightsquigarrow$

$$\boxed{\omega_1^{n_k}} \bullet l_k \oplus \boxed{\omega_1^{n_{k-1}}} \bullet l_{k-1} \oplus \ldots \oplus \boxed{\omega_1^{n_0}} \bullet l_0$$

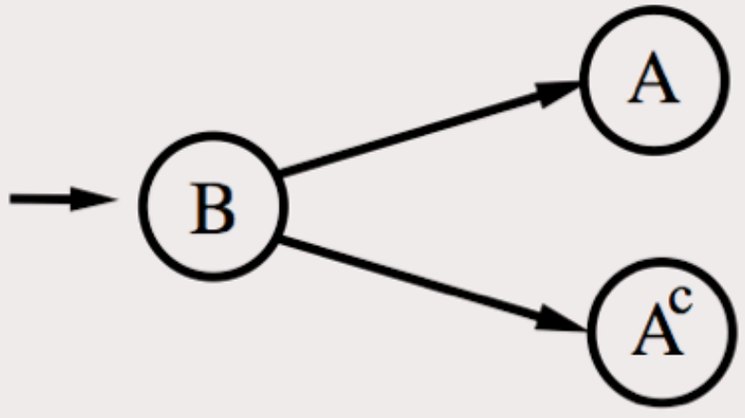
(with $n_0 < n_1 < \ldots < n_k < \omega$ and $\forall i \leq k \;\; 0 < l_i \in \omega$).

The Sum

- Design $+$ such that

$$d^\circ(A+B) = d^\circ A + d^\circ B$$

Definition



The Finite Multiplication

- Design $\cdot$ such that

$$d^\circ A \cdot n = d^\circ A \cdot n$$

Definition

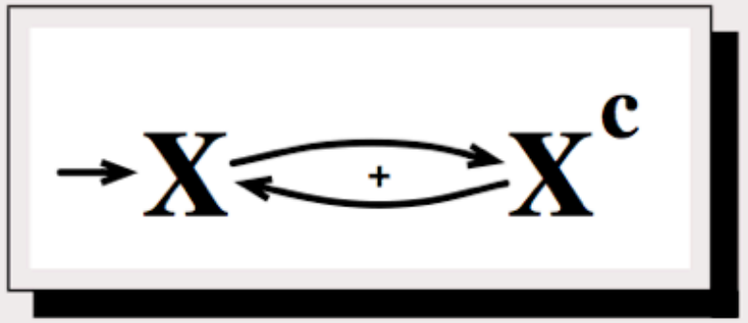
- $A \cdot 1 = A$
- $A \cdot n + 1 = (A \cdot n) + A$

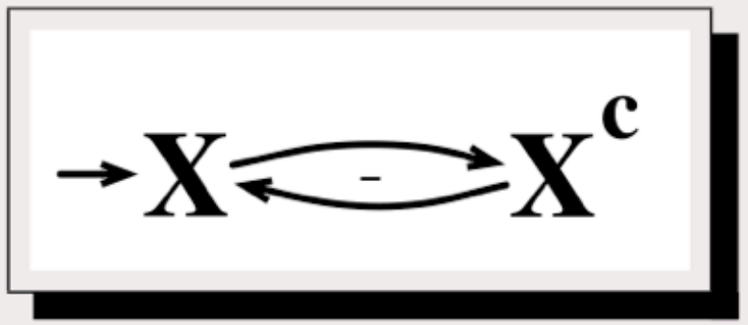
$$A \cdot n = \underbrace{A + A + \ldots + A}_{n}$$

The Multiplication by ω_1

- Design $X \cdot \omega_1$ s.t. $d^\circ\left(X \cdot \omega_1\right) = d^\circ(X) \cdot \omega_1$

Definition

1. 

2. 

The Wadge Hierarchy restricted to ω-regular languages

- ω-words
  - deterministic — (Büchi) $\rightsquigarrow \omega + 1$
  - non-deterministic — (Büchi) $\rightsquigarrow \omega^\omega$
- ω-trees
  - deterministic — $\rightsquigarrow (\omega^\omega)^3 + 3$
  - non-deterministic — $\rightsquigarrow$ *huge!*

Measure of strength : height of hierarchy

automata on words	ω^ω	[Wagner 79]
det. automata on trees	$(\omega^\omega)^3 + 3$	[Murlak 06]
det. pushdown automata	$(\omega^\omega)^\omega$	[D. 03]
det. Turing machines	$(\omega^{\mathsf{CK}})^\omega$	[Selivanov 03]
non-det. pushdown automata	Θ	[Finkel 01, ...]

Reprinted from INFORMATION AND CONTROL
Vol. 43, No. 2, November 1979
All Rights Reserved by Academic Press, New York and London
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER

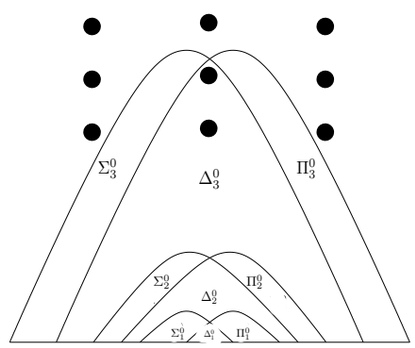
Section of Mathematics, Friedrich Schiller University, Jena, German Democratic Republic

The investigation of the acceptional power of finite automata with respect to several notions of acceptance for ω-sets, done in the literature, has exhibited six subclasses of the class of ω-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of ω-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the m-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

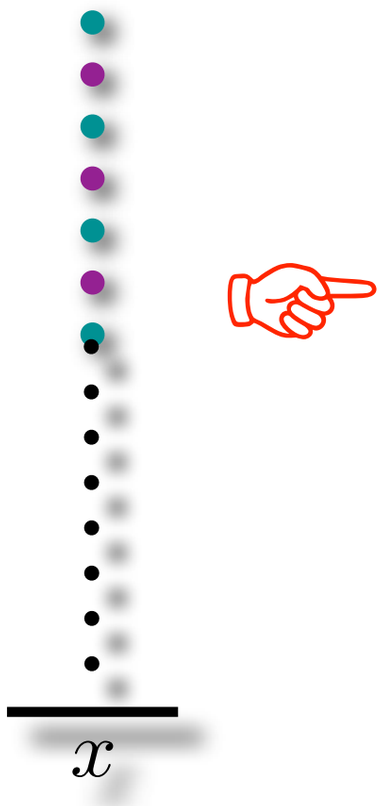


An example: evaluation game for 1st order logic

non determined game

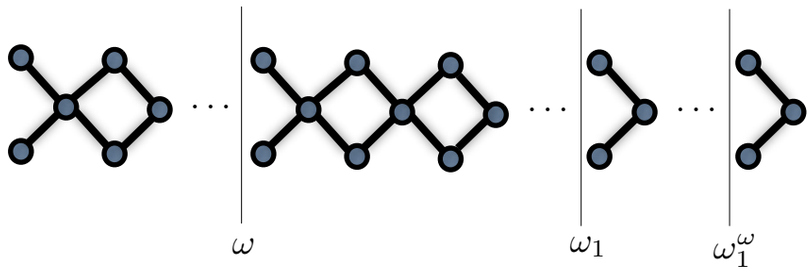


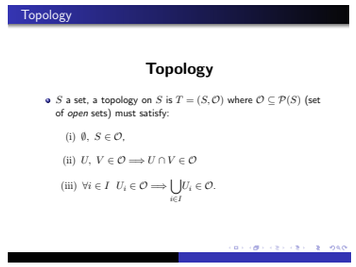
$\mathcal{G}(A)$



– A first approach: games that are essentially finite

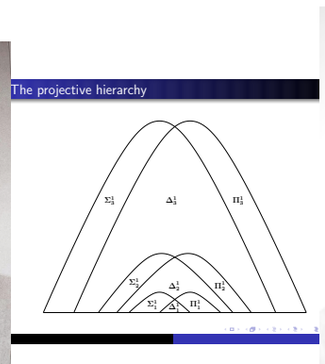
Games that construct mappings



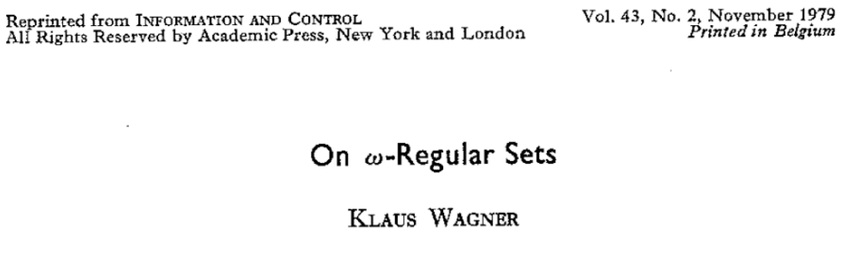










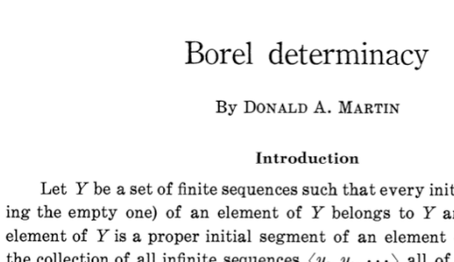


Reprinted from INFORMATION AND CONTROL
All Rights Reserved by Academic Press, New York and London

Vol. 43, No. 2, November 1979
Printed in Belgium

On ω-Regular Sets

KLAUS WAGNER



Annals of Mathematics, **102** (1975), 363–371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initia
ing the empty one) of an element of Y belongs to Y and
element of Y is a proper initial segment of an element of
the collection of all infinite sequences $\langle u_0, u_1, \ldots \rangle$ all of

Blackwell Games

Blackwell Games

Games with slightly imperfect information

$$\begin{array}{lccccccc} I & x_0 & & x_2 & & x_4 & & \ldots\ldots \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ II & x_1 & & x_3 & & x_5 & & \ldots\ldots \end{array} \Bigg\} x \in 2^\omega$$

- Blackwell strategy : $\sigma : 2^{even} \longrightarrow Prob(\{0,1\})$
- Given σ strategy for I, τ strategy for II
  - induces a probability measure on $s^\frown 2^\omega$
  - generates a Borel probability measure $\mu_{\sigma,\tau}$ on 2^ω

Blackwell games

- Given $A \subseteq 2^\omega$, σ for I, τ for II, the *Blackwell value of the strategies* are

$$val_I^A(\sigma) = \inf\left\{\mu^-_{\sigma,\tau}(A) : \tau \text{ Blackwell strategy for II}\right\}$$

$$val_{II}^A(\tau) = \sup\left\{\mu^+_{\sigma,\tau}(A) : \sigma \text{ Blackwell strategy for I}\right\}$$

- Given $A \subseteq X^\omega$, the *Blackwell value sets* are

$$V_I(A) = \left\{val_I^A(\sigma) : \sigma \text{ Blackwell strategy for I}\right\}$$

$$V_{II}(A) = \left\{val_{II}^A(\tau) : \tau \text{ Blackwell strategy for II}\right\}$$

- $V_{II}(A)$ *dominates* $V_I(A)$ iff

$$\forall\ v_{II} \in V_{II}(A), \forall\ v_I \in V_I(A), v_{II} \geq v_I$$

Blackwell games

- If $\inf V_{II}(A) = \sup V_I(A) = p$, the outcome of the game is stochastically determined:

  Both players can approximate the outcome that players I wins with probability p

- A is *Blackwell determined* if $\inf V_{II}(A) = \sup V_I(A)$

$$\sup_{\sigma} \inf_{\tau} \mu^{-}_{\sigma,\tau}(A) = \inf_{\tau} \sup_{\sigma} \mu^{+}_{\sigma,\tau}(A)$$

Blackwell games

Theorem (Martin 98)

$$ZF + DC \vdash \textit{Borel Blackwell determinacy}$$

Theorem (Martin)

Classwise, $Det(\Gamma) \Longrightarrow Blackwell\ Det(\Gamma)$

- $AD \Longrightarrow Blackwell\ Det$

Conjecture (Martin)

$$AD \Longleftrightarrow Blackwell\ Det$$

Theorem (Martin, Neeman, Vervoort 03)

$$(Blackwell\ Det)^{L(\mathbb{R})} \Longrightarrow (AD)^{L(\mathbb{R})}$$