

# Stochastic Pushdown Games

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## Outline

- Preliminaries.
- Infinite-state stochastic games and MDPs.
  - What is different from the finite-state case?
- Stochastic pushdown games.
- Stochastic BPA games and MDPs.
- Stochastic OC games and MDPs.
- Some open problems.

## Stochastic pushdown games (1)

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## PDA games

## Finite-state games

## Infinite-state games

## The value

## Optimal strategies

## Summary

## PDA Games

## Termination

## BPA games

## Termination

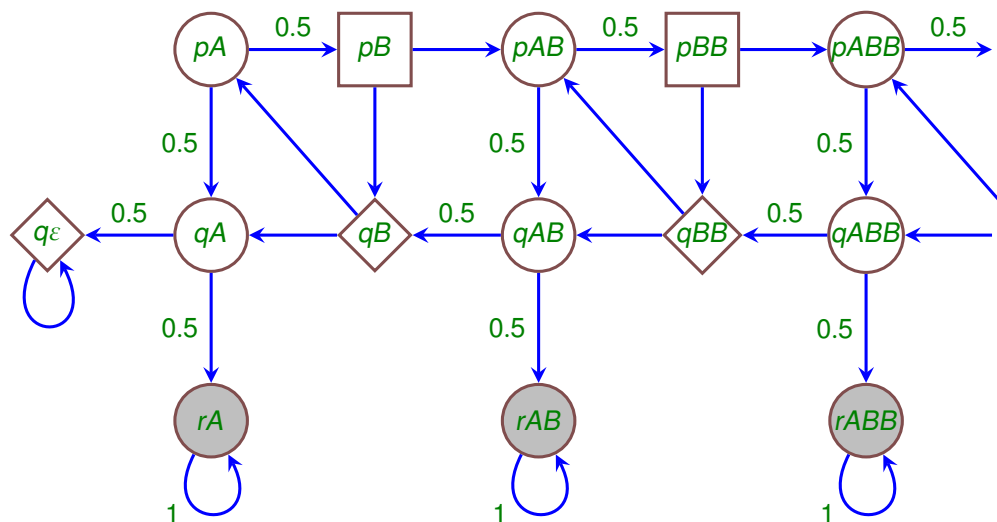
## Reachability

## OC Games

## probabilistic OC

$$Q = \{p, q, r\}, \Gamma = \{A, B\}, H_{\square} = \{pB\}, H_{\diamond} = \{qB\}, H_{\circ} = \{\dots\}$$

$$pA \xrightarrow{0.5} pB, \quad pB \rightarrow pAB, \quad qA \xrightarrow{0.5} rA, \quad qB \rightarrow qA, \quad rA \xrightarrow{1} rA, \\ pA \xrightarrow{0.5} qA, \quad pB \rightarrow qB, \quad qA \xrightarrow{0.5} q\varepsilon, \quad qB \rightarrow pA, \quad rA \xrightarrow{1} rB, \\ qA \xrightarrow{0.5} q\varepsilon, \quad qB \rightarrow pA, \quad rA \xrightarrow{1} rB$$



## Stochastic pushdown games (2)

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- In this lecture we only consider **reachability objectives**.
  - Player  $\square$  is the **maximizer**, player  $\diamond$  is the **minimizer**, player  $\circ$  is the **randomizer**.
- Two important subclasses of PDA games are the following:
  - **BPA games**. The control unit is trivial, i.e., all rules take the form  $X \rightarrow \alpha$  where  $\alpha \in \Gamma^*$ .
  - **One-counter games**. Here  $\Gamma = \{Z, l\}$ , all rules take the form  $pZ \rightarrow ql^k Z$  or  $pl \rightarrow ql^\ell$ , where  $k \in \{0, 1\}$  and  $\ell \in \{0, 1, 2\}$ .
- Stochastic PDA games are equivalent to **recursive stochastic games**, BPA games are equivalent to **1-exit RSG**.
- Closely related models include multi-type branching processes, stochastic grammars, backoff processes, quasi-birth-death processes, solvency games, etc.

Let  $\mathcal{G}$  be a game with **finitely many** vertices  $V$ , and let  $T \subseteq V$ .

- Each vertex  $v \in V$  has a **value**  $val(v)$  given by

$$\sup_{\sigma} \inf_{\pi} \mathcal{P}^{(\sigma, \pi)}(\text{Reach}(v, T)) = \inf_{\pi} \sup_{\sigma} \mathcal{P}^{(\sigma, \pi)}(\text{Reach}(v, T))$$

- Both players have fixed **MD** strategies that are **optimal** in every vertex.
- $val(v)$  is rational.

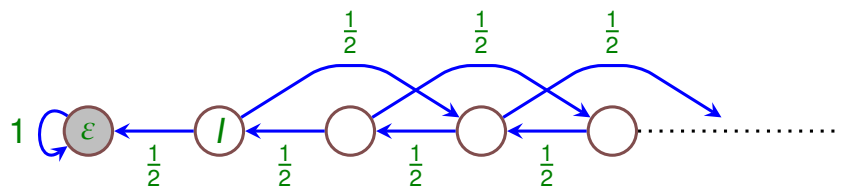
- For MDPs, the value and an optimal strategy are computable in **polynomial time**.
- For games, the value and an optimal strategies are computable in **polynomial space** (for a fixed number of stochastic vertices, there is a **polynomial time** algorithm).
- For games, the sets of all vertices with value **0/1** are computable in **polynomial time** together with strategies that are optimal in these vertices.
- For games, the problem whether  $val(v) > 0.5$  is in **NP**  $\cap$  **coNP**.

# Infinite-state games. The value.

## Theorem 1

Let  $\mathcal{G}$  be a game with *countably many* vertices  $V$ , and let  $T \subseteq V$ . Then every vertex has a value, which can be irrational even in stochastic BPA/OC games where all probabilities are equal to 0.5.

- The existence of  $\text{val}(v)$  follows immediately from Martin's theorem; a simple **direct** proof shows that the tuple of all  $\text{val}(v)$  is the least fixed-point of the Bellman functional  $\Gamma : [0, 1]^{|V|} \rightarrow [0, 1]^{|V|}$ .
- Consider a stochastic BPA (or OC) given by  $I \xrightarrow{0.5} \varepsilon, I \xrightarrow{0.5} III$ .



$\text{val}(I)$  is the **least** solution of  $x = \frac{1}{2} + \frac{1}{2}x^3$  in  $[0, 1]$ , i.e.,  $\frac{\sqrt{5}-1}{2}$

# Infinite-state games. Optimal strategies.

## Observation 2

Since (general) reachability games have a value, both players have  $\varepsilon$ -optimal strategies for every  $\varepsilon > 0$ . However, it is not immediately clear whether  $\varepsilon$ -optimal strategies need memory.

## Theorem 3

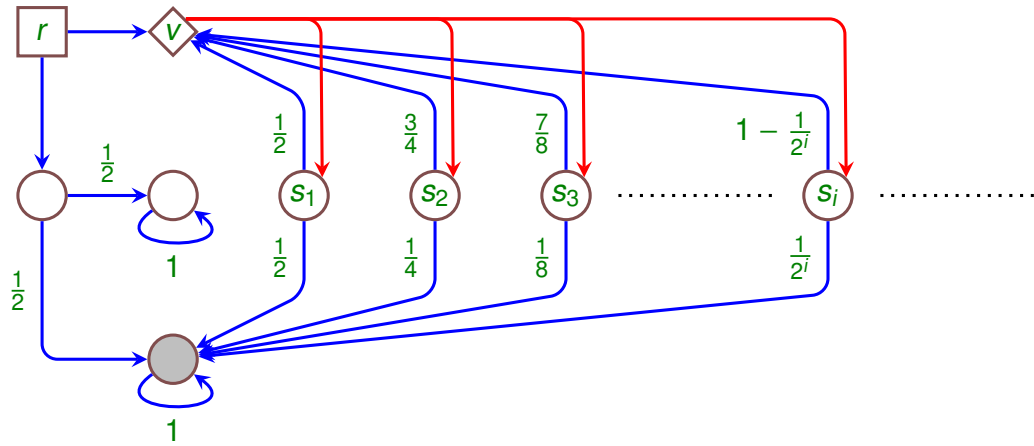
In every *finitely-branching* game, player  $\diamond$  (the minimizer) has an **MD** strategy which is optimal in every vertex.

- Player  $\diamond$  simply selects a successor vertex with the least value. This works also for infinitely-branching games if there is such a successor for every  $\diamond$ -vertex.
- Theorem 3 applies to stochastic pushdown games. But how about general infinitely-branching games?

# Infinite-state games. Optimal strategies (2).

## Theorem 4

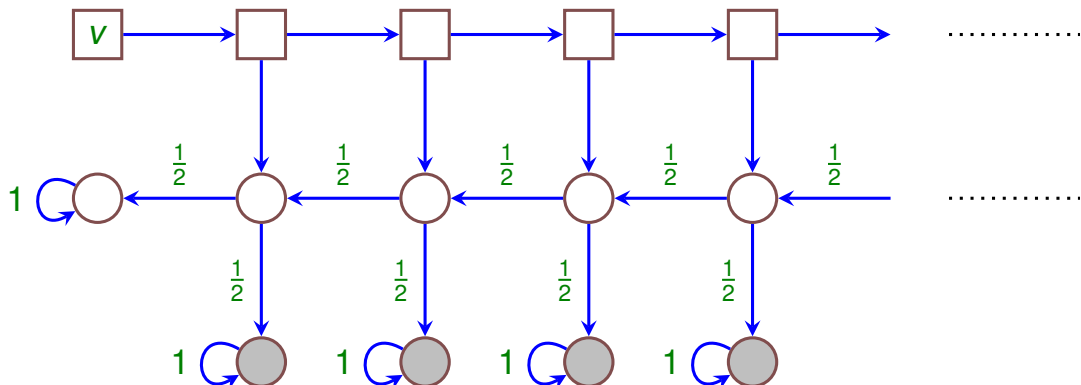
In an *infinitely-branching* game, player  $\diamond$  (the minimizer) does not necessarily have an optimal strategy. An  $\epsilon$ -optimal strategy for player  $\diamond$  may require infinite memory. An optimal strategy for player  $\diamond$  (if it exists) may also require infinite memory.



# Infinite-state games. Optimal strategies (3).

## Theorem 5

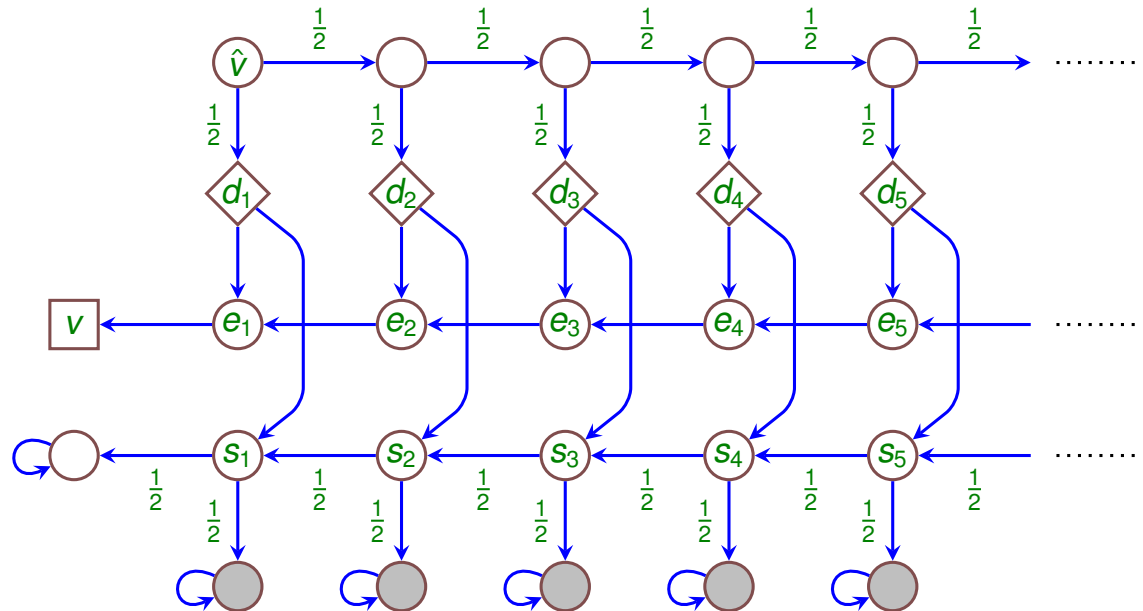
In a stochastic BPA/OC game (or even BPA/OC MDP), player  $\square$  (the maximizer) does not necessarily have an optimal strategy.



# Infinite-state games. Optimal strategies (4).

## Theorem 6

In a stochastic OC game, an optimal strategy for player  $\square$  (the maximizer) may require infinite memory.



# Infinite-state games. Summary.

- The value exists but may be irrational.
- $\varepsilon$ -optimal strategies, where  $\varepsilon > 0$ , exist for both players.
- **Minimizing strategies:**
  - In finitely-branching games, the minimizer has an MD strategy which is optimal in every vertex.
  - In infinitely-branching games, the minimizer does not necessarily have an optimal strategy. Both  $\varepsilon$ -optimal and optimal strategy may require infinite memory.
- **Maximizing strategies:**
  - In stochastic PDA (or even OC) games, the maximizer does not necessarily have an optimal strategy. An optimal strategy (if it exists) may require infinite memory.

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probabilistic OC

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## Theorem 7

Let  $\mathcal{G}$  be a stochastic PDA game,  $T$  be the set of all configurations with empty stack,  $pX$  a configuration of  $\mathcal{G}$ , and  $r \in (0, 1)$ . The following problems are *undecidable*:

- $\text{val}(pX) < r$  (this holds even for PDA MDPs);
- $\text{val}(pX) = 1$  (this holds even for PDA MDPs).

Further, the problem whether

- $\text{val}(pX) = 0$  is **EXPTIME**-complete (even for minimizing PDA MDPs);
- $\text{val}(pX) = 0$  is in **P** for maximizing PDA MDPs.

Finally, there is no algorithm for approximating  $\text{val}(pX)$  up to a given precision  $\frac{1}{2} - \delta$ , where  $\delta > 0$  (this holds even for maximizing PDA MDPs).

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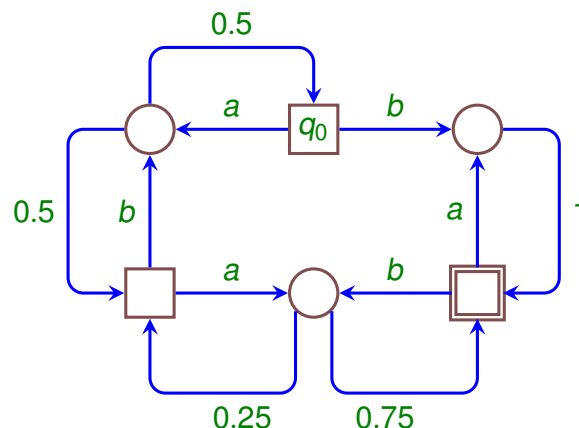
## BPA games

Termination  
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probabilistic OC

The main tool for the undecidability results are **probabilistic finite automata**:



- Given a PFA  $M$  and a threshold  $\lambda$ , the problem whether  $M$  accepts some word with probability greater than  $\lambda$  is **undecidable**.
- Stochastic PDA games and MDPs can simulate PFA (player  $\square$  pushes a word  $w$  onto the stack, and then the computation of  $M$  on  $w$  is simulated).

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## Theorem 8

Let  $\mathcal{G}$  be a stochastic BPA game, and  $T = \{\varepsilon\}$ .

- Each player has a fixed **SMD** strategy, constructible in polynomial space, which is optimal in every configuration.
- For BPA MDPs, the problem whether  $\text{val}(X) = 1$  is in **P**.
- For BPA games, the problems whether  $\text{val}(X) = 1$  and  $\text{val}(X) \leq \varrho$  are in **NP**  $\cap$  **coNP** and **PSPACE**, respectively.



# Stochastic BPA Games. Termination obj. (2)

- The tuple of all  $val(X)$ , where  $X \in \Gamma$ , forms the **least** solution of a system of recursive equations constructed as follows:

- If  $X \hookrightarrow \varepsilon$  and  $X \in \Gamma_{\square}$ , we put  $V_X = 1$ ;

- otherwise, we put

- $V_X = \max \{V_Y \mid X \hookrightarrow Y\}$

- $V_X = \min \{V_Y \mid X \hookrightarrow Y\}$

- $V_X = \sum_{X \xrightarrow{p} \varepsilon} p + \sum_{X \xrightarrow{p} Y} p \cdot V_Y + \sum_{X \xrightarrow{p} YZ} p \cdot V_Y \cdot V_Z$

depending on whether  $X \in \Gamma_{\square}$ ,  $X \in \Gamma_{\diamond}$ , or  $X \in \Gamma_{\circ}$ , respectively.

- The property  $val(X) \leq \varrho$  can be encoded as a closed formula of the existential fragment of in  $(\mathbb{R}, +, *, \leq)$ .

# Stochastic BPA Games. Reachability obj. (1)

Let  $\mathcal{G}$  be a stochastic BPA game and  $T \subseteq \Gamma^*$  a **regular** set of target configurations.

- Player  $\square$  (the maximizer) does not necessarily have an optimal strategy.
- Player  $\diamond$  (the minimizer) has an optimal MD strategy.
- Optimal strategies (if they exist) are not necessarily SMD.

## Stochastic BPA Games. Reachability obj. (2)

## Theorem 9

Consider the sets

- $\mathcal{W}_{\square}^{>0} = \{\alpha \in \Gamma^* \mid \exists \sigma \in \Sigma_{\square} : \forall \pi \in \Sigma_{\diamond} : \mathcal{P}_{\alpha}^{\sigma, \pi}(\text{reach}(T)) > 0\}$
- $\mathcal{W}_{\square}^{=1} = \{\alpha \in \Gamma^* \mid \exists \sigma \in \Sigma_{\square} : \forall \pi \in \Sigma_{\diamond} : \mathcal{P}_{\alpha}^{\sigma, \pi}(\text{reach}(T)) = 1\}$
- $\mathcal{W}_{\diamond}^{=0} = \{\alpha \in \Gamma^* \mid \exists \pi \in \Sigma_{\diamond} : \forall \sigma \in \Sigma_{\square} : \mathcal{P}_{\alpha}^{\sigma, \pi}(\text{reach}(T)) = 0\}$
- $\mathcal{W}_{\diamond}^{<1} = \{\alpha \in \Gamma^* \mid \exists \pi \in \Sigma_{\diamond} : \forall \sigma \in \Sigma_{\square} : \mathcal{P}_{\alpha}^{\sigma, \pi}(\text{reach}(T)) < 1\}$

Then

- these sets are **regular**,  $\mathcal{W}_{\square}^{>0} = \Gamma^* \setminus \mathcal{W}_{\diamond}^{=0}$ , and  $\mathcal{W}_{\square}^{=1} = \Gamma^* \setminus \mathcal{W}_{\diamond}^{<1}$ ;
- winning strategies for both players are **regular**;
- DFA representing  $\mathcal{W}_{\square}^{>0}$ ,  $\mathcal{W}_{\diamond}^{=0}$  are computable in **polynomial time**. The same applies to the winning strategies;
- DFA representing  $\mathcal{W}_{\square}^{=1}$ ,  $\mathcal{W}_{\diamond}^{<1}$  are computable in polynomial time using **NP**  $\cap$  **coNP** oracle. The same applies to the winning strategies.

## BPA MDPs. Reachability obj. (1)

- Let  $\Delta = (\Gamma, \hookrightarrow, (\Gamma_{\square}, \Gamma_{\diamond}), \text{Prob})$  be a BPA Markov decision process, and  $T \subseteq \Gamma^*$  a regular set of target configurations.
- We can safely assume that  $T = \mathcal{R}\Gamma^*$ , where  $\mathcal{R} \subseteq \Gamma$ .
- Consider the sets
  - $\mathcal{W}^{>0} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}_{\alpha}^{\sigma}(\text{reach}(T)) > 0\}$
  - $\mathcal{W}^{=0} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}_{\alpha}^{\sigma}(\text{reach}(T)) = 0\}$
  - $\mathcal{W}^{=1} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}_{\alpha}^{\sigma}(\text{reach}(T)) = 1\}$
  - $\mathcal{W}^{<1} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}_{\alpha}^{\sigma}(\text{reach}(T)) < 1\}$
- We show that these sets are **regular**, the corresponding winning strategies are also **regular**, and the associated DFA are computable in **polynomial time**.

## BPA MDPs. Reachability obj. (2)

## Theorem 10

The set  $\mathcal{W}^{>0}$  is effectively regular.

- Let us consider two sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid X \hookrightarrow^* \varepsilon\}$
  - $\mathcal{B} = \{X \in \Gamma \mid X \hookrightarrow^* R\beta, \text{ where } R \in \mathcal{R} \text{ and } \beta \in \Gamma^*\}$
- We have that  $\mathcal{W}^{>0} = \mathcal{A}^* \mathcal{B} \Gamma^*$ .
- The sets  $\mathcal{A}$  and  $\mathcal{B}$  are (easily) computable in polynomial time.

## BPA MDPs. Reachability obj. (3)

## Theorem 11

The set  $\mathcal{W}^{=0}$  is effectively regular.

- Let us consider two sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow T_\varepsilon)) = 0\}$
  - $\mathcal{B} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow T)) = 0\}$
- We have that  $\mathcal{W}^{=0} = \mathcal{B}^* \cup \mathcal{B}^* \mathcal{A} \Gamma^*$ .
- The pair  $(\mathcal{A}, \mathcal{B})$  is the **greatest** fixed-point of a suitably defined  $\Theta : 2^\Gamma \times 2^\Gamma \rightarrow 2^\Gamma \times 2^\Gamma$ .

## BPA MDPs. Reachability obj. (4)

## Theorem 12

The set  $\mathcal{W}^{=1}$  is effectively regular.

- Let us consider three sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow T)) = 1\}$
  - $\mathcal{B} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow T_\varepsilon)) = 1, \mathcal{P}(\text{reach}(X \rightarrow T)) > 0\}$
  - $\mathcal{C} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow \varepsilon)) = 1\}$
- We have that  $\mathcal{W}^{=1} = (\mathcal{B} \cup \mathcal{C})^* \mathcal{A} \Gamma^*$ .
- Moreover, the set  $\mathcal{W}_\varepsilon^{=1} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(\alpha \rightarrow T_\varepsilon)) = 1\}$  is equal to  $(\mathcal{B} \cup \mathcal{C})^* \cup (\mathcal{B} \cup \mathcal{C})^* \mathcal{A} \Gamma^*$ .
- The set  $\mathcal{C}$  is computable in polynomial time by applying the results for termination objectives.
- The sets  $\mathcal{A}, \mathcal{B}$  are again computable as the greatest fixed-point of a suitably defined  $\Theta : 2^\Gamma \times 2^\Gamma \rightarrow 2^\Gamma \times 2^\Gamma$ .

## BPA MDPs. Reachability obj. (5)

## Theorem 13

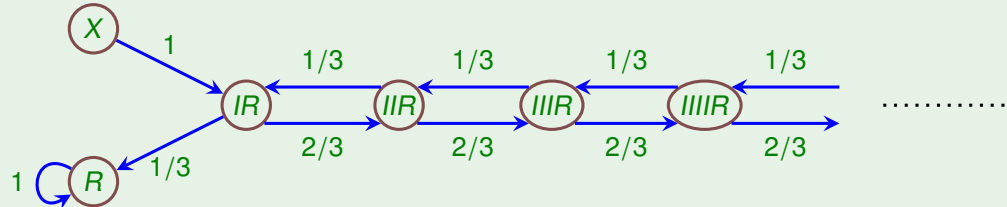
The set  $\mathcal{W}^{<1}$  is effectively regular.

- Let us consider two sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow \varepsilon)) > 0\}$
  - $\mathcal{B} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}^\sigma(\text{reach}(X \rightarrow T_\varepsilon)) < 1\}$
- We have that  $\mathcal{W}^{<1} = \mathcal{A}^* \cup (\mathcal{A}^* \mathcal{B} \Gamma^*)$ .
- The set  $\mathcal{A}$  is computable by the previous results.
- The challenge is to compute the set  $\mathcal{B}$ . The membership  $X \in \mathcal{B}$  is witnessed in **two** ways.
  - There is strategy  $\sigma$  such that  $\mathcal{P}^\sigma(\text{reach}(X \rightarrow \mathcal{W}_\varepsilon^{=0})) > 0$ .
  - There is a “closed” family of stack symbols disjoint with  $\mathcal{R}$  (this family forms a BPA MDP  $\Delta'$ ) such that  $X$  can be forced to enter  $\Delta'$  with a positive probability, and there is a strategy  $\sigma$  in  $\Delta'$  such that  $\mathcal{P}^\sigma(\text{reach}(X \rightarrow \varepsilon)) < 1$ .

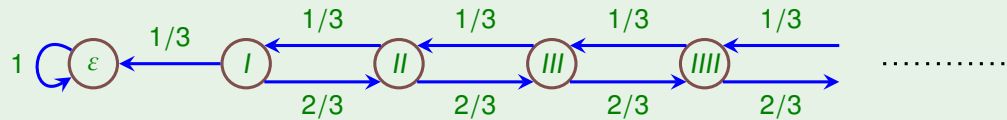
# BPA MDPs. Reachability obj. (6)

## Example 14

- Let  $\Gamma = \{X, R, I\}$ , where  $\Gamma_{\square} = \emptyset$ ,  $\Gamma_{\circ} = \{X, R, I\}$ , and  
 $X \xrightarrow{1} IR$ ,  $R \xrightarrow{1} R$ ,  $I \xrightarrow{2/3} II$ ,  $I \xrightarrow{1/3} \varepsilon$
- Let  $\mathcal{R} = \{R\}$ . Then  $\mathcal{A} = \emptyset$ ,  $\mathcal{B} = \{X, I\}$ , hence  $\mathcal{W}^{<1} = \{X, I\}\Gamma^*$ .



- The membership  $I \in \mathcal{B}$  is witnessed by a BPA MDP  $\Delta'$  where  
 $\Gamma' = \Gamma'_{\circ} = \{I\}$  and  $I \xrightarrow{2/3} II$ ,  $I \xrightarrow{1/3} \varepsilon$ .



# Stochastic PDA/BPA games and MPDs. Literature.

- K. Etessami, M. Yannakakis. *Efficient Qualitative Analysis of Classes of Recursive Markov Decision Processes and Simple Stochastic Games*. Proc. STACS 2006, pp. 634–645, LNCS 3884, Springer 2006.
- T. Brázdil, V. Brožek, V. Forejt, and A. Kučera, and J. Obdržálek. *Reachability in Recursive Markov Decision Processes*. Proc. CONCUR 2006, pp. 358-374, LNCS 4137. (Also Inf.&Comp., 206(5):520-537, 2008).
- T. Brázdil, V. Brožek, A. Kučera, and J. Obdržálek. *Qualitative Reachability in Stochastic BPA Games*. Proc. STACS 2009, pp. 207–218, 2009. (Also Inf.&Comp., 209(8):1160-1183, 2011.).

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- **One-counter automaton (OC)** is a PDA with two stack symbols  $\{I, Z\}$ . All rules are of the form
  - $pZ \rightarrow qI^k Z$ , where  $k \in \{0, 1\}$ ;
  - $pI \rightarrow qI^k$ , where  $k \in \{0, 1, 2\}$ .

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**Theorem 15**

Let  $\mathcal{G}$  be a maximizing OC MDP,  $T = \{p(0) \mid p \in Q\}$ . The problem whether  $\text{val}(q(i)) = 1$  is in **P**. If  $\text{val}(q(i)) = 1$ , then player  $\square$  has an optimal CMD strategy computable in polynomial time.

**Theorem 16**

Let  $\mathcal{G}$  be a maximizing OC MDP,  $T = \{p(0) \mid p \in Q\}$ , where  $Q' \subseteq Q$ . Let  $q(i)$  be a configuration of  $\mathcal{G}$ . The problem whether player  $\square$  has a strategy  $\sigma$  such that  $\mathcal{P}^\sigma(\text{reach}(q(i)) \rightarrow T) = 1$  is **PSPACE-hard** and in **EXPTIME**. The strategy  $\sigma$  (if it exists) is computable in exponential time.

# Results about stochastic OC games

## Theorem 17

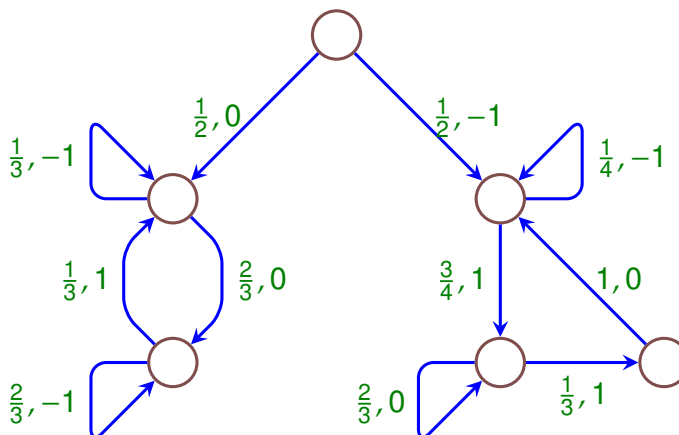
Let  $\mathcal{G}$  be a stochastic OC game,  $T = \{p(0) \mid p \in Q\}$ . The problem whether  $\text{val}(q(i)) = 1$  is in  $\mathbf{NP} \cap \mathbf{coNP}$ . If  $\text{val}(q(i)) = 1$ , then player  $\square$  has an optimal strategy computable in polynomial time using  $\mathbf{NP} \cap \mathbf{coNP}$  oracle.

## Theorem 18

Let  $\mathcal{G}$  be a stochastic OC game,  $T = \{p(0) \mid p \in Q\}$ . Then  $\text{val}(q(i))$  can be effectively approximated up to an arbitrarily small additive error  $\varepsilon > 0$ , and  $\varepsilon$ -optimal strategies for both players are effectively computable. The algorithms run in non-deterministic time which is exponential in the size of  $\mathcal{G}$  an polynomial in  $\log(1/\varepsilon)$  and  $\log(i)$ .

# Stochastic OC machines

- The analysis of OC games and MDPs is based on understanding the underlying stochastic process, particularly its “positive subprocess”.



- $change(s) = \sum_{s \xrightarrow{x, \delta} t} x \cdot \delta$
- $trend(C) = \sum_{s \in C} \mu(s) \cdot change(s)$

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- A **martingale** is a stochastic process  $m^{(0)}, m^{(1)}, m^{(2)} \dots$  such that
  - $\mathbb{E}(|m^{(i)}|) < \infty$  for all  $i \geq 0$ ;
  - $\mathbb{E}(m^{(i+1)} \mid m^{(0)}, \dots, m^{(i)}) = m^{(i)}$  almost surely.
- We have that  $\mathbb{E}(m^{(i)}) = \mathbb{E}(m^{(0)})$ .

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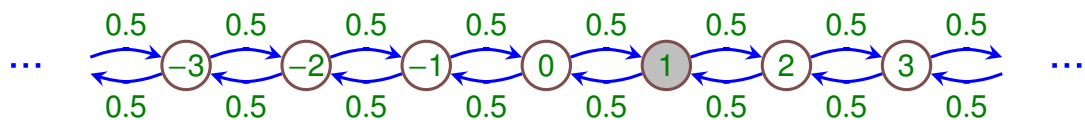
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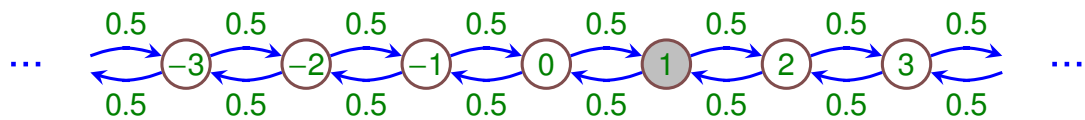
probabilistic OC



- Let  $m^{(i)} : Run(1) \rightarrow \mathbb{N}_0$  be a r.v. assigning to each  $w \in Run(1)$  the value  $w(i)$ .
- $\mathbb{E}(m^{(i+1)} \mid m^{(i)}=k) = \frac{1}{2}(k+1) + \frac{1}{2}(k-1) = k$ .

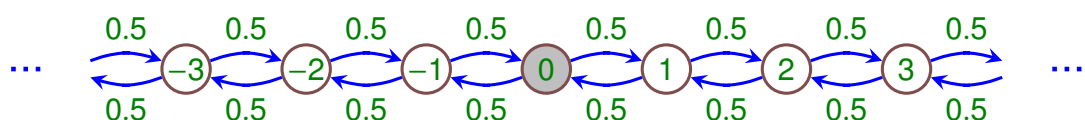


## Martingales. First example (2).



- What is the probability  $p$  of visiting the state  $k > 1$  without visiting the state 0 before?
- Optional stopping time theorem: Let  $m^{(0)}, m^{(1)}, \dots$  be a martingale (over a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ ), such that  $|m^{(i)} - m^{(i-1)}| < c$  for some fixed  $c \in \mathbb{N}$ . Let  $X : \Omega \rightarrow \mathbb{N}_0$  a r.v. where  $\mathbb{E}(X) < \infty$ . Then  $\mathbb{E}(m^{(X)}) = \mathbb{E}(m^{(0)})$ .
- For every  $w \in \text{Run}(1)$ , let  $X(w)$  be the least index  $\ell$  such that  $w(\ell) = 0$  or  $w(\ell) = k$  (if there is no such  $\ell$ , then  $X(w) = 5$ ).
- By optional stopping theorem,  $\mathbb{E}(m^{(X)}) = 1$ . Now observe  $\mathbb{E}(m^{(X)}) = p \cdot k + (1 - p) \cdot 0$ . Hence,  $p = 1/k$ .

## Martingales. First example (3).



- What is the probability  $p$  that after performing  $k$  steps we end up outside the interval  $(-\frac{k}{2}, \frac{k}{2})$ ?
- Azuma's inequality: Let  $|m^{(i)} - m^{(i-1)}| < c_i$  almost surely. Then

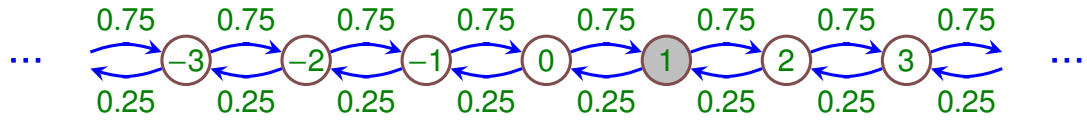
$$\mathcal{P}(|m^{(N)} - m^{(0)}| \geq t) \leq 2 \exp\left(\frac{-t^2}{\sum_{i=1}^N c_i^2}\right)$$

- In our case,  $|m^{(i)} - m^{(i-1)}| \leq 1$ ,  $N = k$ ,  $t = \frac{k}{2}$ . Thus, we obtain

$$p \leq 2a^k$$

where  $a = \exp(\frac{-1}{8}) < 1$ .

# Martingales. Second example.



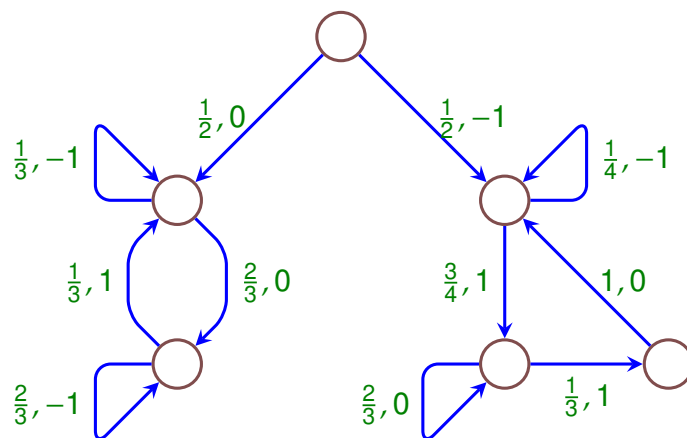
- Let  $m^{(i)} : Run(1) \rightarrow \mathbb{N}_0$  be a r.v. assigning to each  $w \in Run(1)$  the value  $w(i)$ .
- $\mathbb{E}(m^{i+1} | m^{(i)}=k) = \frac{3}{4}(k+1) + \frac{1}{4}(k-1) = k + \frac{1}{2} \neq k$ .

A slight modification helps:

- Let  $m^{(i)} : Run(1) \rightarrow \mathbb{N}_0$  be a r.v. assigning to each  $w \in Run(1)$  the value  $w(i) - i \cdot \frac{1}{2}$ .
- $\mathbb{E}(m^{i+1} | m^{(i)}=k) = \frac{3}{4}(k + \frac{i}{2} + 1 - \frac{i+1}{2}) + \frac{1}{4}(k + \frac{i}{2} - 1 - \frac{i+1}{2}) = k$

# Martingales. Third example (1).

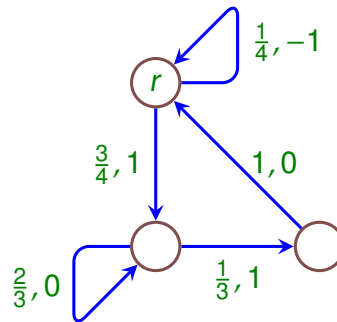
Let us consider again the “positive subprocess” of a given probabilistic OC.



- $change(s) = \sum_{s \rightarrow t} x_{s,t} \cdot \delta$
- $trend(C) = \sum_{s \in C} \mu(s) \cdot change(s)$

## Martingales. Third example (2).

Now consider a given BSCC  $C$ .



- If  $change(s) = trend(C)$  for every  $s \in C$ , we can setup a martingale over  $Run(r(\ell))$  as follows:

$$m(i) = counter^{(i)} - i \cdot trend(C)$$

- Otherwise, the difference among the individual control states can be “compensated” by suitable constants  $v_s$ , where  $s \in C$ .

$$m(i) = counter^{(i)} - i \cdot trend(C) + v_{s(i)}$$

## Stochastic OC games. Literature.

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## Preliminaries

PDA games

Finite-state games

Infinite-state  
games

The value

Optimal strategies

Summary

## PDA Games

Termination

## BPA games

Termination

Reachability

## OC Games

probabilistic OC

# Some open problems.

- Given a configuration  $\alpha$  of a BPA/OC game, decide if  $val(\alpha) = 1$ .
- Given a configuration  $\alpha$  of a BPA game, approximate  $val(\alpha)$  upto a given precision.
- Can we characterize/compute optimal strategies for the minimizer in stochastic BPA games?
- Can we decide whether the maximizer has an optimal strategy in a given BPA/OC configuration?
- How about more general objectives?